NP complete problems

$P = \{ L \mid$ there is a DTM M and a polynomial $p(n)$ such that M is of time complexity $p(n)$ and $\mathcal{L}(M) = L \}$

$NP = \{ L \mid$ " NDTM " $\}$

A Language $L_0 \in NP$ is <u>NP-complete</u> if the following condition is satisfied:
If we are given a deterministic algorithm of time complexity $T(n) \geq n$ to recognize L_0, then for every language $L \in NP$ we can effectively find a deterministic algorithm of time complexity $T(p_L(n))$ where p_L is a polynomial

that depends on L. We say L is <u>reducible</u> to L_0.

A language L is <u>polynomially transformable</u> to L_0 iff there is a deterministic polynomial-time bounded Turing machine M which will convert each string w in the alphabet of L into a string w_0 in the alphabet of L_0 such that $w \in L$ iff w_0 is in L_0.

If L is polynomially transformable to L_0 and there is a polynomial algorithm for L_0, then there is also one for L.

<u>Theorem</u> The problem of determining whether a boolean formula in CNF (conjunctive normal form) is <u>satisfiable</u> is NP-complete.

CNF: $(x_1 + x_2)(x_2 + \bar{x}_1 + \bar{x}_3)$

"product of sums"

<u>proof</u>

Let $L \in NP$ and let M be a non-deterministic polynomially time bdd. Turing machine that accepts L. Let w be some input tape of M. We will show how to polynomially transform the pair (M, w) into a boolean formula $\mathcal{F}$ such that $w \in T(M)$ iff $\mathcal{F}$ is satisfiable.

States of M : $q_1, q_2, \ldots q_s$

tape symbols: $X_1, X_2, \ldots X_m$

$p(n)$ - time complexity of M

$|w| = n$

If M accepts w, then there must be a sequence of ID's $I_0, I_1, \ldots I_q$ such that

1. I_0 is the initial ID.
2. I_q is the final accepting ID.

3. $I_j \Rightarrow I_{j+1}$ corresponds to a valid move
4. $q \le p(n)$
5. No ID has more than $p(n)$ tape symbols.

Proposositions used in boolean formula

1. c_{ijt} $1 \le i \le p(n)$, $1 \le j \le m$, $0 \le t \le p(n)$

   c_{ijt} will be true iff the i^{th} cell on M's input tape contains X_j at time t.

2. S_{kt} $1 \le k \le s$, $0 \le t \le p(n)$

   S_{kt} will be true iff M is in state q_k at time t.

3. H_{it} $1 \le i \le p(n)$, $0 \le t \le p(n)$

H_{it} will be true if the tape head of M is scanning cell i at time t.

Total number of propositions is $O(p^2(n))$.

Exactly one of $x_1, \ldots, x_K$

$U(x_1, \ldots x_K) = (x_1 + x_2 + \ldots + x_K) \prod_{i \neq j} (\neg x_i + \neg x_j)$

length of $U(x_1, \ldots x_K) = O(K^2)$.

a. The tape head is scanning exactly one symbol in each ID:

$A = A_0 A_1 A_2 \ldots A_{p(n)}$ where

$A_t = U(H_{1t}, H_{2t}, \ldots H_{p(n)t})$

Note that $|A| = O(p^3(n))$.

b. Each tape cell contains exactly one symbol at each time:

$$B = \prod_{i,t} B_{it} \quad \text{where}$$

$$B_{it} = U(c_{i1t}, c_{i2t}, \ldots c_{imt})$$

Note that $|B| = O(p^2(n))$.

c. M is in exactly one state at time t:

$$C = C_0 C_1 \ldots C_{p(n)} \quad \text{where}$$

$$C_t = U(S_{1t}, S_{2t}, \ldots S_{st})$$

Note that $|C| = O(p(n))$

d. At most one tape cell can change at each time:

$$D = \prod_{ijt} \left(\left(C_{ijt} \equiv C_{ij(t+1)} \right) + H_{it} \right)$$

$(x \equiv y) + z$ is not in CNF.

$$((x \rightarrow y) \wedge (y \rightarrow x) + z)$$

$$((x \rightarrow y) + z)((y \rightarrow x) + z)$$

$$(\bar{x} + y + z)(\bar{y} + x + z)$$

Hence, $|D| = O(p^2(n))$

e. Transitions satisfy next move function δ of M:

$$E = \prod_{ijkt} E_{ijkt}$$

$$= \left[\left(c_{ijt} \wedge H_{it} \wedge S_{kt} \right) \Rightarrow \sum_{\ell} c_{ij_\ell(t+1)} S_{k_\ell(t+1)} H_{i_\ell(t+1)} \right]$$

$$E_{ijkt} = \neg c_{ijt} + \neg H_{it} + \neg S_{kt} + \sum_{\ell} c_{ij_\ell(t+1)} S_{k_\ell(t+1)} H_{i_\ell(t+1)}$$

Machine may be nondeterministic.

$$\delta(q_k, x_j) = \{ (q_{k_1}, x_{j_1}, d_1), \ldots, (q_{k_r}, x_{j_r}, d_r) \}$$

and $i_\ell = i + d_\ell$ where $d_\ell = \begin{cases} 0 & \text{stay} \\ -1 & \text{move left} \\ +1 & \text{move right} \end{cases}$

Can show that $|E| = O(p^3(n))$

f. The initial conditions are satisfied:

$$F = S_{10} H_{10} \prod_{1 \le i \le n} c_{ij_i 0} \prod_{n < i \le p(n)} c_{i10}$$

X_1 is the blank symbol,

$w_i = X_{j_i}$ for $1 \le i \le n$

Note that $|F| = O(p(n))$

g. M eventually enters its accepting state

$G = S_{s\,p(n)}$

(we assume that M has been modified so that once it enters its accepting state, it stays there)

$\mathcal{F} = ABCDEFG$

Clearly, $|\mathcal{F}| = O(p^4(n))$

Note also that $\mathcal{F}$ is in CNF so we obtain the result that satisfiability is NP complete for boolean expressions in CNF. □