Binary Decision Diagrams Part 1

15-414 Bug Catching: Automated Program Verification and Testing

Sagar Chaki
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Software Engineering Institute | Carnegie Mellon

BDDs in a nutshell

Typically mean Reduced Ordered Binary Decision Diagrams (ROBDDs)

Canonical representation of Boolean formulas

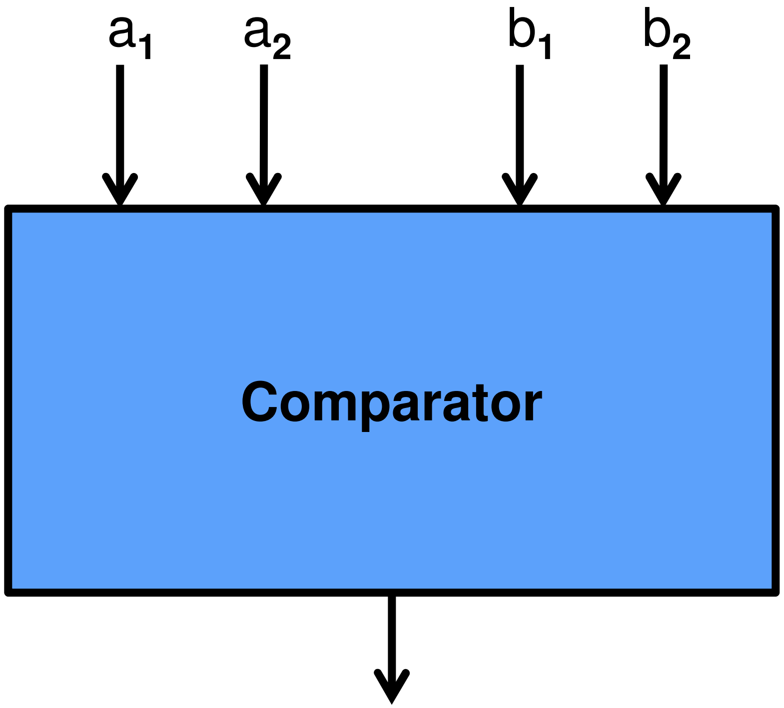
Often substantially more compact than a traditional normal form

Can be manipulated very efficiently

- Conjunction, Disjunction, Negation, Existential Quantification

R. E. Bryant. Graph-based algorithms for boolean function manipulation. *IEEE Transactions on Computers, C-35(8), 1986.*

Running Example: Comparator



$$f = 1 \Leftrightarrow a_1 = b_1 \wedge a_2 = b_2$$

Conjunctive Normal Form

$$f$$

$$a_1 = b_1 \quad \wedge \quad a_2 = b_2$$

$$a_1 \Rightarrow b_1 \wedge b_1 \Rightarrow a_1 \quad \wedge \quad a_2 \Rightarrow b_2 \wedge b_2 \Rightarrow a_2$$

$$(\neg\, a_1 \vee b_1\,) \wedge (\neg\, b_1 \vee a_1) \wedge (\neg\, a_2 \vee b_2\,) \wedge (\neg\, b_2 \vee a_2)$$

$$(\neg\, b_1 \vee a_1\,) \wedge (\neg\, a_1 \vee b_1) \wedge (\neg\, a_2 \vee b_2\,) \wedge (\neg\, b_2 \vee a_2)$$

Not Canonical

Truth Table (1)

a_1	b_1	a_2	b_2	f
0	0	0	0	1
0	0	0	1	0
0	0	1	0	0
0	0	1	1	1
0	1	0	0	0
0	1	0	1	0
0	1	1	0	0
0	1	1	1	0
1	0	0	0	0
1	0	0	1	0
1	0	1	0	0
1	0	1	1	0
1	1	0	0	1
1	1	0	1	0
1	1	1	0	0
1	1	1	1	1

