Nerode's Theorem

A binary relation $R \subseteq S \times S$ is an <u>equivalence relation</u> iff

(1) $\forall s \in S \; [s R s]$

(2) $\forall s,t \in S \; [s R t \Rightarrow t R s]$

(3) $\forall r,s,t \in S \; [r R s \;\&\; s R t \Rightarrow r R t]$

An equivalence relation $R \subseteq S \times S$ partitions S into disjoint subsets called <u>equivalence classes</u>. Two elements elements $a, b \in S$ are in the same equivalence class iff $a R b$. The equivalence class of a will be denoted by $[a]$, i.e. $[a] = \{b \mid b R a\}$. For any a and b in S, either $[a] = [b]$ or $[a]$ and $[b]$ are disjoint. The <u>index</u> of an equivalence relation is the number of equivalence classes.

example : $R = \{(x, y) \in \mathbb{N}^2 \mid x$ and y have the same remainder when divided by $3\}$. Is R an equivalence relation? what are the equivalence classes? What is the index of R?

Let E be an equivalence relation on Σ^*, then E is *right invariant* iff $\forall u, v, w\, [\, u E v \Rightarrow uw E vw\,]$.

Theorem The following statements are equivalent:

(1) L is accepted by a DFA;

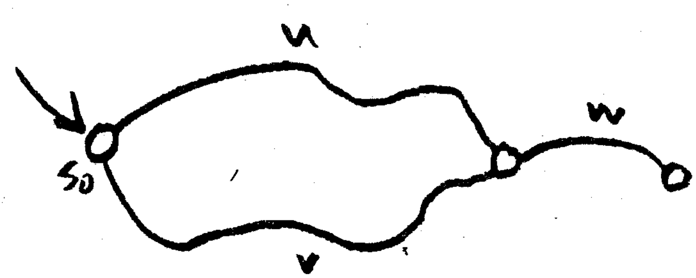
(2) L is the union of some equivalence classes of a right invariant equivalence relation of finite index.

(3) Let the equivalence relation R be defined by $x R y$ iff for all $z \in \Sigma^*$

$xz \in L$ iff $yz \in L$. Then R is of finite index.

proof

(1) $\Rightarrow$ (2) Let $L = T(M)$ where $M = (S, \Sigma, \delta, s_0, F)$. Define an equivalence relation E on Σ^* by $u E v$ iff $\delta(s_0, u) = \delta(s_0, v)$. Now show that E is right invariant.



$$\delta(s_0, u) = \delta(s_0, v) \Rightarrow \delta(s_0, uw) = \delta(s_0, vw).$$

Hence $u E v \Rightarrow uw\, E\, vw$.

What are the equivalence classes of E?

Why does E have finite index?

Show that $L = \bigcup_{t \in F} \{ w \mid \delta(s_0, w) = t \}$

(2) $\Rightarrow$ (3) Any E satisfying (2) must be a <u>refinement</u> of R, i.e. each equivalence class of E is contained entirely within some equivalence class of R. Consequently, if E has finite index so does R.

(3) $\Rightarrow$ (1) Define a DFA M' as follows:

$$M' = (\{ [w] \mid w \in \Sigma^* \}, \Sigma, [\lambda], \delta, \{ [w] \mid w \in L \})$$

Let $\delta([w], x) = [wx]$. Show $T(M') = L$.

<u>Reduced Automata</u>

<u>Theorem</u>: M' constructed in (3) $\Rightarrow$ (1) above is the DFA with the minimum number of states such that $T(M') = L$.

proof Given M such that $T(M) = L$. construct E as in the proof of (1) $\Rightarrow$ (2) above. From the proof of (2) $\Rightarrow$ (3) we know that E is a refinement of R.

Theorem If M and M' are two minimal finite automata accepting the same language L, then $|S| = |S'|$ and M' can be obtained from M by simply relabelling the states of M (M & M' are isomorphic).

proof

How can I determine which state of M' corresponds to a given state of M?