Binary Decision Diagrams Part 2

15-414 Bug Catching: Automated Program Verification and Testing

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September 14, 2011

BDDs Recap

Typically mean Reduced Ordered Binary Decision Diagrams (ROBDDs)

- Can be viewed as reduced forms of Ordered Binary Decision Trees
  - Obtained by eliminating duplicate nodes and redundant nodes
- Often substantially smaller than the OBDT

Canonical representation of Boolean formulas

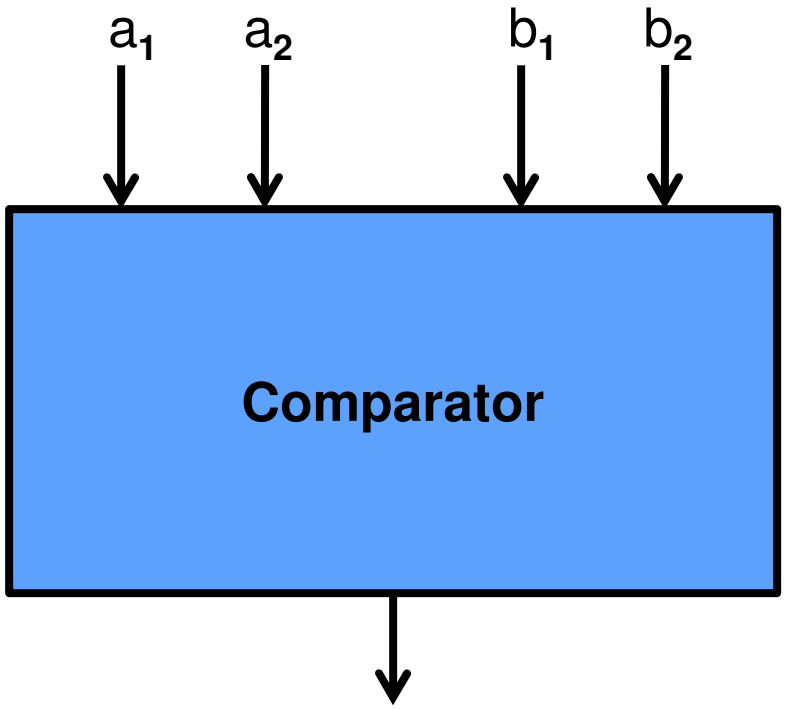
- Unlike other normal forms like CNF and DNF

Size of BDD depends critically on variable ordering

In practice, BDDs are built up from their components

- Via (efficient) Boolean operations
- Dynamic variable ordering used to manage BDD size

Running Example: Comparator



(A) $f = 1 \Leftrightarrow a_1 = b_1 \wedge a_2 = b_2$ ✓

(B) $f = 1 \Leftrightarrow a_1 = b_1 \vee a_2 = b_2$

Conjunctive Normal Form

$$(b_1 \vee a_1) \wedge (\neg a_1 \vee \neg b_1) \wedge (\neg a_2 \vee b_2) \wedge (\neg b_2 \vee a_2)$$

$$(\neg b_1 \vee a_1) \wedge (\neg a_1 \vee b_1) \wedge (\neg a_2 \vee b_2) \wedge (\neg b_2 \vee a_2)$$

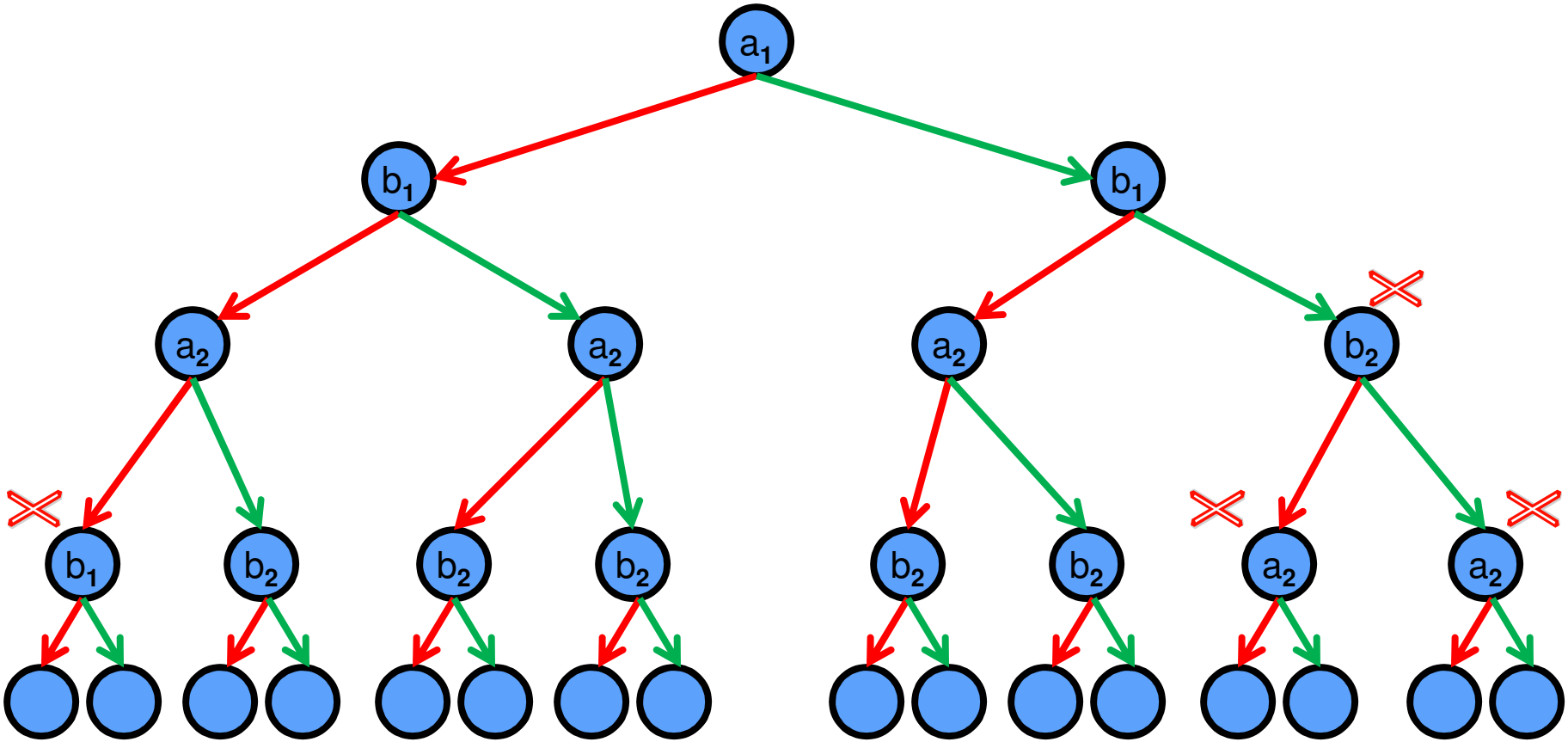
$$(\neg a_1 \vee b_1) \wedge (\neg b_1 \vee a_1) \wedge (\neg a_2 \vee b_2) \wedge (\neg b_2 \vee a_2)$$

$$(\neg a_1 \vee b_1) \wedge (\neg b_1 \vee a_1) \wedge (\neg a_2 \vee \neg b_2) \wedge (b_2 \vee a_2)$$

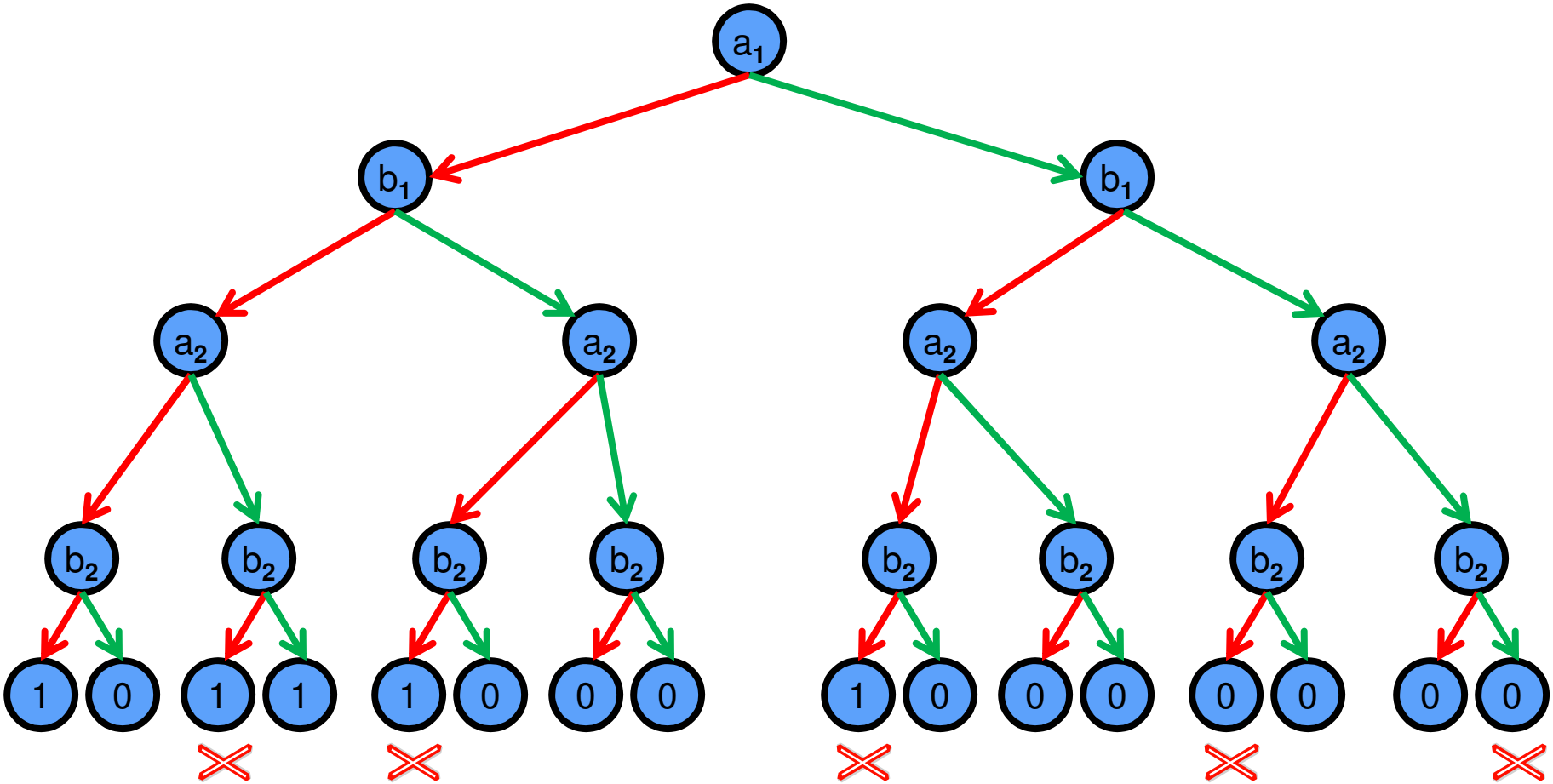
Truth Table

Row#	a_1	b_1	a_2	b_2	f
0	0	0	0	0	0
1	0	0	0	1	1
2	0	0	1	0	0
3	0	0	1	1	1
4	0	1	0	0	0
5	0	1	0	1	1
6	0	1	1	0	0
7	0	1	1	1	0
8	1	0	0	0	0
9	1	0	0	1	1
10	1	0	1	0	1
11	1	0	1	1	0
12	1	1	0	0	1
13	1	1	0	1	0
14	1	1	1	0	0
15	1	1	1	1	0

Representing a Truth Table using a Graph



Representing a Truth Table using a Graph



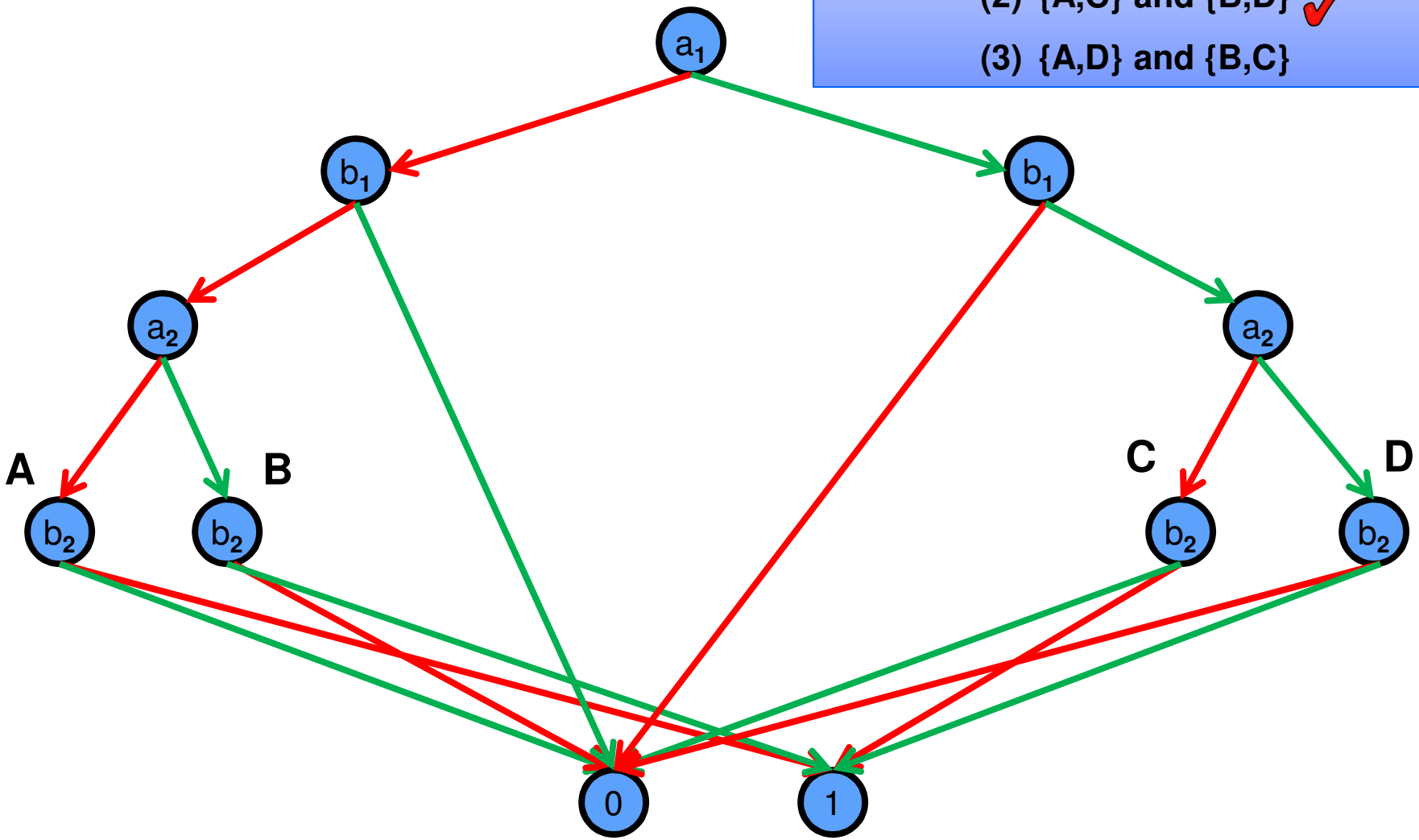
OBDT to ROBDD

Which pairs are isomorphic?