Still Not Canonical

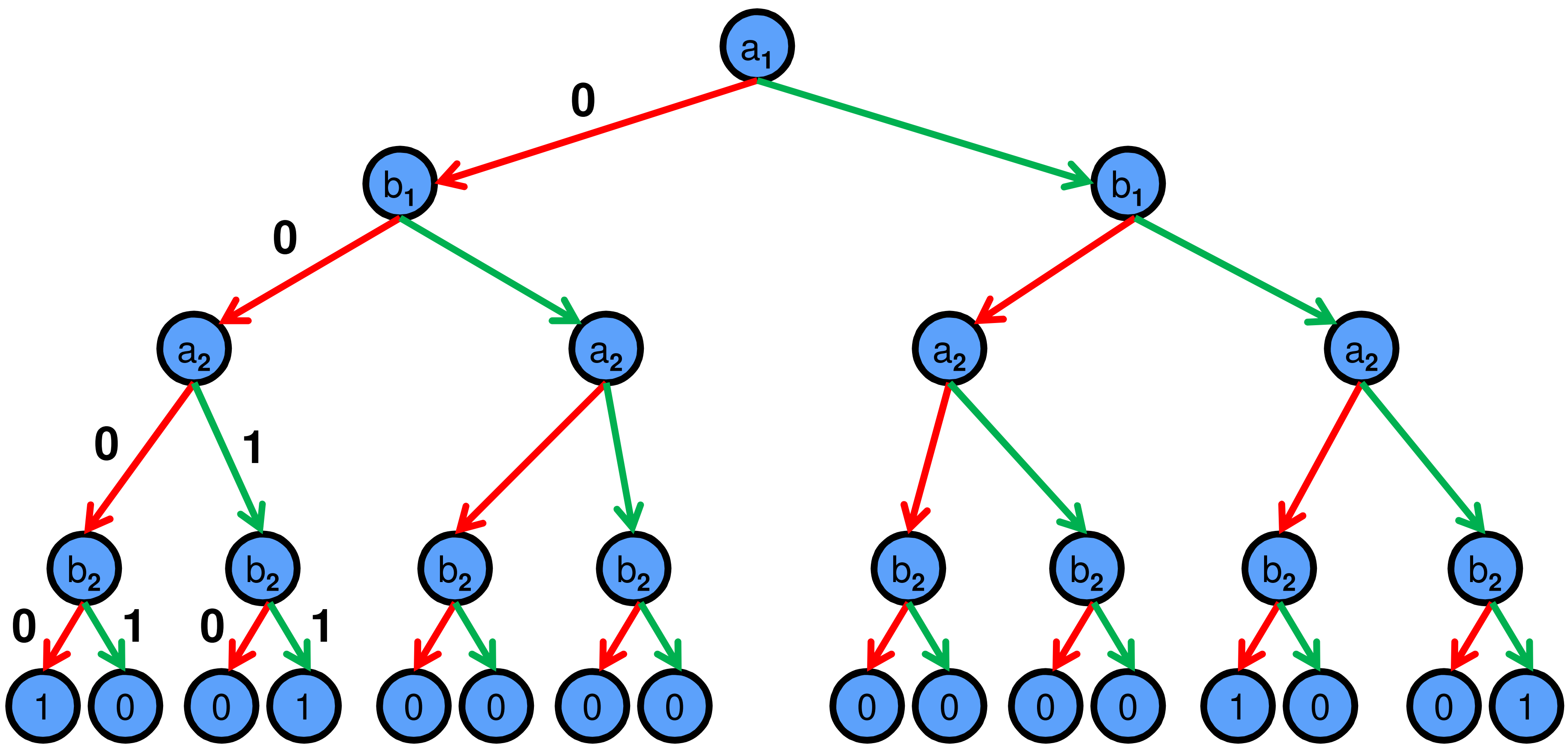
Truth Table (2)

a_1	a_2	b_1	b_2	f
0	0	0	0	1
0	0	0	1	0
0	0	1	0	0
0	0	1	1	0
0	1	0	0	0
0	1	0	1	1
0	1	1	0	0
0	1	1	1	0
1	0	0	0	0
1	0	0	1	0
1	0	1	0	1
1	0	1	1	0
1	1	0	0	0
1	1	0	1	0
1	1	1	0	0
1	1	1	1	1

Canonical if you fix variable order.

But always exponential in # of variables. Let's try to fix this.

Representing a Truth Table using a Graph



Binary Decision Tree (in this case ordered)

Binary Decision Tree: Formal Definition

Balanced binary tree. Length of each path = # of variables

Leaf nodes labeled with either 0 or 1

Internal node v labeled with a Boolean variable var(v)

- Every node on a path labeled with a different variable

Internal node v has two children: low(v) and high(v)

