

15414/614 Optional Lecture 1: Propositional Logic

Qinsi Wang

Logic is the study of information encoded in the form of logical sentences. We use the language of Logic to state observations, to define concepts, and to formalize theories. We use logical reasoning to derive conclusions from existing information. We use logical proofs to convince others of these conclusions. The goal of logic in computer science is to develop languages to model the situations we encounter as computer science professionals, in order to reason about them formally.

1 Introduction

Propositional Logic is concerned with propositions and their interrelationships. Propositions are declarative sentences, which one can, in principle, argue as being true or false.

Problem 1. Among the following sentences, which are propositions?

- The sum of the numbers 3 and 5 equals 8.
- Ready, go!
- May fortune come your way.
- Every even natural number >2 is the sum of two prime numbers.

2 Syntax

2.1 Atomic Propositions

In Propositional Logic, there are two types of declarative sentences - simple sentences and compound sentences. Simple sentences express simple facts about the world. Compound sentences express logical relationships between the simpler sentences of which they are composed.

Simple sentences in Propositional Logic are often called atomic propositions. For example, "5 is a prime" and "program P terminates".

2.2 Logical Connectives

Propositional formulas are constructed from atomic propositions by using logical connectives.

- \neg ...not [negation]
- \wedge ...and [conjunction]
- \vee ...or [disjunction]
- \rightarrow ...implies [implication / conditional]
- \leftrightarrow ...is equivalent [Equivalence / biconditional]

The \neg operator has higher precedence than \wedge ; \wedge has higher precedence than \vee ; and \vee has higher precedence than \rightarrow and \leftrightarrow . \rightarrow and \leftrightarrow are right-associative.

2.3 Well-formed Propositional Formulas

The well-formed formulas of propositional logic are obtained by using the construction rules below:

- An atomic proposition ϕ is a well-formed formula.
- If ϕ is a well-formed formula, then so is $\neg\phi$.
- If ϕ and ψ are well-formed formulas, then so are $\phi \wedge \psi$, $\phi \vee \psi$, $\phi \rightarrow \psi$, and $\phi \leftrightarrow \psi$.
- If ϕ is a well-formed formula, then so is (ϕ) .
- A formula results from a finite number of applications of the above rules.

3 Semantics

3.1 Truth Assignments

The meaning or semantics of a formula determines its interpretation. Given the truth values of all atomic propositions in a formula, it can be evaluated to determine its truth value (True or False).

A truth assignment is a mapping that associates a truth value with each of the atomic propositions.

3.2 Truth Tables

A truth table shows whether a propositional formula is true or false for each possible truth assignment. If we know how the five basic logical connectives work, it is easy (in principle) to construct a truth table.

X	$\neg X$
0	1
1	0

X	Y	$X \wedge Y$	$X \vee Y$	$X \rightarrow Y$	$X \Leftrightarrow Y$
0	0	0	0	1	1
0	1	0	1	1	0
1	0	0	1	0	0
1	1	1	1	1	1

- If the truth value of a sentence is 1, the truth value of its negation is 0. If the truth value of a sentence is 0, the truth value of its negation is 1.
- The truth value of a conjunction is true if and only if the truth value of its conjuncts are both true; otherwise, the truth value is false.
- The truth value of a disjunction is true if and only if the truth value of at least one its conjuncts is true; otherwise, the truth value is false. Note that this is the inclusive or interpretation of the \vee operator and is differentiated from the exclusive or interpretation in which a disjunction is true if and only if an odd number of its disjuncts are false.
- The truth value of an implication is false if and only if its antecedent is true and its consequent is false; otherwise, the truth value is true. This is called material implication.
- An equivalence is true if and only if the truth values of its constituents agree, i.e. they are either both true or both false.

For example, a more complex truth table. Let S be the formula $((A \wedge B) \rightarrow C) \wedge (A \rightarrow B) \rightarrow (A \rightarrow C)$. To construct the truth table for S we must consider all possible truth assignments for A , B , and C . In this case there are $2^3 = 8$ such truth assignments. Hence, the table for S will have 8 rows.