(1) {A,B} and {C,D}

(2) {A,C} and {B,D} ✓

(3) {A,D} and {B,C}

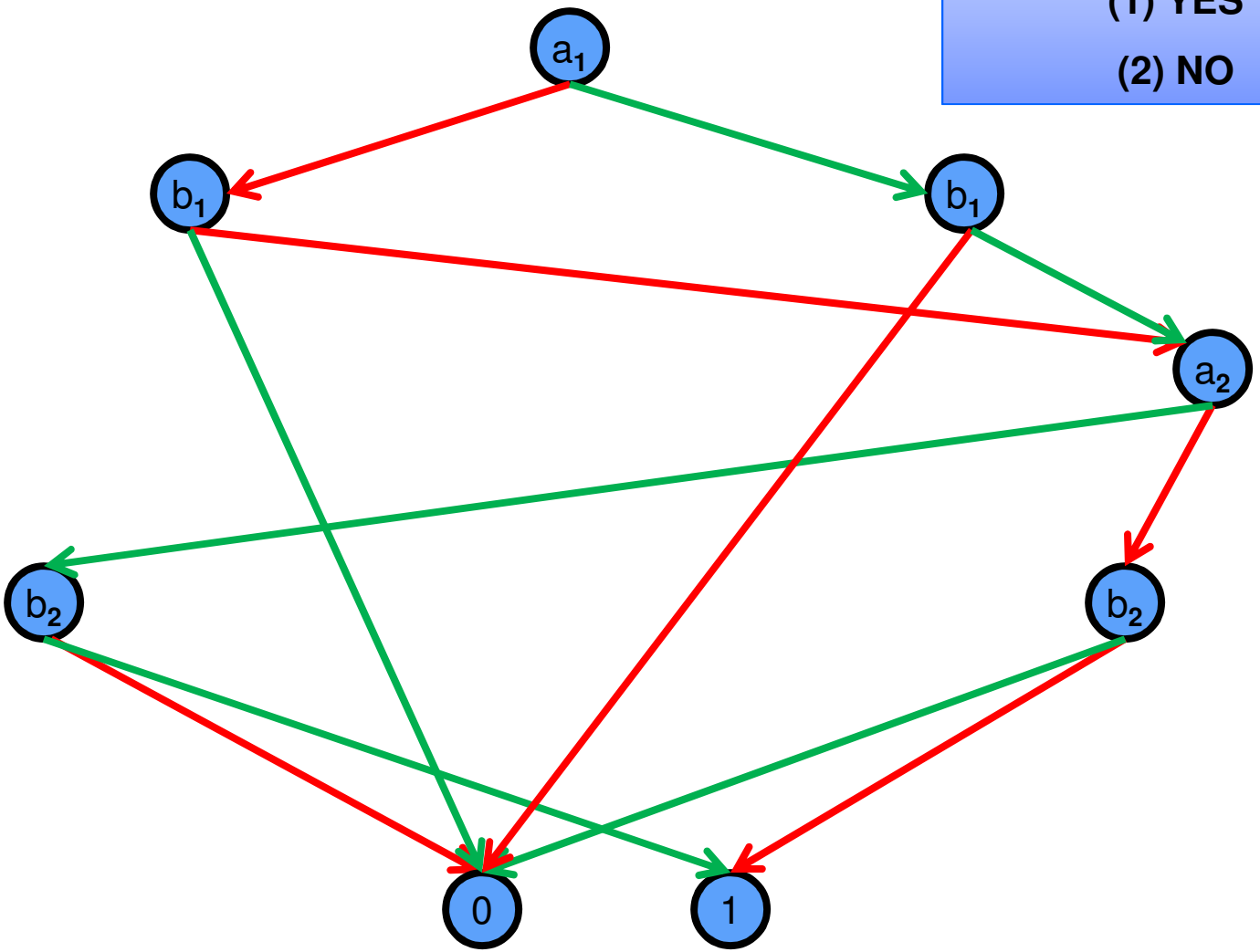


OBDT to ROBDD

Is this a ROBDD?

(1) YES ✓

(2) NO



OBDT to ROBDD

What function does X represent?

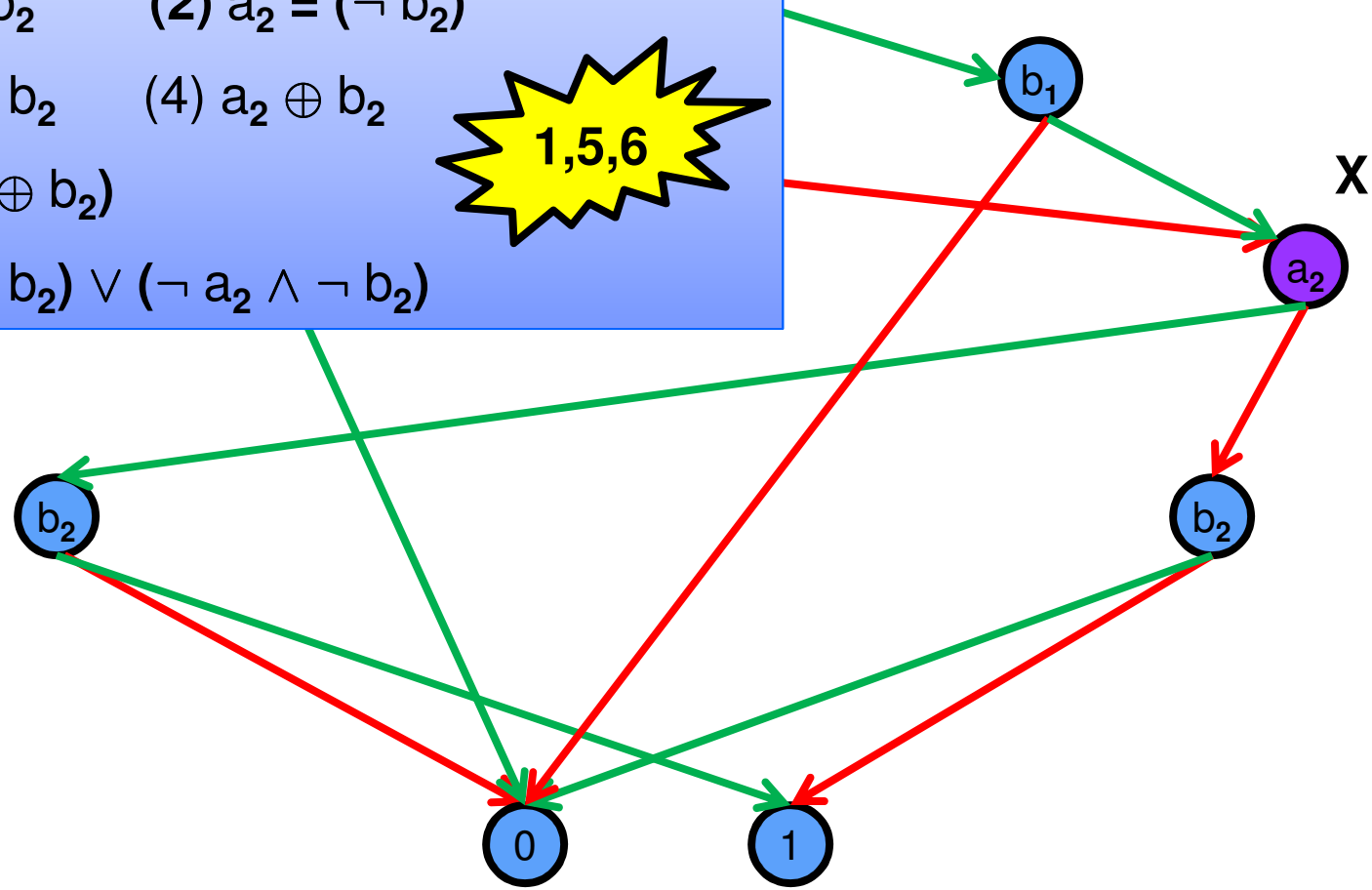
(1) $a_2 = b_2$ **(2)** $a_2 = (\neg b_2)$

(3) $a_2 \Rightarrow b_2$ (4) $a_2 \oplus b_2$

(5) $\neg(a_2 \oplus b_2)$

(6) $(a_2 \wedge b_2) \vee (\neg a_2 \wedge \neg b_2)$

1,5,6



ROBDD (a.k.a. BDD) Summary

If BDD(f_1) and BDD(f_2) are isomorphic then:

✓ 1. $f_1 = f_2$
✓ 2. f_1 and f_2 have the same variables
✓ 3. BDD(f_1) and BDD(f_2) have the same variable ordering

If BDD(f) is the leaf node "1" then f is:

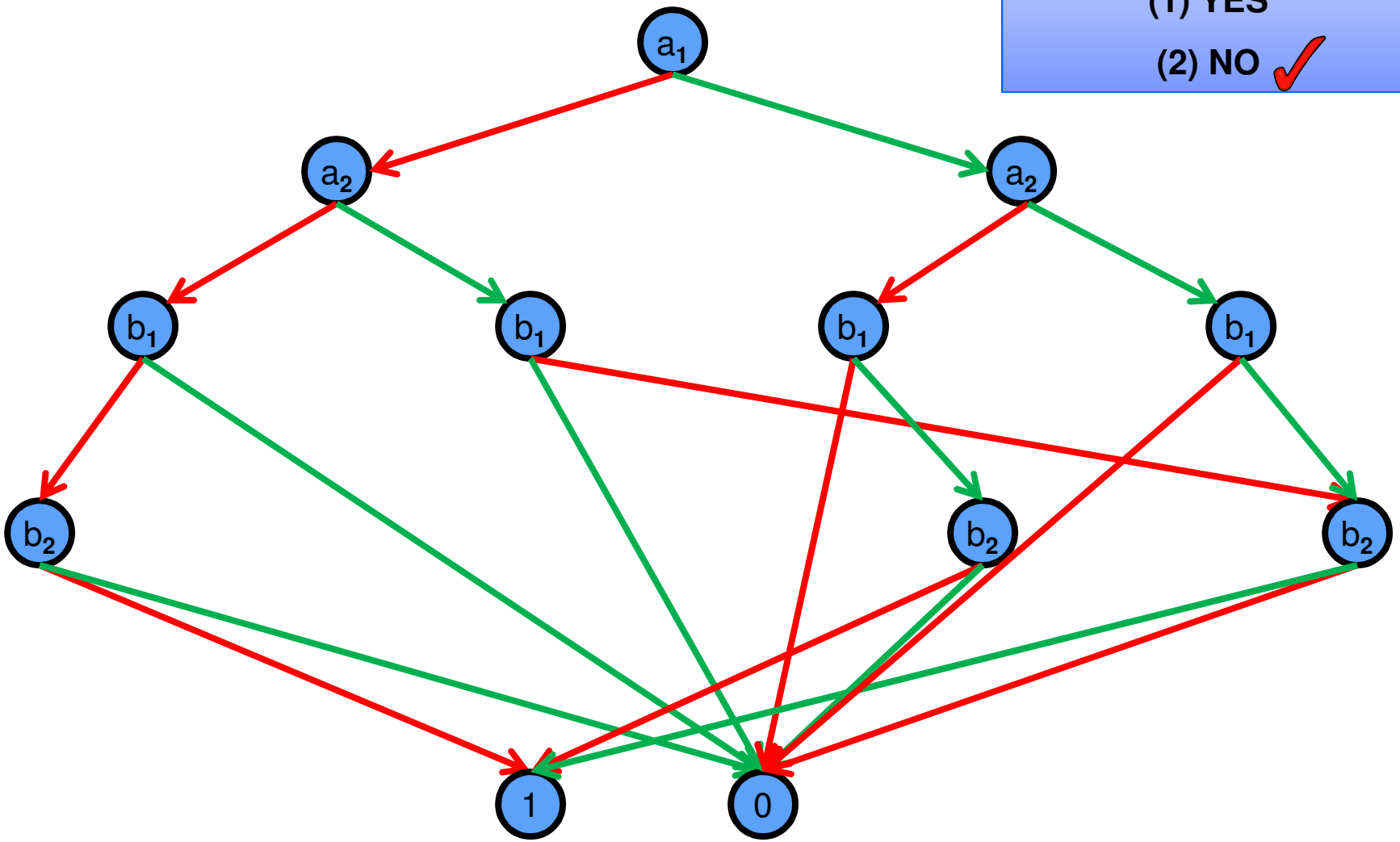
1. Satisfiable
2. Unsatisfiable
✓ 3. Valid

ROBDD and variable ordering

Is this a ROBDD?