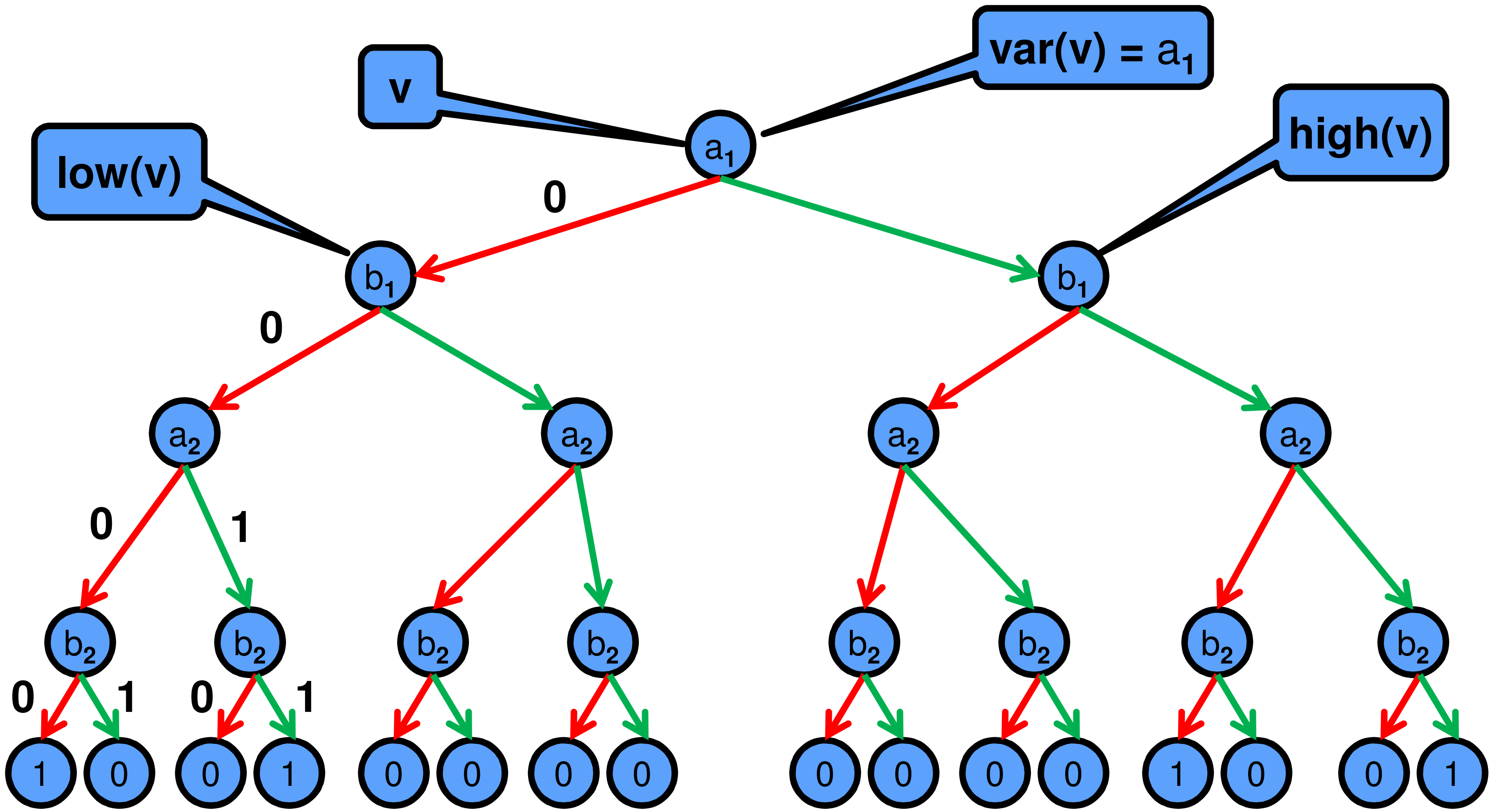
Each path corresponds to a (partial) truth assignment to variables