A	B	C	$(A \wedge B) \rightarrow C$	$A \rightarrow B$	$A \rightarrow C$	S
0	0	0	1	1	1	1
0	0	1	1	1	1	1
0	1	0	1	1	1	1
0	1	1	1	1	1	1
1	0	0	1	0	0	1
1	0	1	1	0	1	1
1	1	0	0	1	0	1
1	1	1	1	1	1	1

In general, if the truth of a formula depends on n propositions, its truth table will have 2^n rows.

3.3 Special Formulas

- A formula is valid or a tautology if and only if it is satisfied by every truth assignment. For example, the sentence $p \vee \neg p$ is valid.
- A formula is unsatisfiable or a contradiction if and only if it is not satisfied by any truth assignment. For example, $p \wedge \neg p$ is unsatisfiable. No matter what truth assignment we take, the sentence is always false.
- A formula is satisfiable if and only if there is some truth assignment that satisfies it.

Note that, A is a contradiction iff $\neg A$ is a tautology. And, A is satisfiable iff $\neg A$ is not a tautology.

4 Inference Rules - Natural Deduction

In natural deduction, we have such a collection of proof rules. They allow us to infer formulas from other formulas. By applying these rules in succession, we may infer a conclusion from a set of premises.

Suppose we have a set of formulas $\phi_1, \phi_2, \phi_3, \dots, \phi_n$, which we will call premises, and another formula, ψ , which we will call a conclusion. By applying proof rules to the premises, we hope to get some more formulas, and by applying more proof rules to those, to eventually obtain the conclusion. This intention we denote by

$$\phi_1, \phi_2, \phi_3, \dots, \phi_n \vdash \psi$$

.This expression is called a sequent. Note that, premises are not always needed. If we can prove A by showing $\vdash A$, A is called a theorem.

4.1 The rules for conjunction

Rule 1: and-introduction

$$\frac{\phi \quad \psi}{\phi \wedge \psi} \wedge i$$

Note that, for all the rules, above the line are the premises of the rule. Below the line goes the conclusion.

Rule 2: and-elimination1

$$\frac{\phi \wedge \psi}{\phi} \wedge e_1$$

Rule 3: and-elimination2

$$\frac{\phi \wedge \psi}{\psi} \wedge e_2$$

Problem 2. Use these rules to prove $p \wedge q, r \vdash q \wedge r$.

Problem 3. Use these rules to prove $(p \wedge q) \wedge r, s \wedge t \vdash q \wedge s$.

Problem 4. Use these rules to prove $(p \wedge q) \wedge r \vdash p \wedge (q \wedge r)$.

4.2 The rules for double negation

Rule 4: double negation-introduction (derived rule ¹)

$$\frac{\phi}{\neg\neg\phi} \neg\neg i$$

Rule 5: double negation-elimination

$$\frac{\neg\neg\phi}{\phi} \neg\neg e$$

Problem 5. Use rules introduced so far to prove $p, \neg\neg(q \wedge r) \vdash \neg\neg p \wedge r$.

4.3 The rules for implication

In natural deduction, to prove $\phi \rightarrow \psi$, we first assume ϕ and prove ψ under the assumption, by applying other rules of inference. Finally, we must remember to *discharge* the assumption and conclude that $\phi \rightarrow \psi$.

Rule 6: assumption

$$\frac{}{x : \phi} \text{ assum}$$

We are always allowed to make assumptions from which to prove other statements. An assumption must be given a name (e.g., x above). As said above, we don't get to make assumptions for free. To get a valid proof, assumptions must be eventually discharged, giving rise to the following rule.

Rule 7: implies-introduction

$$\frac{\boxed{\begin{array}{c} x : \phi \\ \vdots \\ \psi \end{array}}}{\phi \rightarrow \psi} \rightarrow i/x$$

¹A *derived* rule is one which can be derived from other rules and hence, not necessary to be included. The main purpose of derived rules is to avoid re-proving things again and again. In this particular case, we can derive this rule by using Rules 6, 7, 13 and 14.

Here, under a prior assumption ϕ (with name x) we could prove ψ . So, we can discharge the assumption named x and conclude that $\phi \rightarrow \psi$. Note that the name of the rule mentions the assumption being discharged.