(1) YES

(2) NO ✓

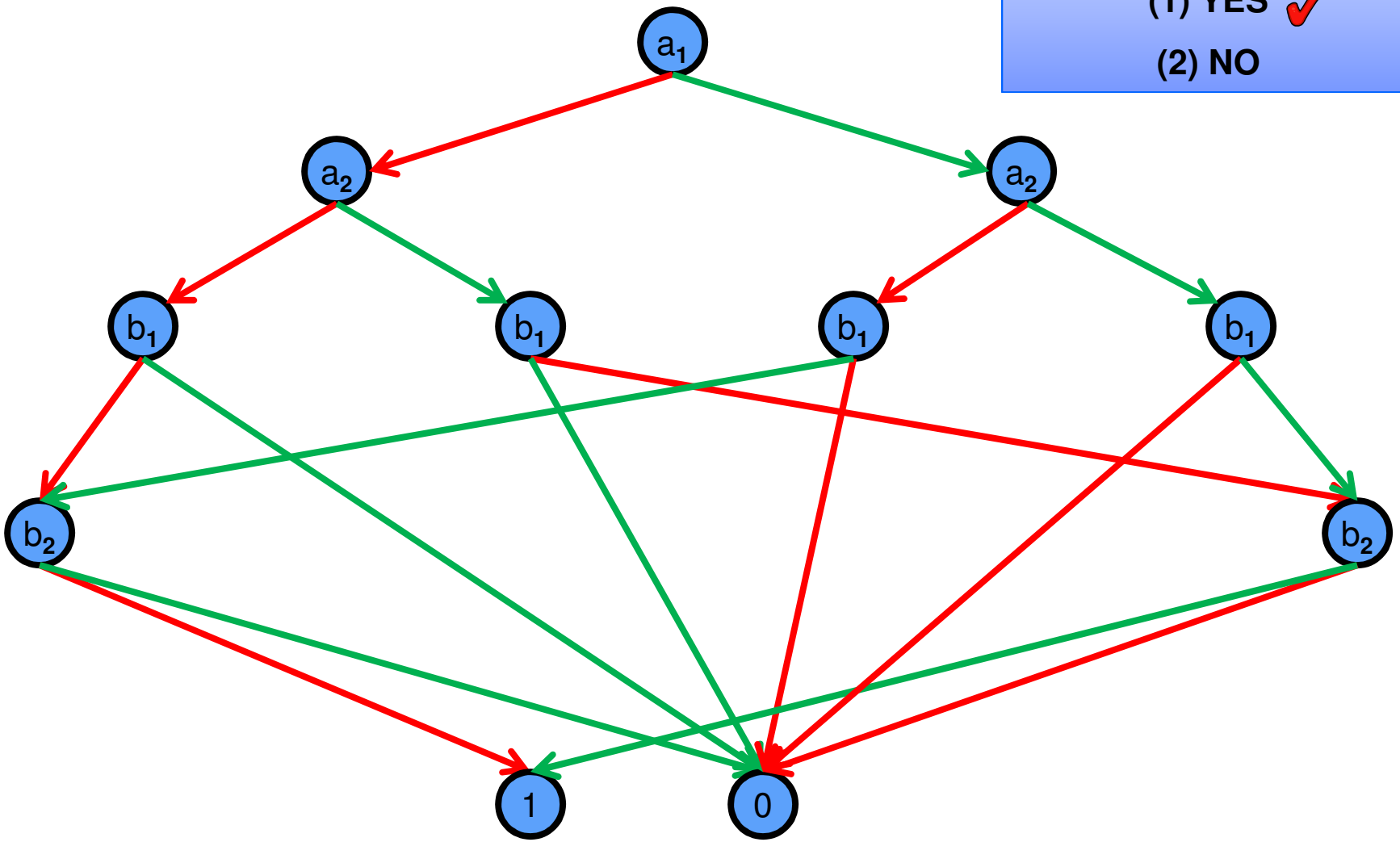


ROBDD and variable ordering

Is this a ROBDD?

(1) YES ✓

(2) NO



ROBDD and variable ordering

There exists a function whose BDD grows polynomially in the number of variables for some ordering and exponentially for others?

- TRUE

There exists a function whose BDD grows exponentially for all variable orderings?

- TRUE

There exists a function whose BDD grows linearly for all variable orderings?

- TRUE

BDD Operations

True : BDD(TRUE)

False: BDD(FALSE)

Var : $v \mapsto$ BDD(v)

Not : BDD(f) $\mapsto$ BDD($\neg$f)

And : BDD(f_1) $\times$ BDD(f_2) $\mapsto$ BDD($f_1 \wedge f_2$)

Or : BDD(f_1) $\times$ BDD(f_2) $\mapsto$ BDD($f_1 \vee f_2$)

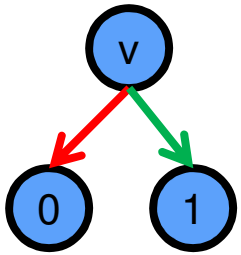
Exist : BDD(f) $\times$ v $\mapsto$ BDD($\exists$ v. f)

Basic BDD Operations

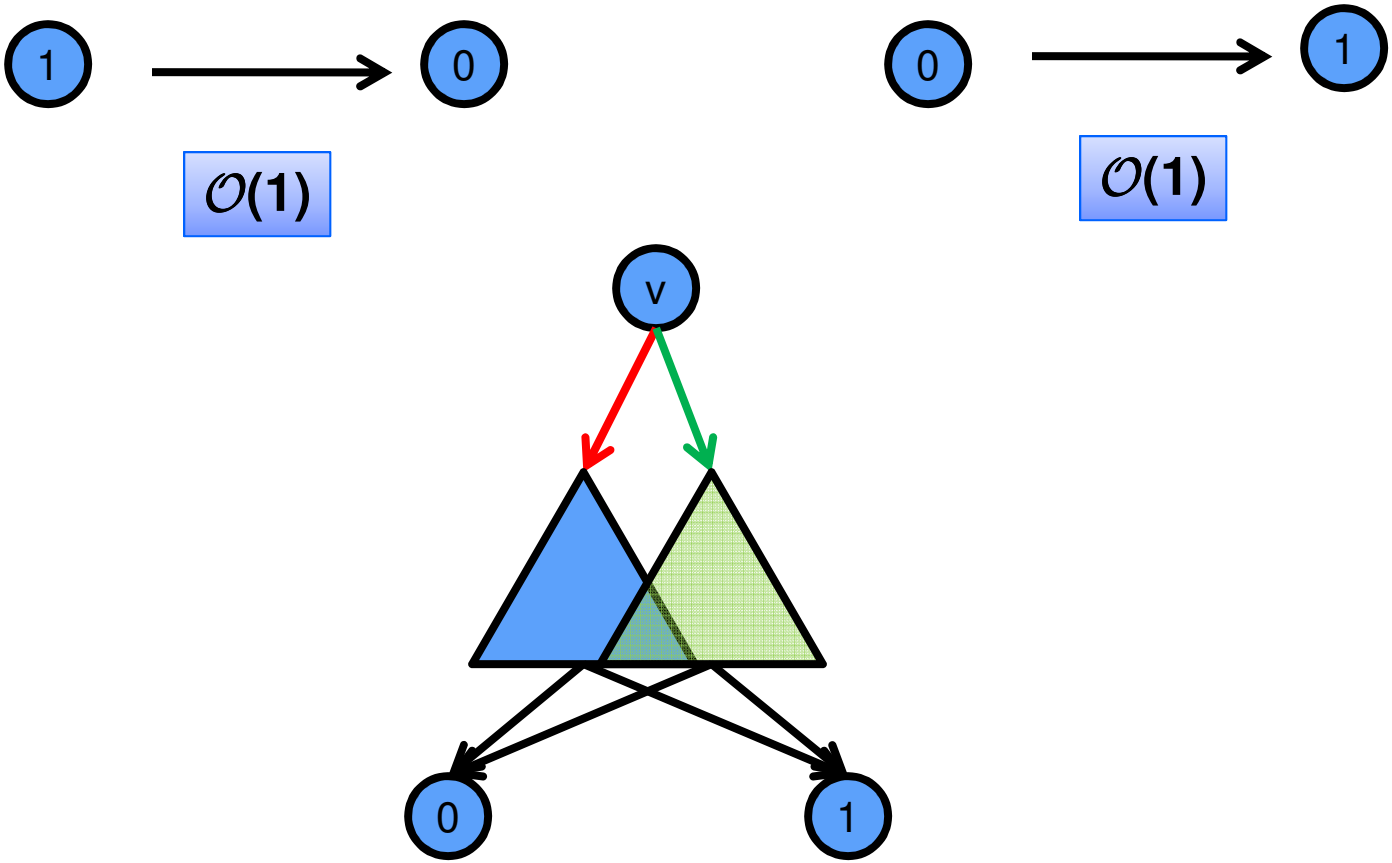
True

False

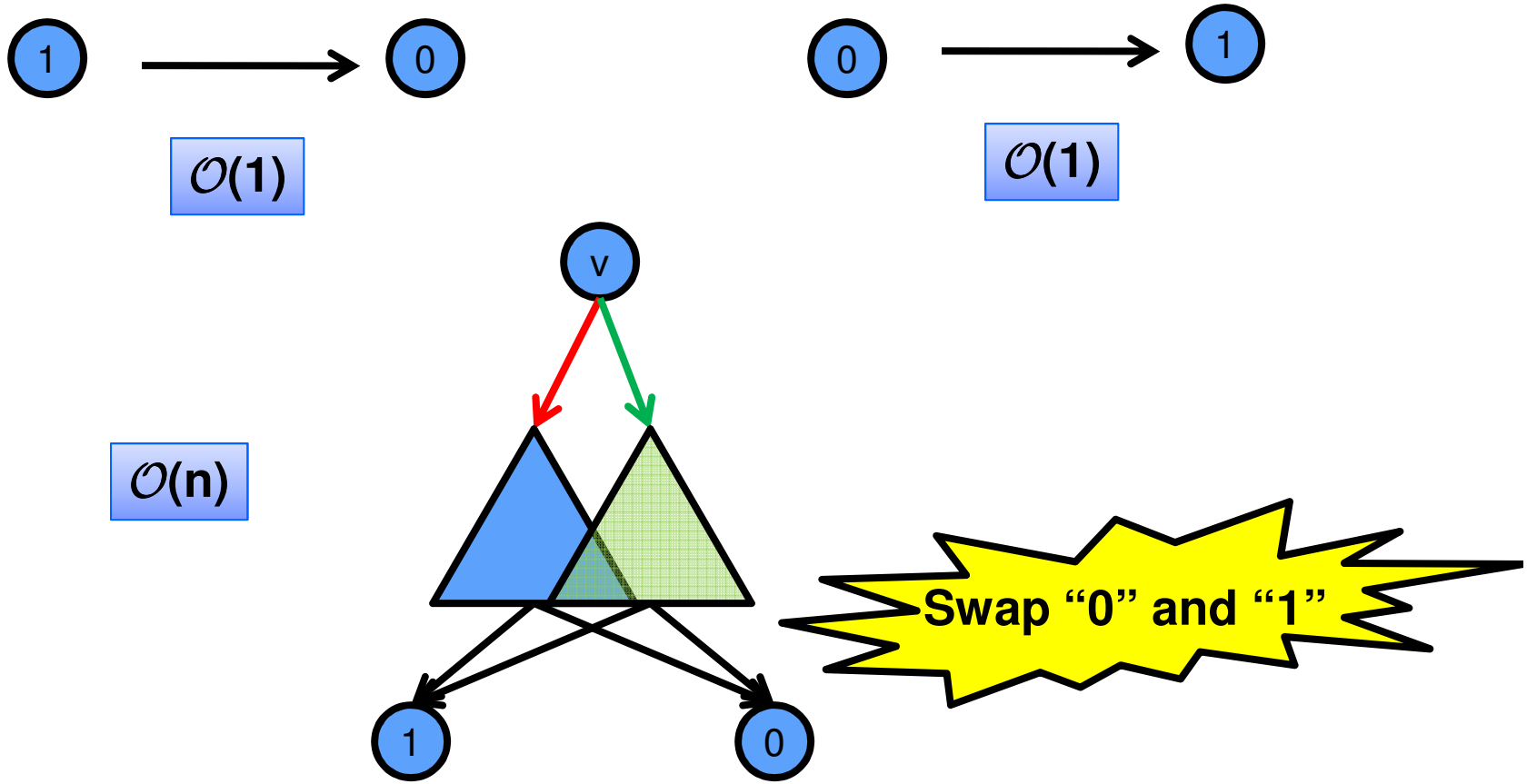
Var(v)