- Assign 0 to var(v) if low(v) is in the path, and 1 if high(v) is in the path

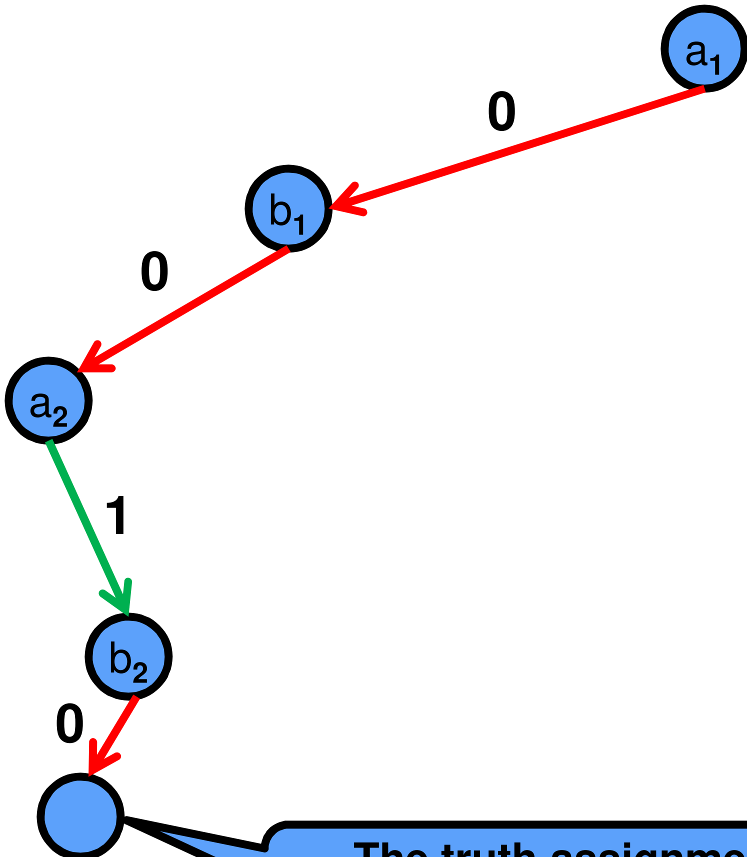
Value of a leaf is determined by:

- Constructing the truth assignment for the path leading to it from the root
- Looking up the truth table with this truth assignment