A proof is valid only if every assumption is eventually discharged. This must happen below the point where an assumption has been made, in the proof tree. If an assumption is used more than once, it must be discharged in all those paths in the proof tree.

Rule 8: implies-elimination (modus ponens)

$$\frac{\phi \quad \phi \rightarrow \psi}{\psi} \rightarrow e$$

Rule 9: modus tollens (derived rule)

$$\frac{\phi \rightarrow \psi \quad \neg\psi}{\neg\phi} MT$$

Problem 6. Use rules introduced so far to prove $p \rightarrow (q \rightarrow r)$, p , $\neg r \vdash \neg q$.

Problem 7. Use rules introduced so far to prove $\neg q \rightarrow \neg p \vdash p \rightarrow \neg\neg q$.

Problem 8. Use rules introduced so far to prove $p \wedge q \rightarrow r \vdash p \rightarrow (q \rightarrow r)$.

4.4 The rules for disjunction

Rule 10: or-introduction1

$$\frac{\phi}{\phi \vee \psi} \vee i_1$$

Rule 11: or-introduction2

$$\frac{\psi}{\phi \vee \psi} \vee i_2$$

Rule 12: or-elimination

$$\frac{\phi \vee \psi \quad \phi \rightarrow \chi \quad \psi \rightarrow \chi}{\chi} \vee e$$

Problem 9. Use rules introduced so far to prove $(p \vee q) \vee r \vdash p \vee (q \vee r)$.

4.5 The rules for single negation

Contradictions are expressions of the form $\phi \wedge \neg\phi$ or $\neg\phi \wedge \phi$, where ϕ is any formula. We usually use symbol \perp to denote a contradiction.

Note that,

- As far as logic is concerned, all contradictions are equivalent.

- It is not just that contradictions can be derived from contradictions: any formula can be derived from a contradiction.
For example, let p be "It is raining.", and q be "I like to eat watermelon."
We have that $p \wedge \neg p \vdash q$ is valid, because my personal interest has nothing to do with today's weather.

Rule 13: not-elimination

$$\frac{\phi \quad \neg\phi}{\perp} \neg e$$

Rule 14: not-introduction

$$\frac{\phi \rightarrow \perp}{\neg\phi} \neg i$$

Rule 15: bottom-elimination

$$\frac{\perp}{\phi} \perp e$$

Problem 10. Use rules introduced so far to prove $\neg p \vee q \vdash p \rightarrow q$.

Problem 11. Use rules introduced so far to prove $p \rightarrow q, p \rightarrow \neg q \vdash \neg p$.

4.6 Derived Rules

I have already introduced two derived rules so far.

Rule 4: double negation-introduction

$$\frac{\phi}{\neg\neg\phi} \neg\neg i$$

Rule 9: modus tollens

$$\frac{\phi \rightarrow \psi \quad \neg\psi}{\neg\phi} MT$$

There are two additional rules. Rule 16: reductio ad absurdum (RAA/ proof by contradiction)

$$\frac{\neg\phi \rightarrow \perp}{\phi} RAA$$

Rule 17: tertium non datur (/ the law of the excluded middle (LEM))

$$\frac{}{\phi \vee \neg\phi} LEM$$

Problem 12: Show how rules 9, 16 and 17 are derived from other rules.

5 Intuitionistic Natural Deduction

In intuitionistic logic, a statement/formula is true only if it can be *proved directly*, i.e. a witness can be shown for the formula. As such, it does not allow Rule 5 mentioned above. Here is a simple high level reasoning. The rule says

that if $\neg\neg\phi$ is provable, then ϕ is also provable. But intuitionistic logic disagrees. If $\neg\neg\phi$ is provable, it only says that *there is no proof for $\neg\phi$* . And that, it *does not* imply that a proof of ϕ can be found. In this framework, the absence of a proof for the negation of a formula is not the presence of a proof for the formula. On the other hand, the absence of a proof for a formula is a legitimate proof of its negation.

If rule 5 is not considered, the derived rules which use rule 5 are also not considered (rules 16 and 17 above - see Problem 12). In particular, LEM does not hold in the intuitionistic framework.

This is what Coq uses by default. There is a way to use *classical* natural deduction in Coq – we will see it later in the class.