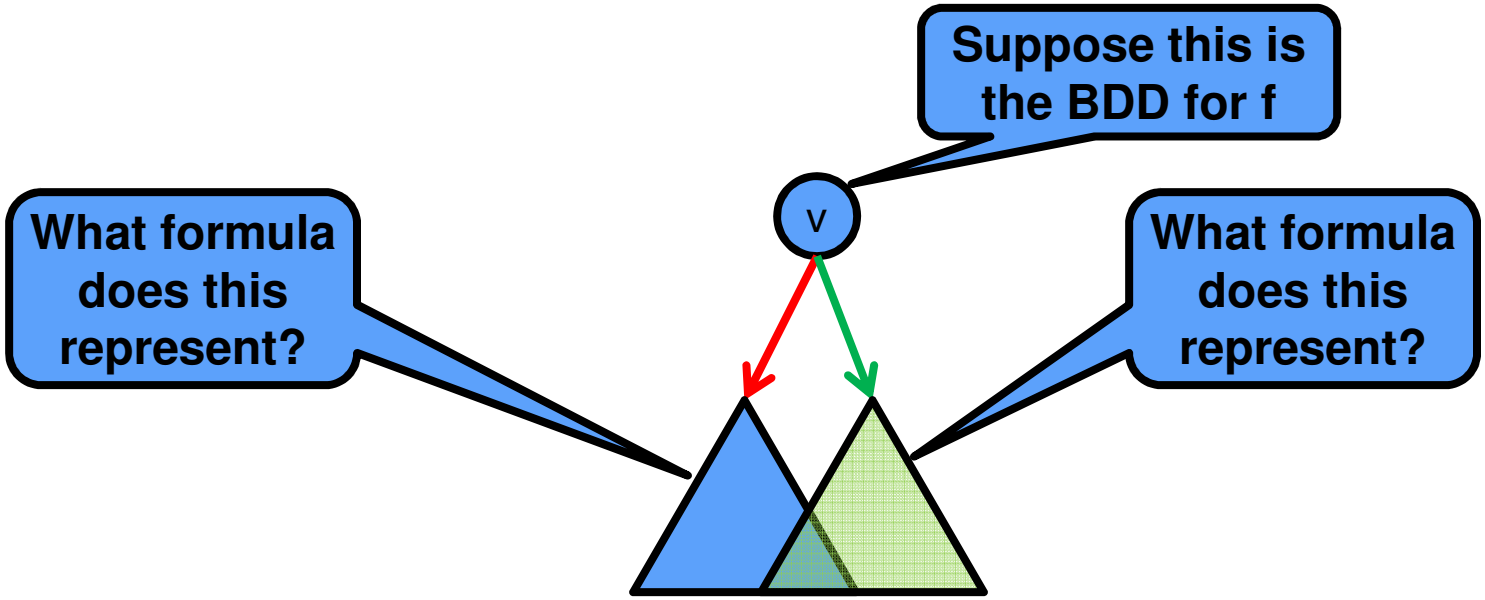
BDD Operations: Not



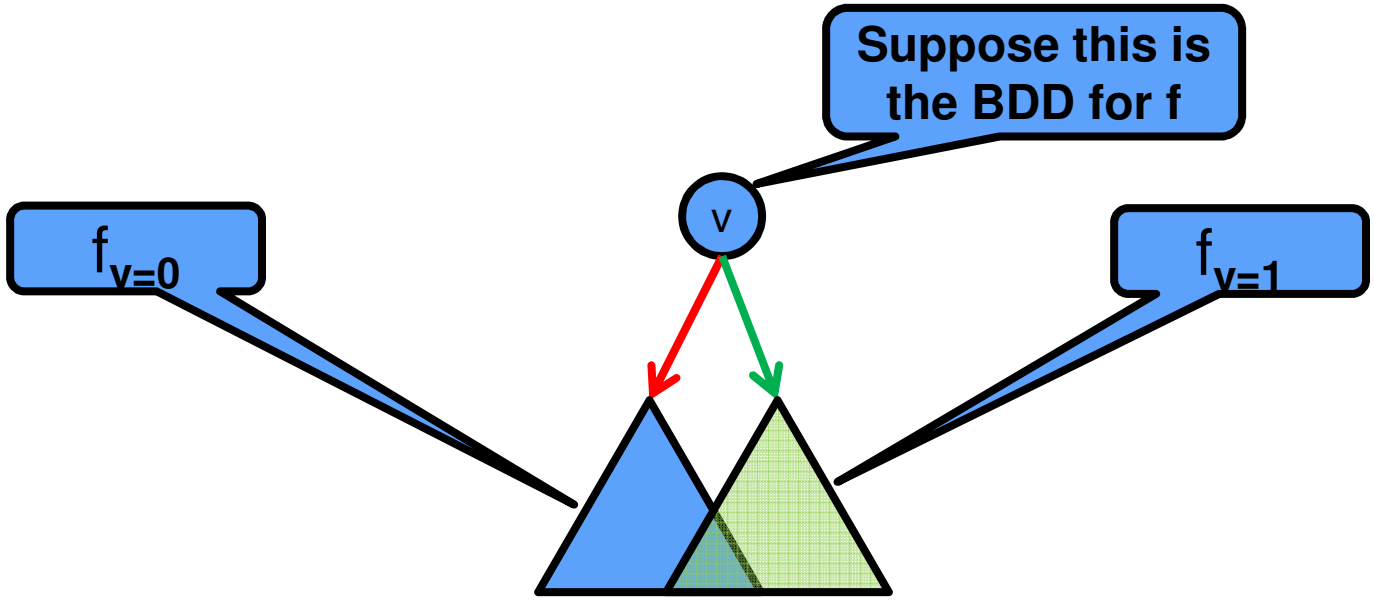
BDD Operations: Not



BDD Operations: And



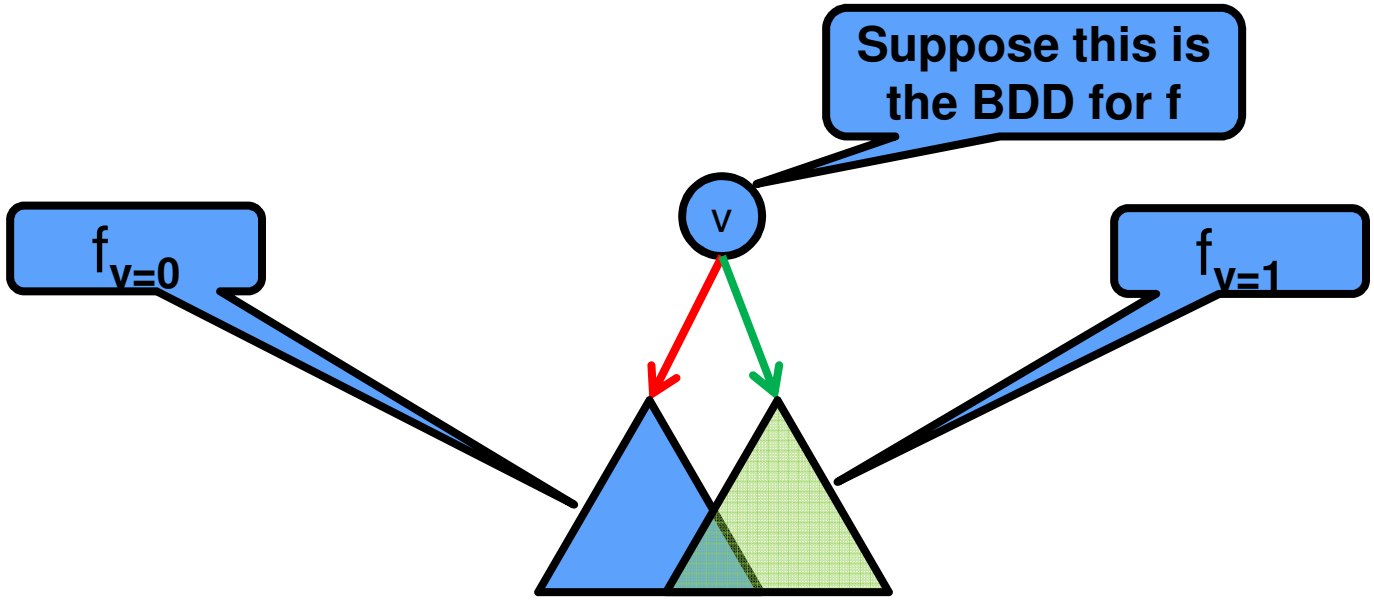
BDD Operations: And



$f_{v=0}$ and $f_{v=1}$ are known as the co-factors of f w.r.t. v

$$f = (X \wedge f_{v=0}) \vee (Y \wedge f_{v=1})$$

BDD Operations: And



$f_{v=0}$ and $f_{v=1}$ are known as the co-factors of f w.r.t. v

$$f = (\neg v \wedge f_{v=0}) \vee (v \wedge f_{v=1})$$

BDD Operations: And (Simple Cases)

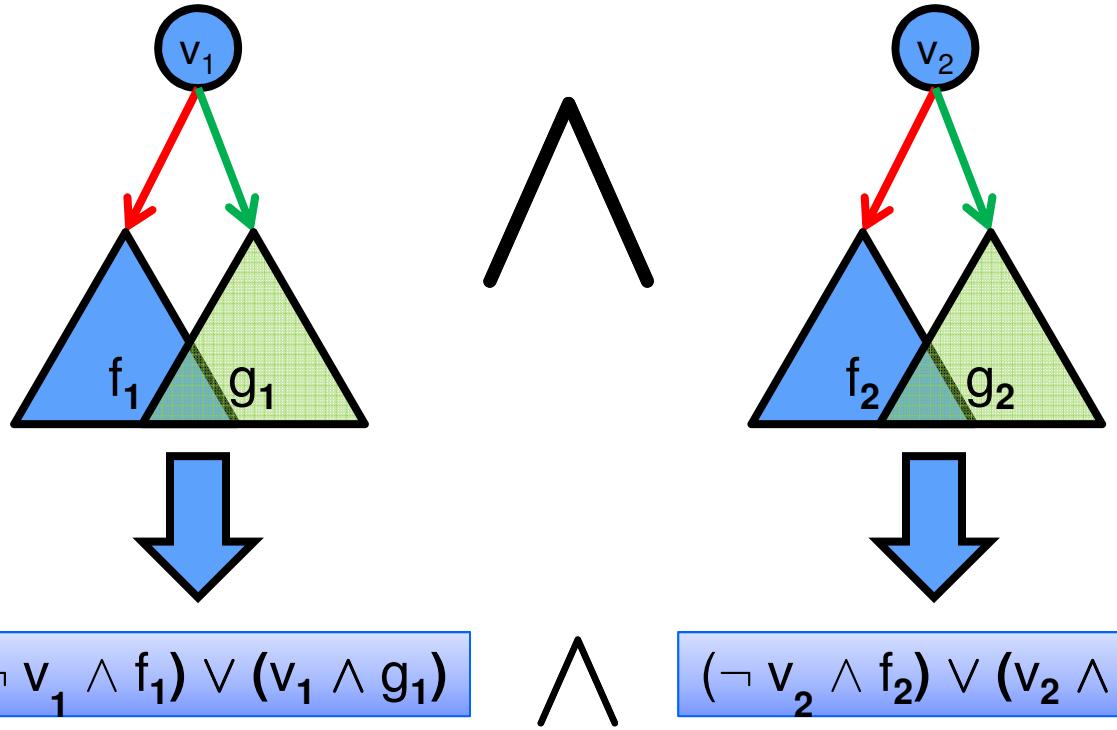
And (f, 0) = 0

And (f, 1) = f

And (1 ,f) = f

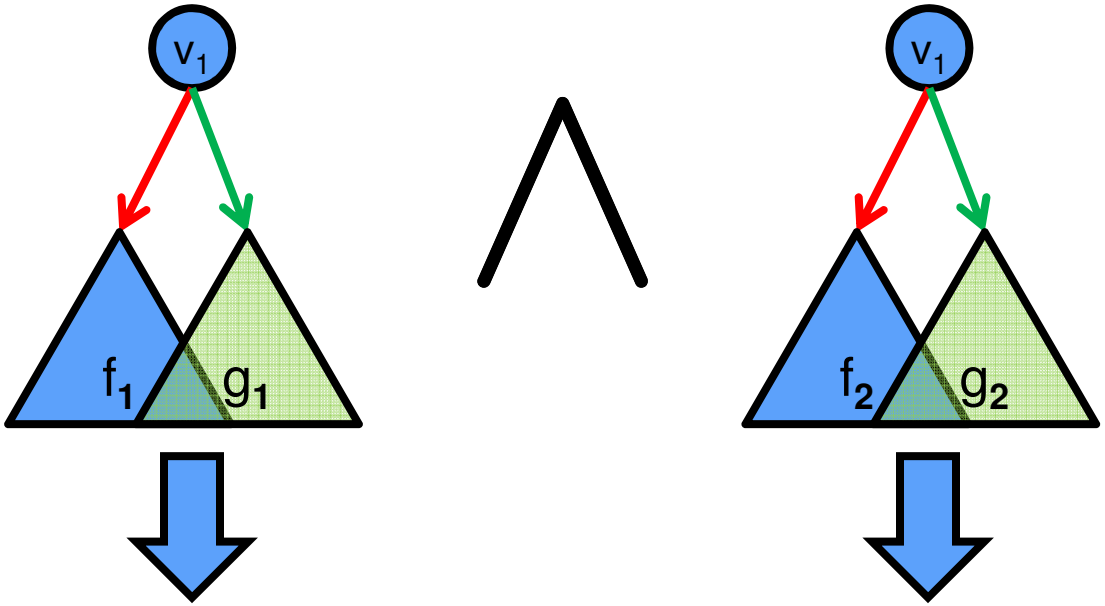
And (0 ,f) = 0

BDD Operations: And (Complex Case)



BDD Operations: And (Complex Case 1)

$v_1 = v_2$



$(\neg v_1 \wedge f_1) \vee (v_1 \wedge g_1)$ $\wedge$ $(\neg v_1 \wedge f_2) \vee (v_1 \wedge g_2)$

BDD Operations: And (Complex Case 1)

$v_1 = v_2$

$(\neg v_1 \wedge X) \vee (v_1 \wedge Y)$

$(\neg v_1 \wedge f_1) \vee (v_1 \wedge g_1)$ $\wedge$ $(\neg v_1 \wedge f_2) \vee (v_1 \wedge g_2)$

BDD Operations: And (Complex Case 1)

$v_1 = v_2$

Compute recursively

$(\neg v_1 \wedge (f_1 \wedge f_2)) \vee (v_1 \wedge (g_1 \wedge g_2))$

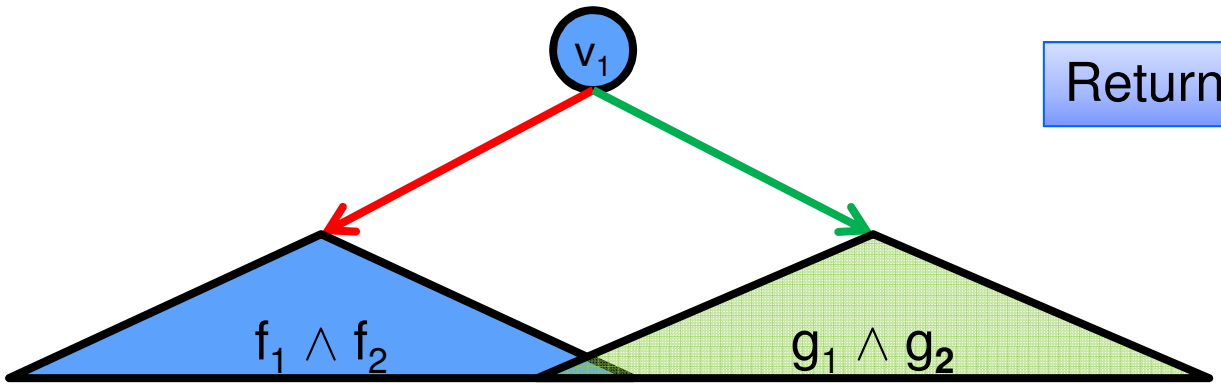
$(\neg v_1 \wedge f_1) \vee (v_1 \wedge g_1)$ $\wedge$ $(\neg v_1 \wedge f_2) \vee (v_1 \wedge g_2)$

BDD Operations: And (Complex Case 1)

$v_1 = v_2$

What if $f_1 \wedge f_2 = g_1 \wedge g_2$?