Binary Decision Tree



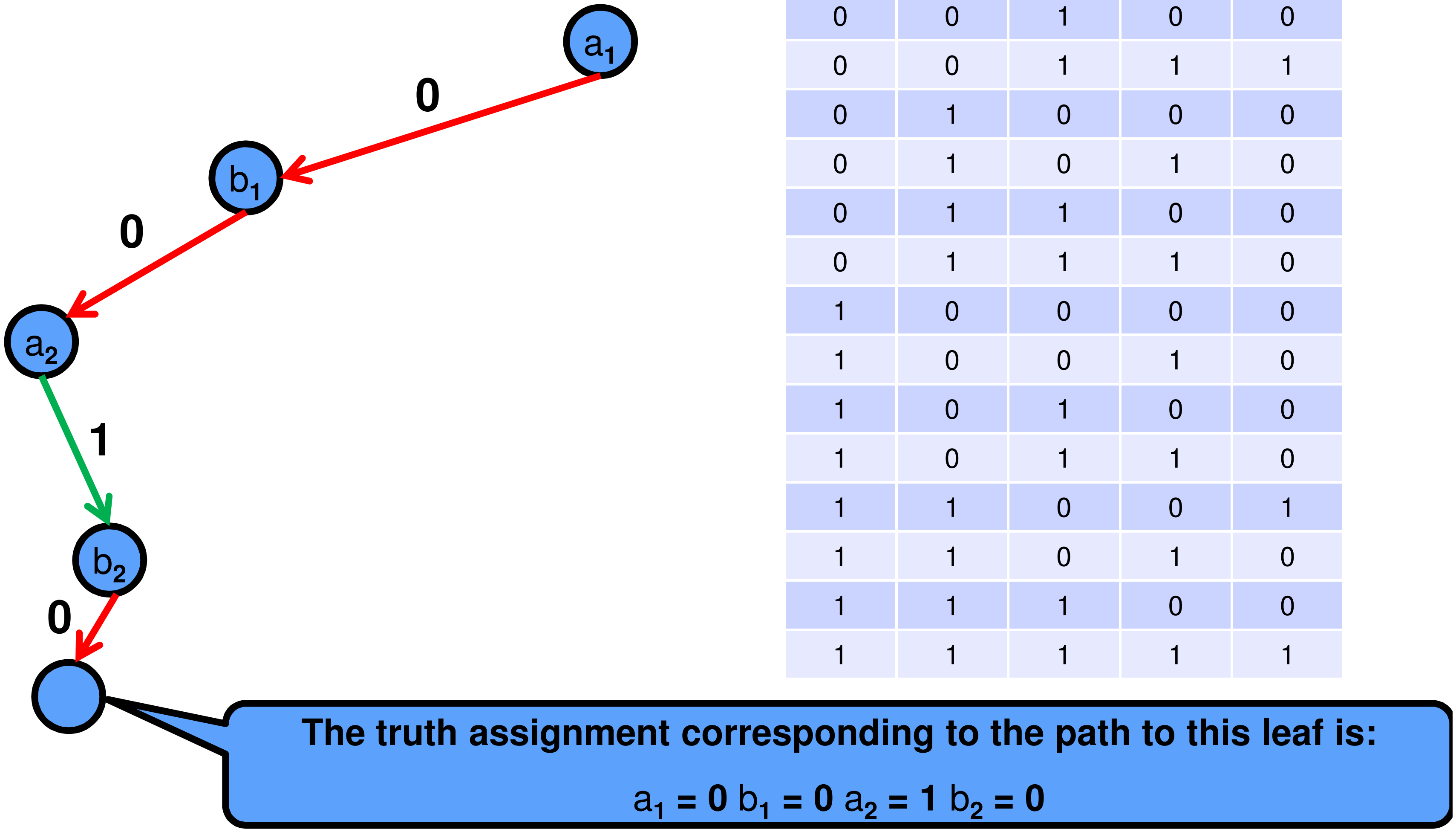
Binary Decision Tree



The truth assignment corresponding to the path to this leaf is:

a_1 = ? b_1 = ? a_2 = ? b_2 = ?

Binary Decision Tree



a_1	b_1	a_2	b_2	f
0	0	0	0	1
0	0	0	1	0
0	0	1	0	0
0	0	1	1	1
0	1	0	0	0
0	1	0	1	0
0	1	1	0	0
0	1	1	1	0
1	0	0	0	0
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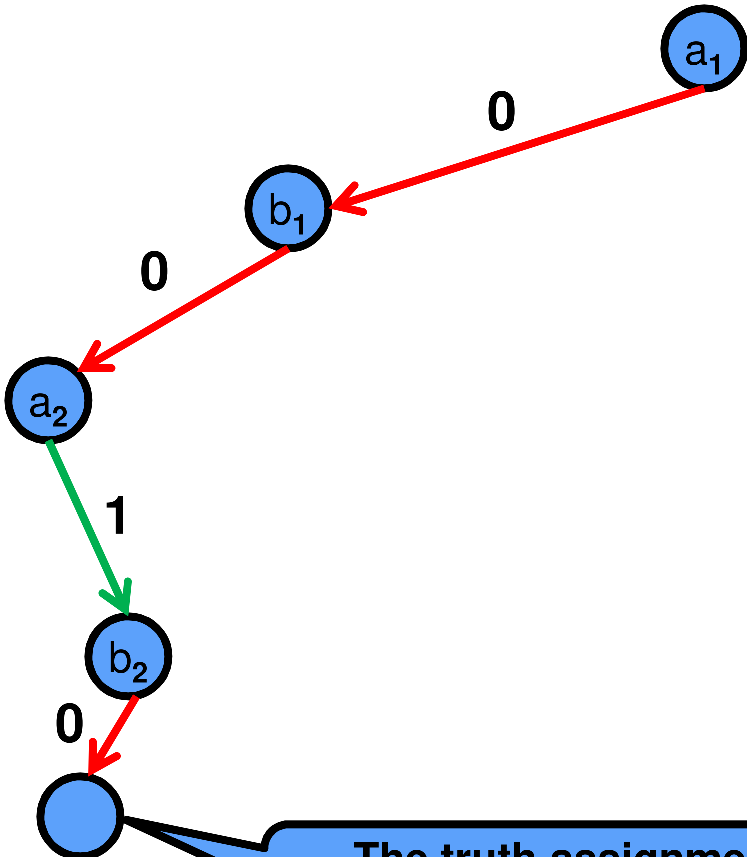
Binary Decision Tree



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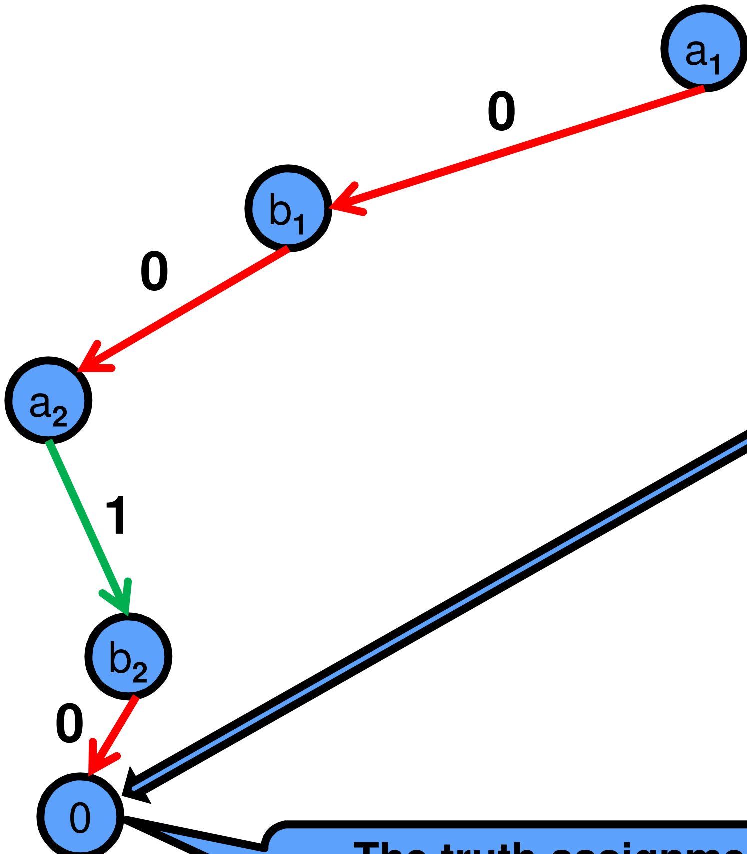
0	1	1	1	0
1	0	0	0	0
1	0	0	1	0
1	0	1	0	0
1	0	1	1	0
1	1	0	0	1
1	1	0	1	0
1	1	1	0	0
1	1	1	1	1