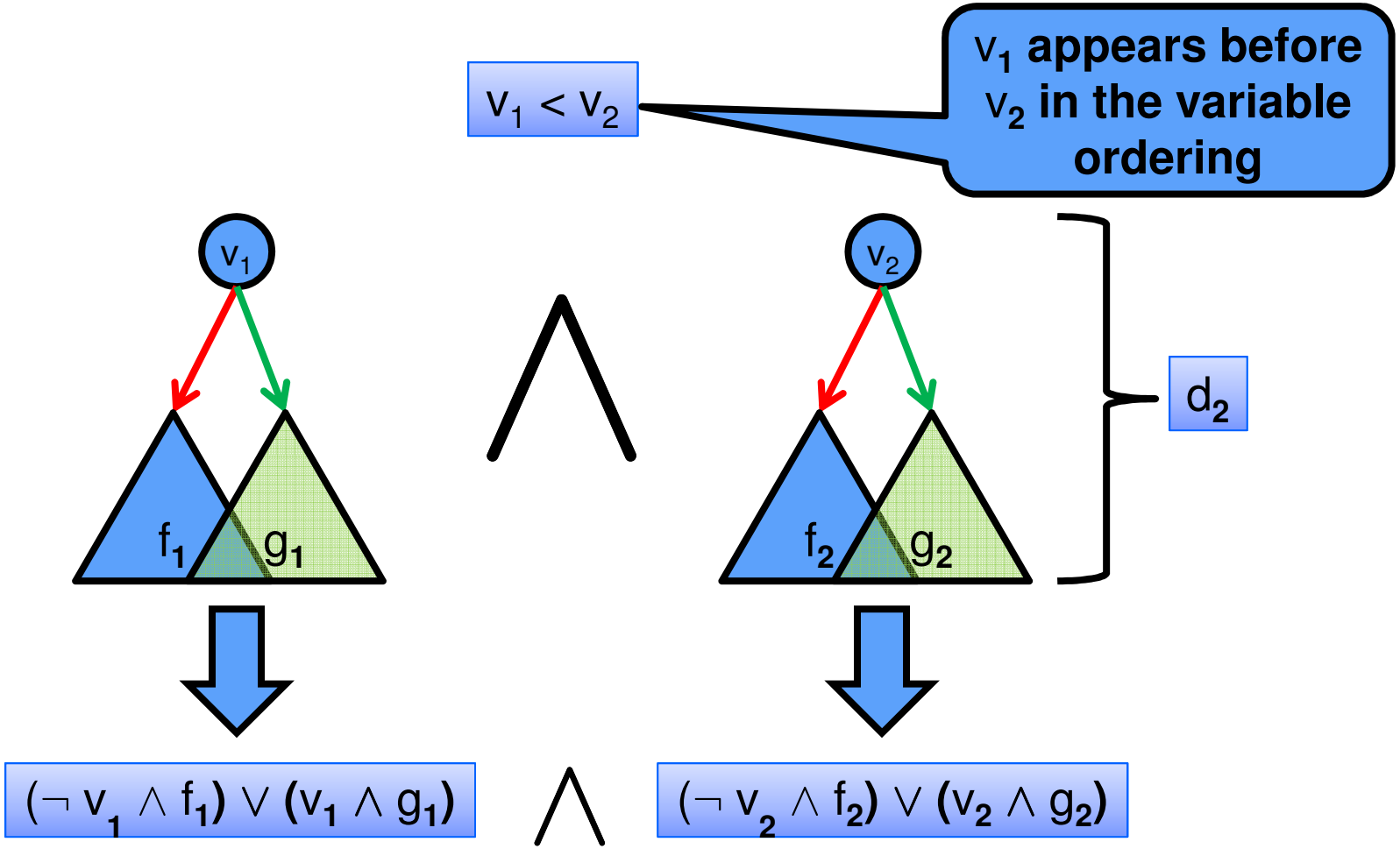
Return $f_1 \wedge f_2$



$(\neg v_1 \wedge (f_1 \wedge f_2)) \vee (v_1 \wedge (g_1 \wedge g_2))$

$(\neg v_1 \wedge f_1) \vee (v_1 \wedge g_1)$ $\wedge$ $(\neg v_1 \wedge f_2) \vee (v_1 \wedge g_2)$

BDD Operations: And (Complex Case 2)

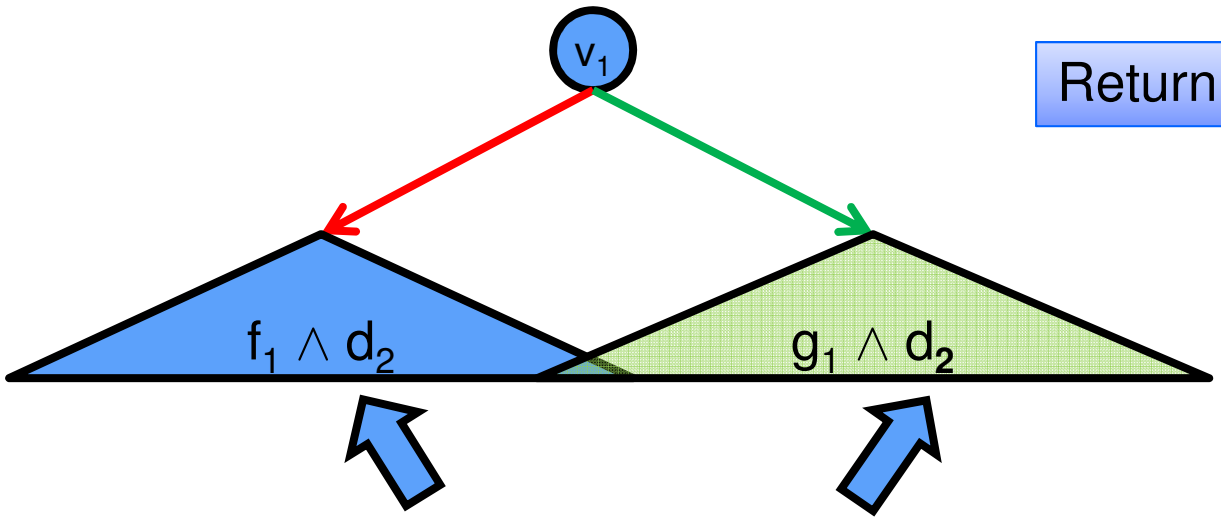


BDD Operations: And (Complex Case 2)

$v_1 < v_2$

What if $f_1 \wedge d_2 = g_1 \wedge d_2$?

Return $f_1 \wedge d_2$



$(\neg v_1 \wedge (f_1 \wedge d_2)) \vee (v_1 \wedge (g_1 \wedge d_2))$

$(\neg v_1 \wedge f_1) \vee (v_1 \wedge g_1)$ $\wedge$ d_2

$\mathcal{O}(n_1 \times n_2)$

BDD Operations: Or

$$\mathrm{Or}(d1, d_2)$$

$$=$$

$$\textbf{Not (And (} \mathrm{Not}(d_1), \mathrm{Not}(d_2) \textbf{))}$$

$$\mathcal{O}(n_1 \times n_2)$$

BDD Operations: Exist

Exist(“0”,v) = ?

BDD Operations: Exist

Exist(“0”,v) = “0”

Exist(“1”,v) = ?

BDD Operations: Exist

$$\text{Exist}(\text{“0”}, v) = \text{“0”}$$

$$\text{Exist}(\text{“1”}, v) = \text{“1”}$$

$$\text{Exist}((\neg v \wedge f) \vee (v \wedge g), v) = ?$$

BDD Operations: Exist

$$\text{Exist}(\text{“0”}, v) = \text{“0”}$$

$$\text{Exist}(\text{“1”}, v) = \text{“1”}$$

$$\text{Exist}((\neg v \wedge f) \vee (v \wedge g), v) = \text{Or}(f,g)$$

$$\text{Exist}((\neg v' \wedge f) \vee (v' \wedge g), v) = ?$$

BDD Operations: Exist

$\mathcal{O}(n^2)$

$$\text{Exist}(\text{“0”},v) = \text{“0”}$$

$$\text{Exist}(\text{“1”},v) = \text{“1”}$$

$$\text{Exist}((\neg v \wedge f) \vee (v \wedge g), v) = \text{Or}(f,g)$$

$$\text{Exist}((\neg v' \wedge f) \vee (v' \wedge g), v) =$$

$$(\neg v' \wedge \text{Exist}(f,v)) \vee (v' \wedge \text{Exist}(g,v))$$

But f is SAT iff $\exists$ V. f is not “0”. So why doesn’t this imply P = NP?

Because the BDD size changes!

BDD Applications

SAT is great if you are interested to know if a solution exists

BDDs are great if you are interested in the set of all solutions

- How many solutions are there?
- How do you do this on a BDD?

Or if your problem involves computing a fixed point

- Set of nodes reachable from a given node in a graph

BDD Application: Counting Sudoku Solutions

	1	2	3	4
1			1	
2				
3	3		2	
4				

How many ways can you solve this puzzle?

BDD Application: Counting Sudoku Solutions

	1	2	3	4
1	0	3	1	2
2	2	1	0	3
3	3	0	2	1
4	1	2	3	0

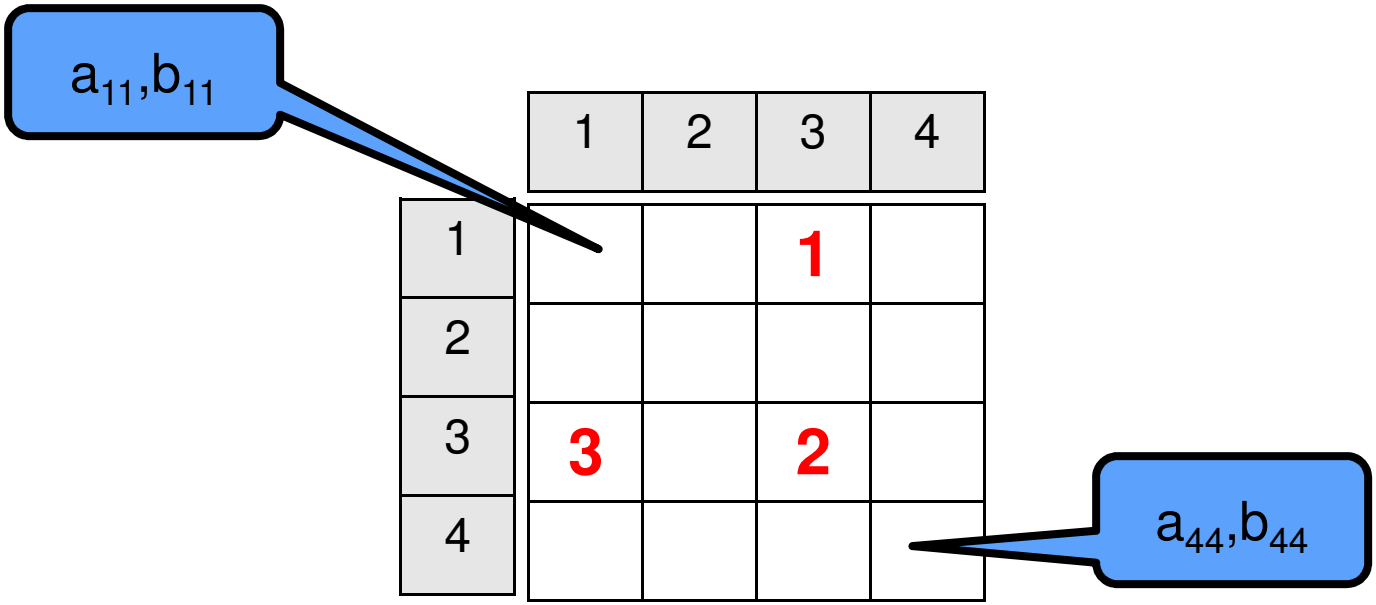
How many ways can you solve this puzzle? At least 2.