The truth assignment corresponding to the path to this leaf is:

a_1 **= 0** b_1 **= 0** a_2 **= 1** b_2 **= 0**

Binary Decision Tree



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0	0	1	0	0
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0	1	0	1	0
0	1	1	0	0
0	1	1	1	0
1	0	0	0	0
1	0	0	1	0
1	0	1	0	0
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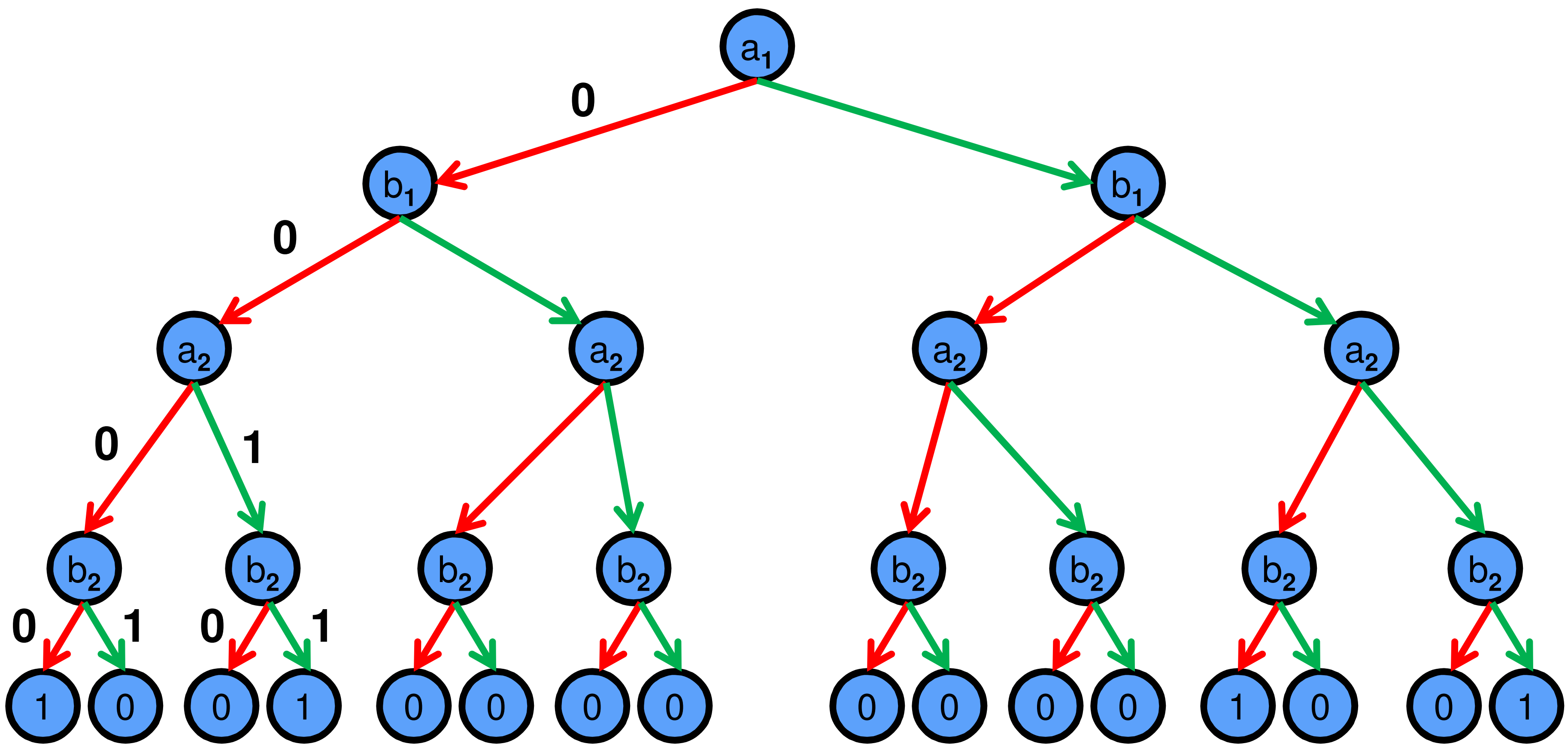
Binary Decision Tree (BDT)



Canonical if you fix variable order (i.e., use ordered BDT)

But still exponential in # of variables. Let's try to fix this.

Reduced Ordered BDD

Conceptually, a ROBDD is obtained from an ordered BDT (OBDT) by eliminating redundant sub-diagrams and nodes

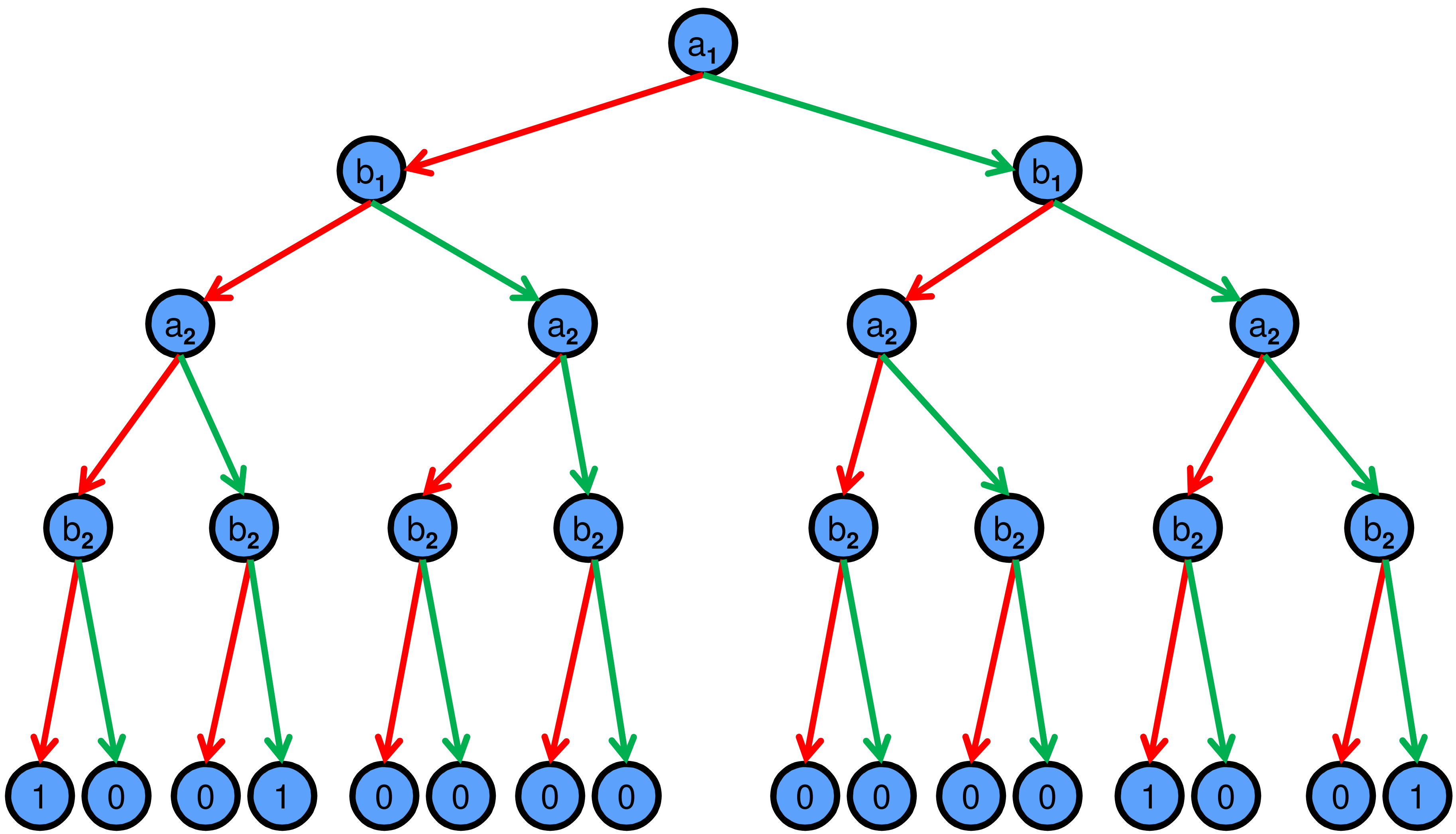
Start with OBDT and repeatedly apply the following two operations as long as possible:

1. Eliminate duplicate sub-diagrams. Keep a single copy. Redirect edges into the eliminated duplicates into this single copy.
2. Eliminate redundant nodes. Whenever low(v) = high(v), remove v and redirect edges into v to low(v).