BDD Application: Counting Sudoku Solutions

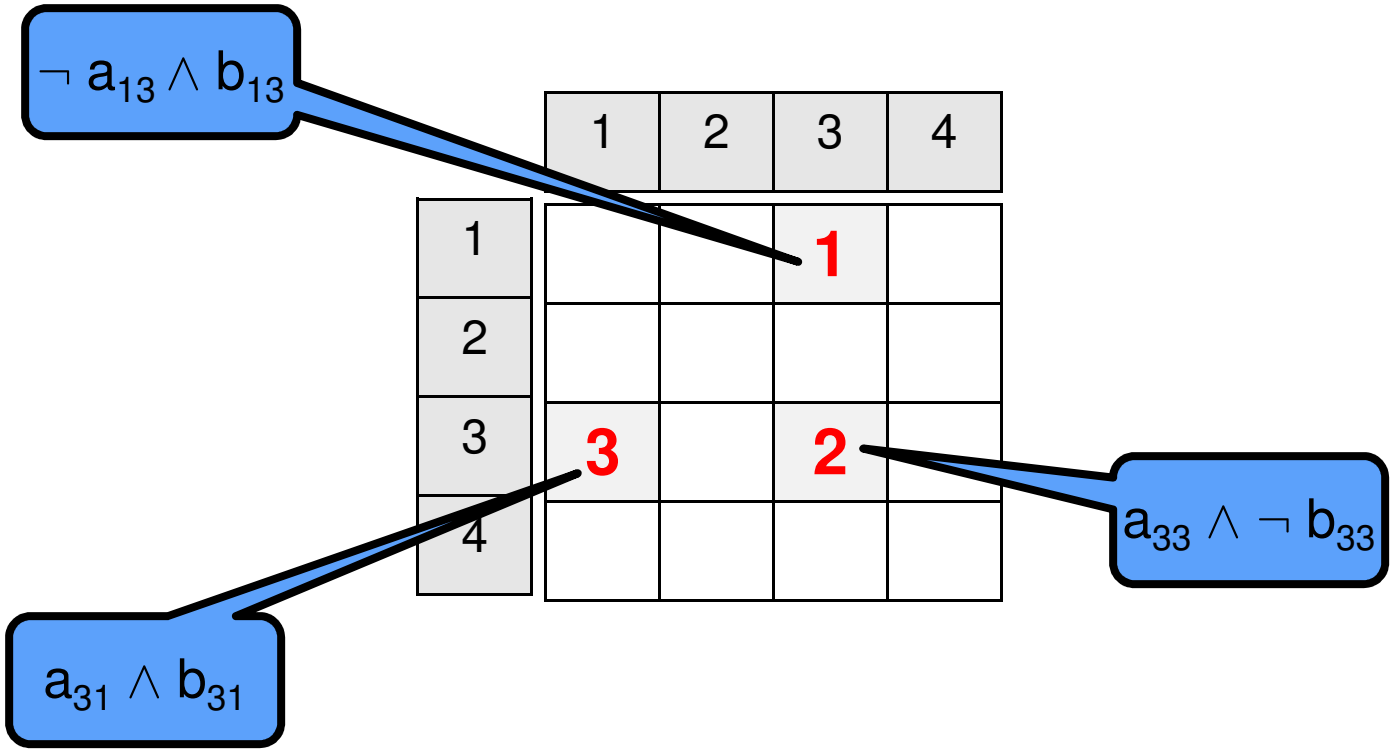
	1	2	3	4
1	0	3	1	2
2	1	2	0	3
3	3	0	2	1
4	2	1	3	0

How many ways can you solve this puzzle? At least 2.

BDD Application: Counting Sudoku Solutions



BDD Application: Counting Sudoku Solutions



BDD Application: Counting Sudoku Solutions

$$a_{11} \oplus a_{12} \vee b_{11} \oplus b_{12}$$
$$\wedge$$
$$a_{11} \oplus a_{13} \vee b_{11} \oplus b_{13}$$
$$\wedge$$
$$a_{11} \oplus a_{14} \vee b_{11} \oplus b_{14}$$
$$\wedge$$
$$a_{12} \oplus a_{13} \vee b_{12} \oplus b_{13}$$
$$\wedge$$
$$a_{12} \oplus a_{14} \vee b_{12} \oplus b_{14}$$
$$\wedge$$
$$a_{13} \oplus a_{14} \vee b_{13} \oplus b_{14}$$

	1	2	3	4
1			1	
2				
3	3		2	
4				

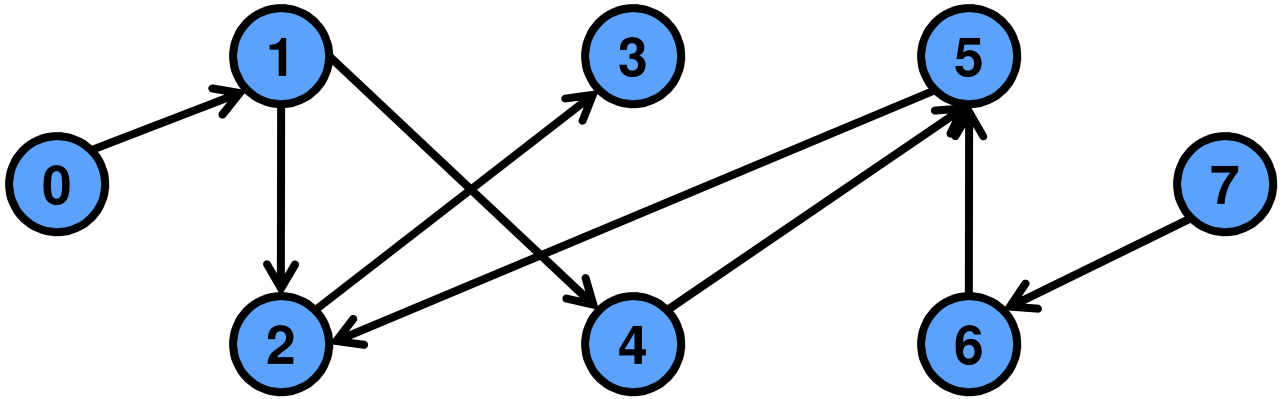
Distinct Elements

Repeat for each row, column and sub-square

Construct BDD

Count number of solutions

Graph Reachability

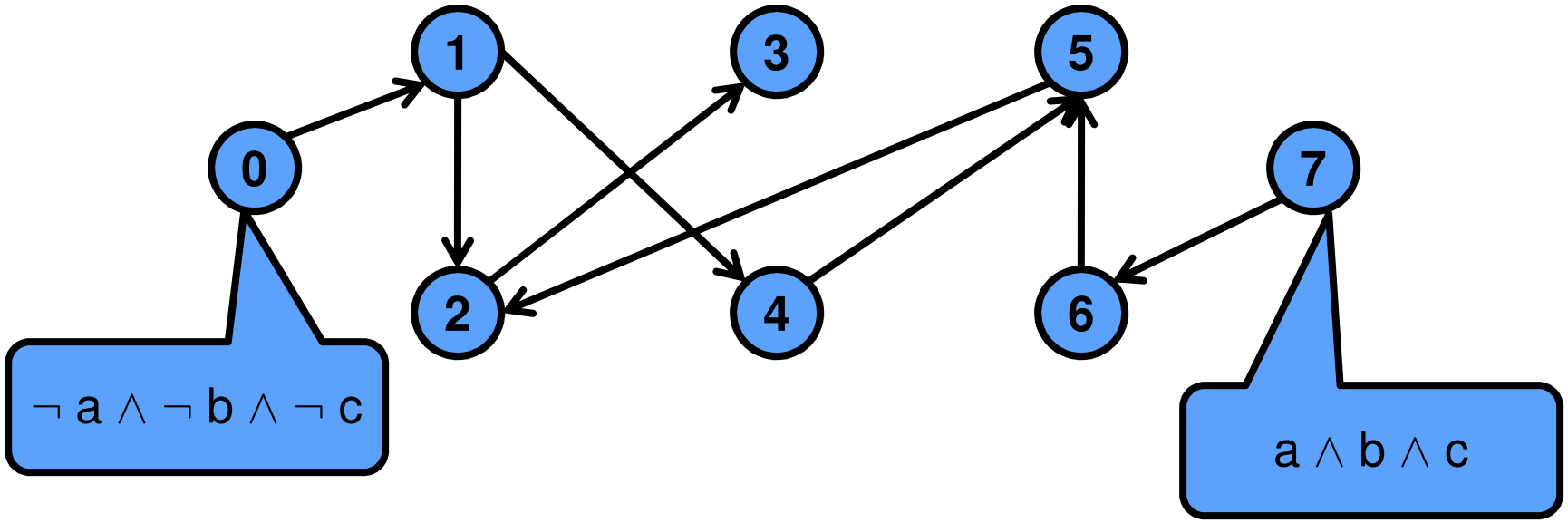


Which nodes are reachable from “7”?

{2,3,5,6,7}

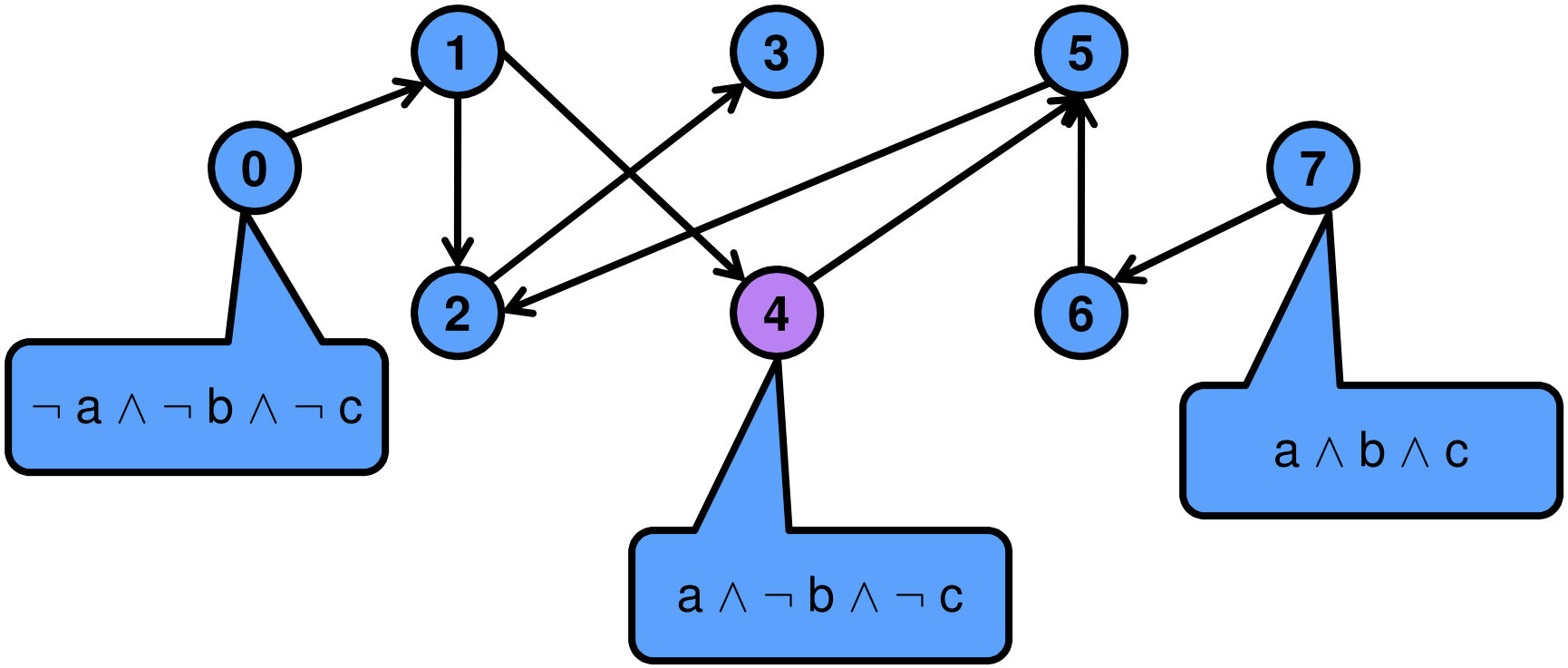
But what if the graph has trillions of nodes?

Graph Reachability



Use three Boolean variables (a,b,c) to encode each node?

Graph Reachability



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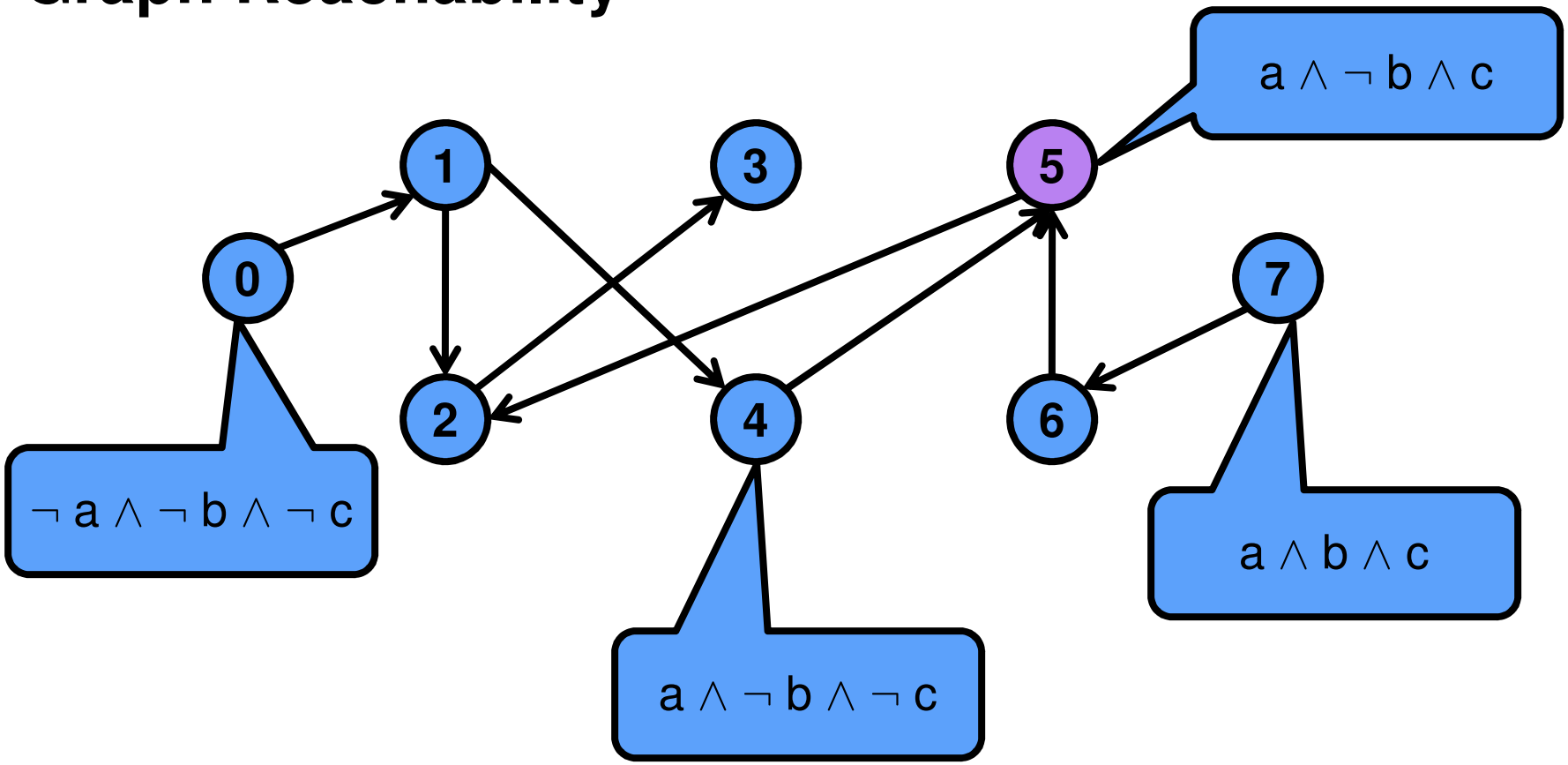
Graph Reachability



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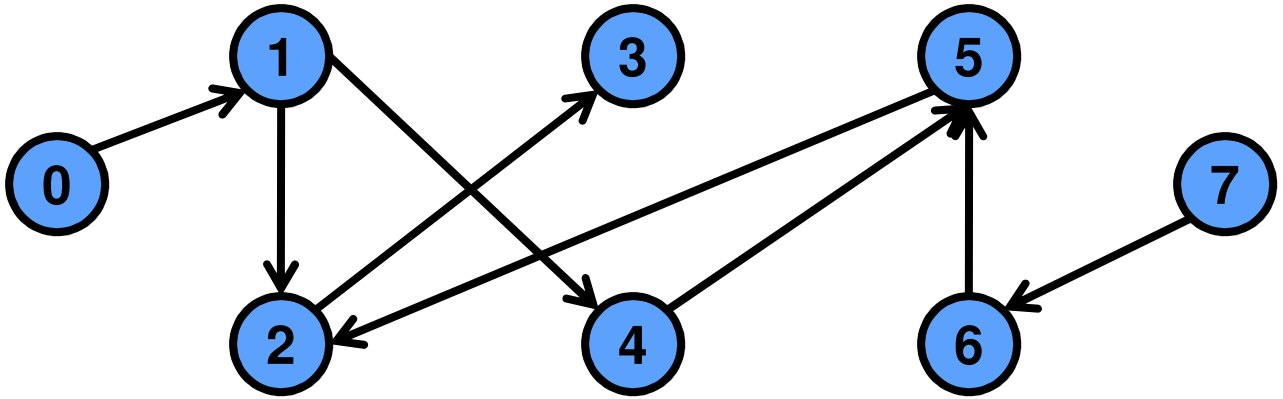
Graph Reachability

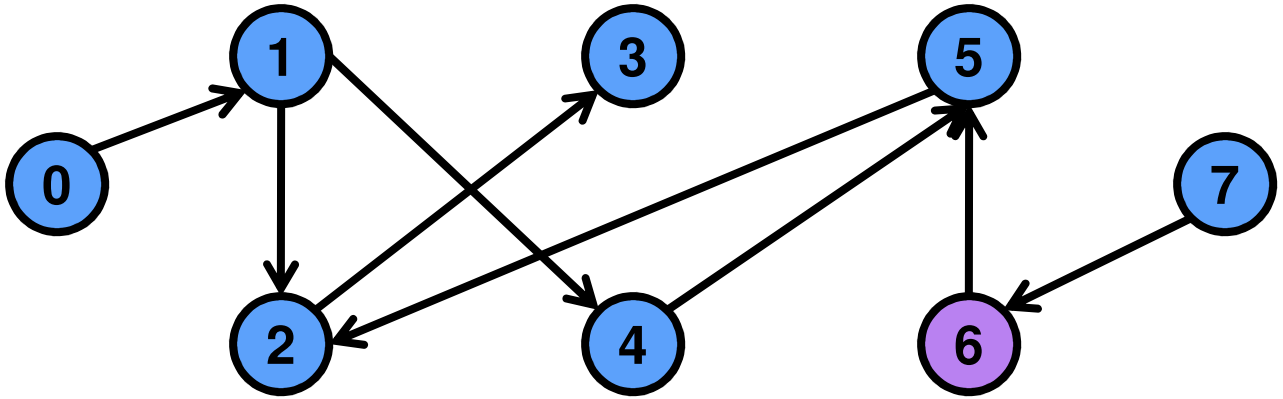


$a \wedge b \wedge \neg c = ?$

Key Idea 1: Every Boolean formula represents a set of nodes!

The nodes whose encodings satisfy the formula.

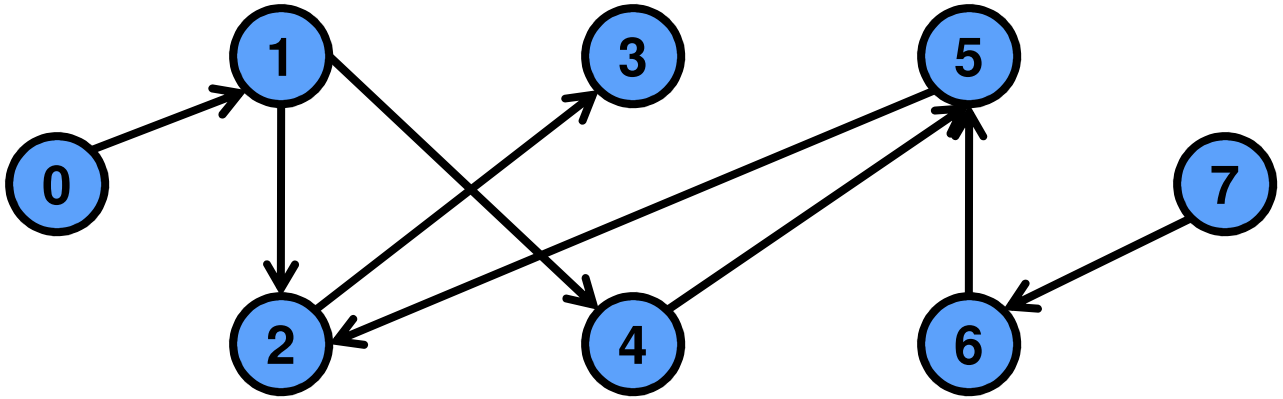
Graph Reachability



$a \wedge b \wedge \neg c = \{6\}$

Key Idea 1: Every Boolean formula represents a set of nodes!

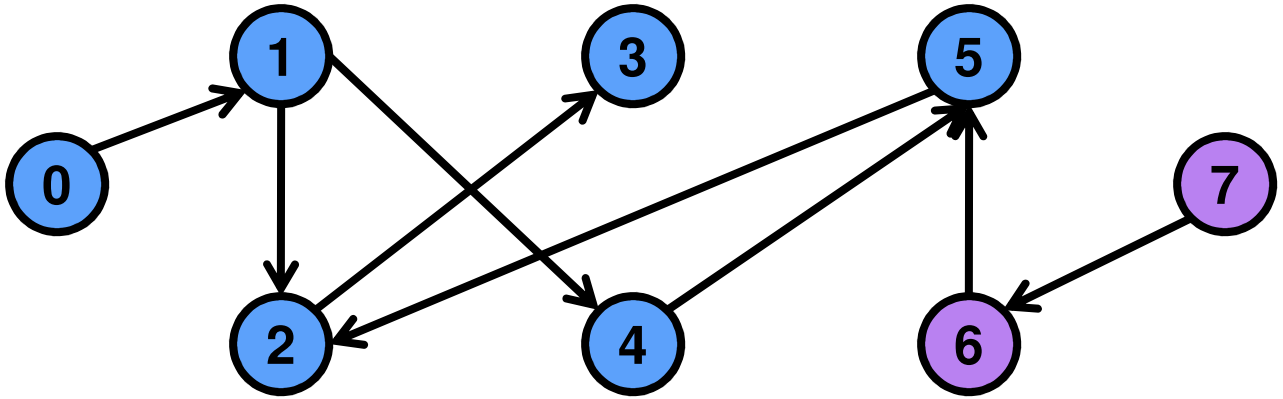
Graph Reachability



$a \wedge b = ?$

Key Idea 1: Every Boolean formula represents a set of nodes!

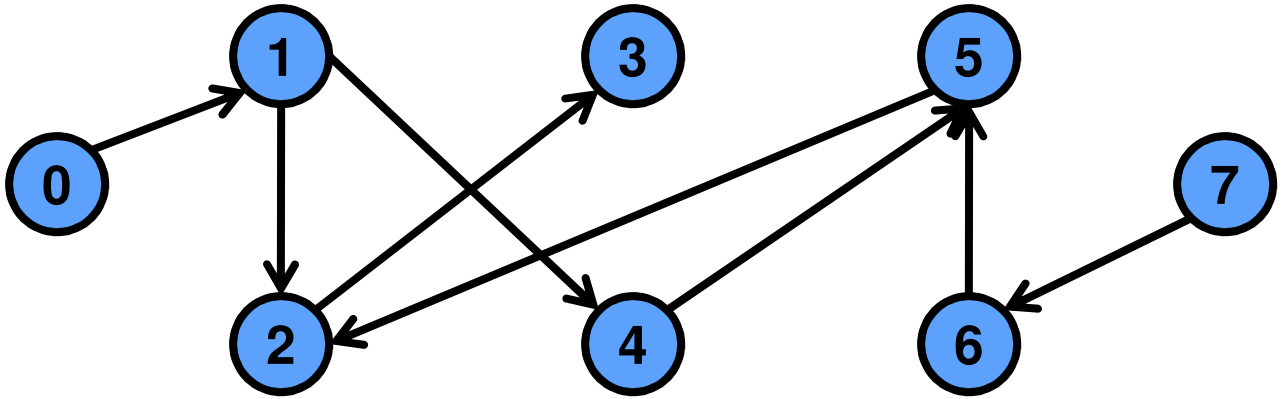
Graph Reachability



$a \wedge b = \{6,7\}$

Key Idea 1: Every Boolean formula represents a set of nodes!

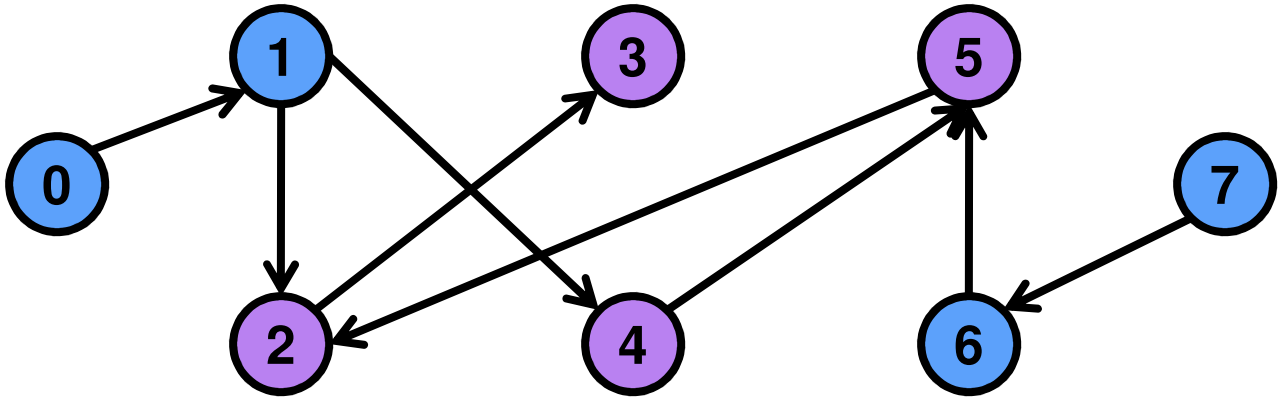
Graph Reachability



$a \oplus b = ?$

Key Idea 1: Every Boolean formula represents a set of nodes!

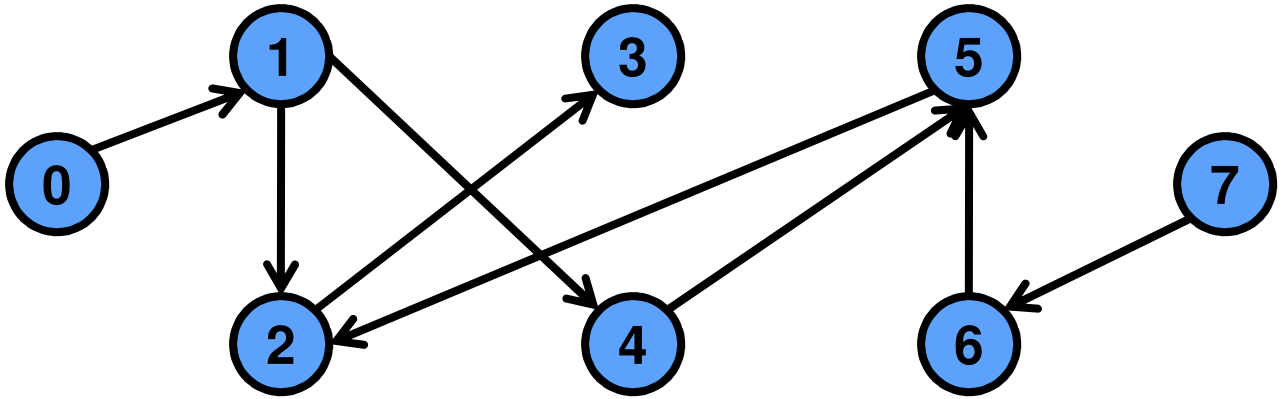
Graph Reachability



$a \oplus b = \{2,3,4,5\}$

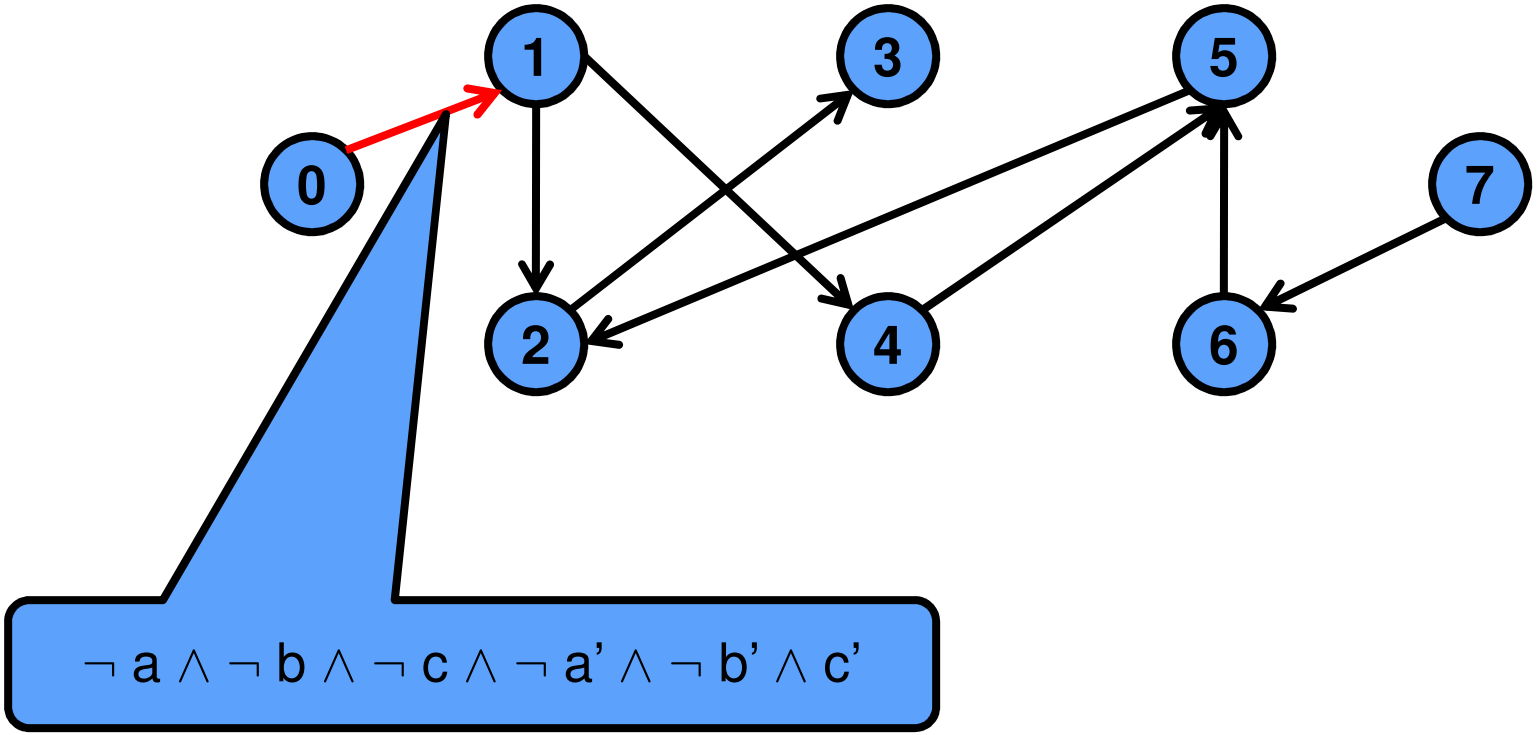
Key Idea 1: Every Boolean formula represents a set of nodes!

Graph Reachability



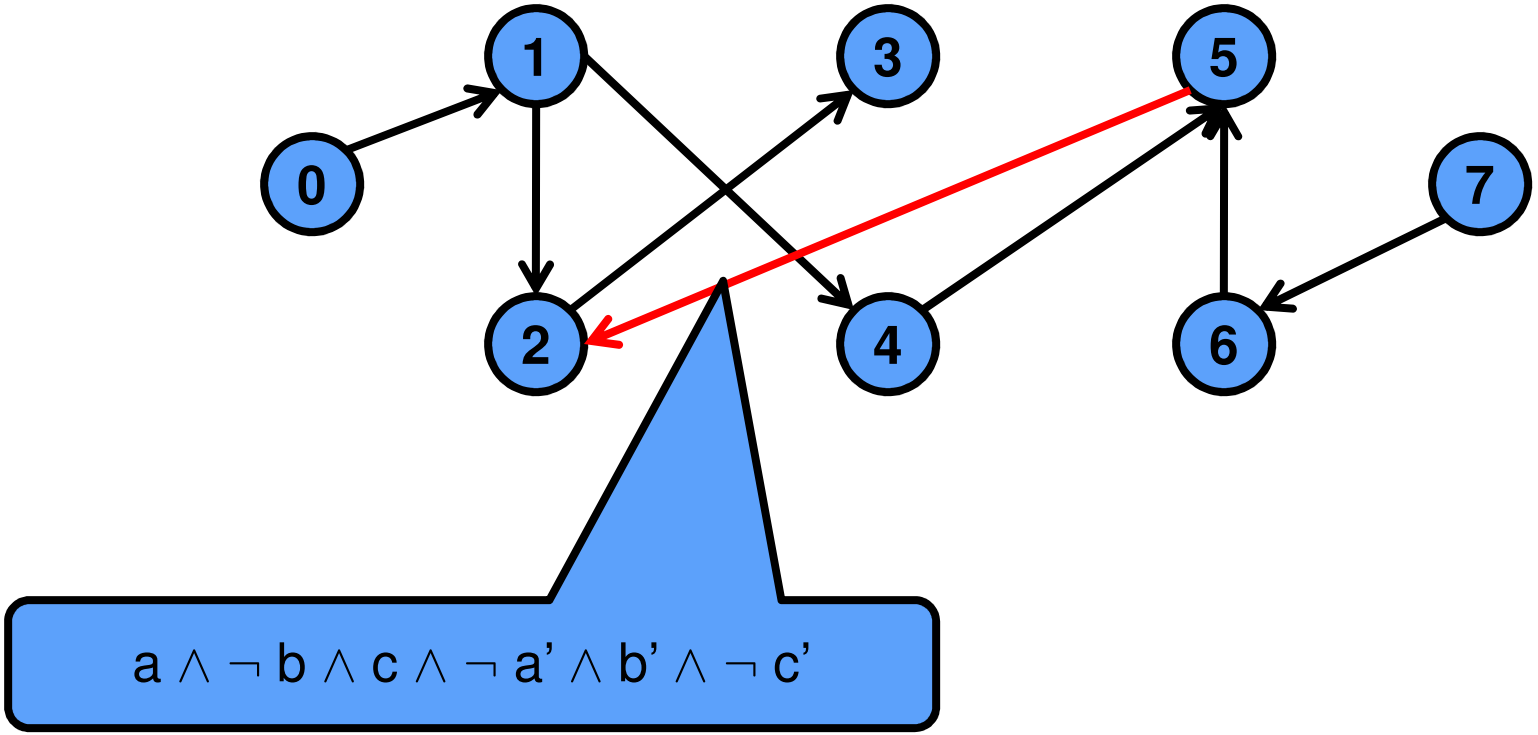
- Key Idea 2: Edges can also be represented by Boolean formulas
- An edge is just a pair of nodes
- Introduce three new variables: a', b', c'
- Formula Φ represents all pairs of nodes (n,n') that satisfy Φ when n is encoded using (a,b,c) and n' is encoded using (a',b',c')

Graph Reachability



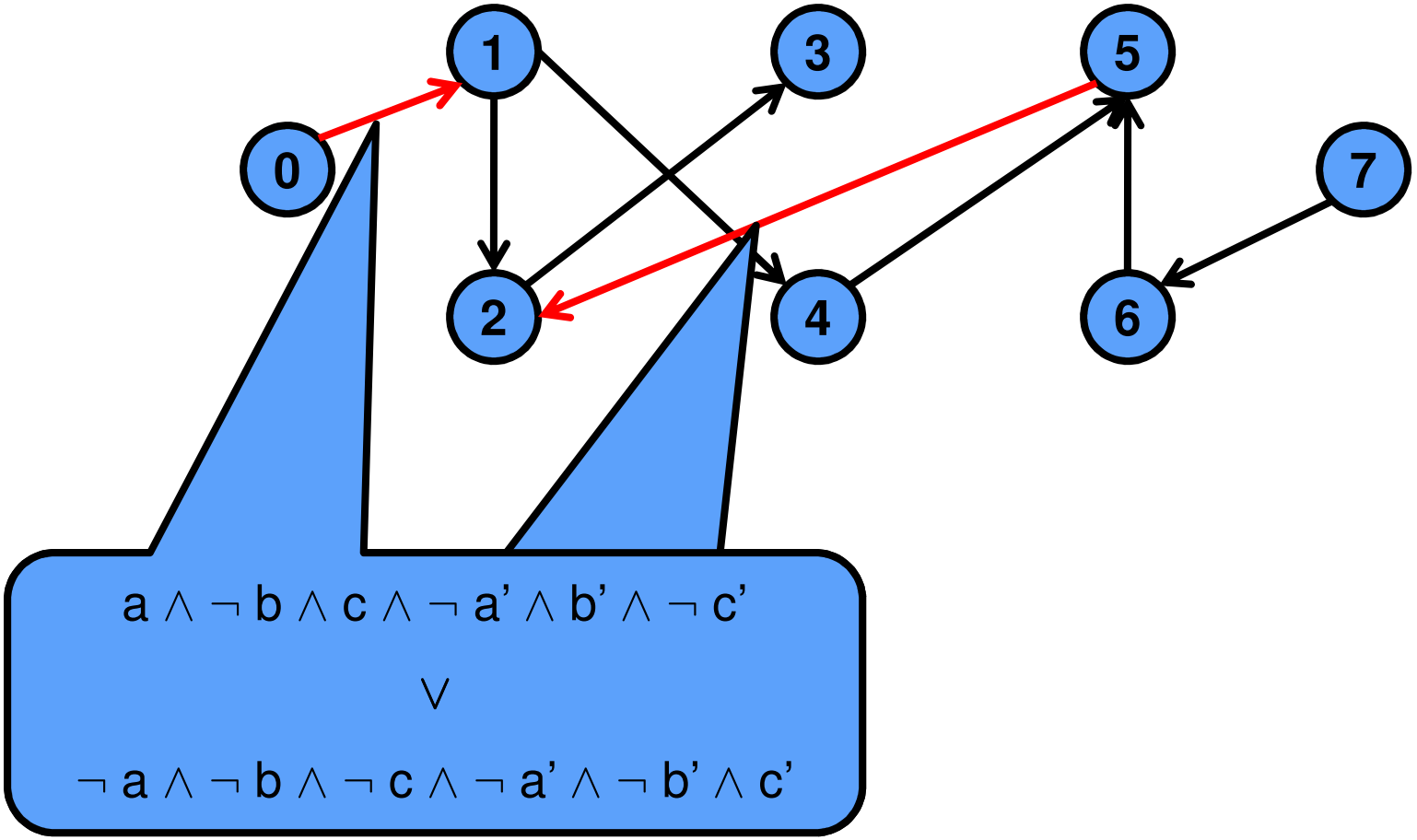
Key Idea 2: Edges can also be represented by Boolean formulas

Graph Reachability



Key Idea 2: Edges can also be represented by Boolean formulas

Graph Reachability



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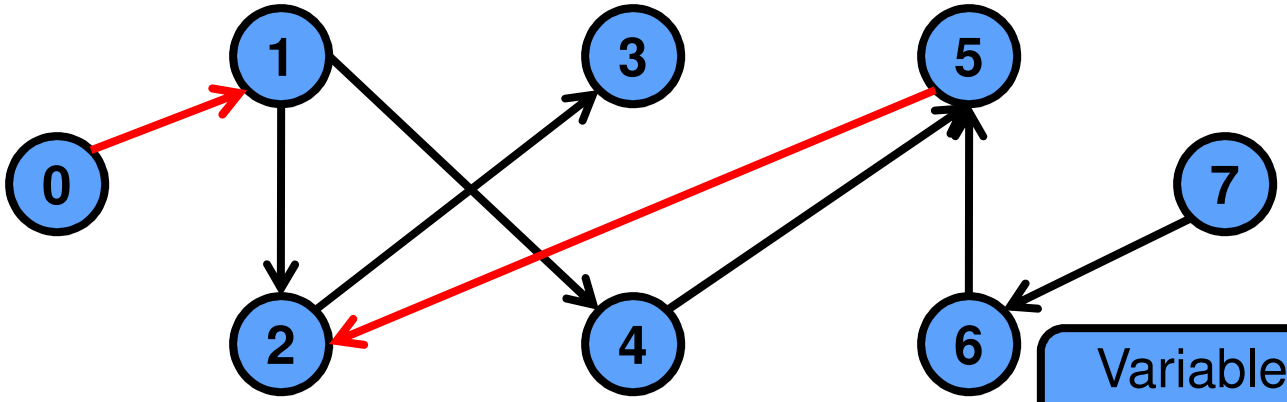

Variable renaming : replace a' with a

$$\text{Image}(S,R) =$$

$$(\exists\, a,b,c \,.\, (S \wedge R))\, [\, a \setminus a',\ b \setminus b',\ c \setminus c' \,]$$

Key Idea 3: Given the BDD for a set of nodes S, and the BDD for the set of all edges R, the BDD for all the nodes that are adjacent to S can be computed using the BDD operations

Graph Reachability Algorithm

```
S = BDD for initial set of nodes;
R = BDD for all the edges of the graph;

while (true) {
   I = Image(S,R); //compute adjacent nodes to S
   if (And(Not(S),I) == False) //no new nodes found
      break;
   S = Or(S,I); //add newly discovered nodes to result
}

return S;
```

Symbolic Model Checking. Has been done for graphs with 10^{20} nodes.

Questions?

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