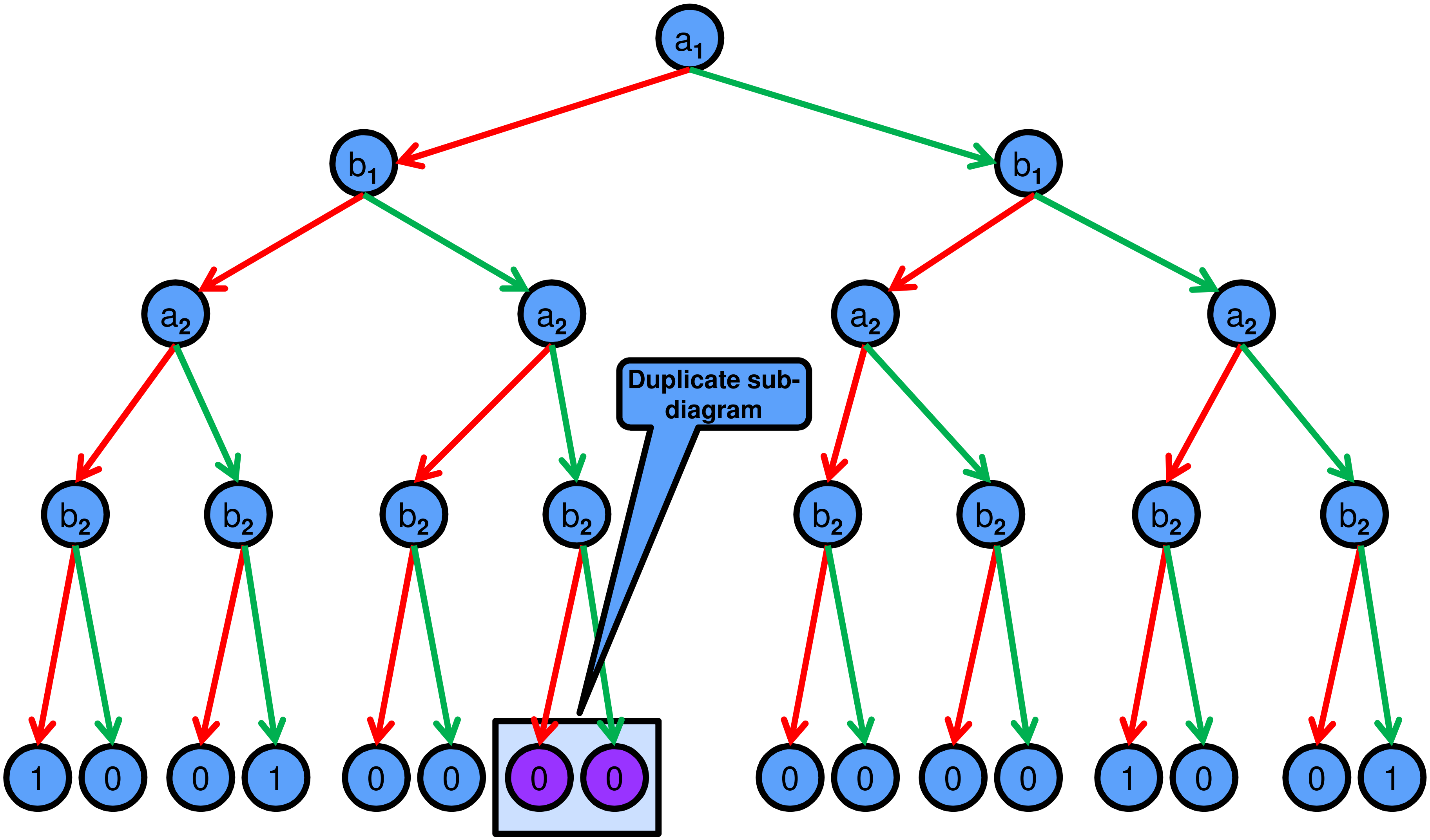
- Why does this terminate?

ROBDD is often exponentially smaller than the corresponding OBDT

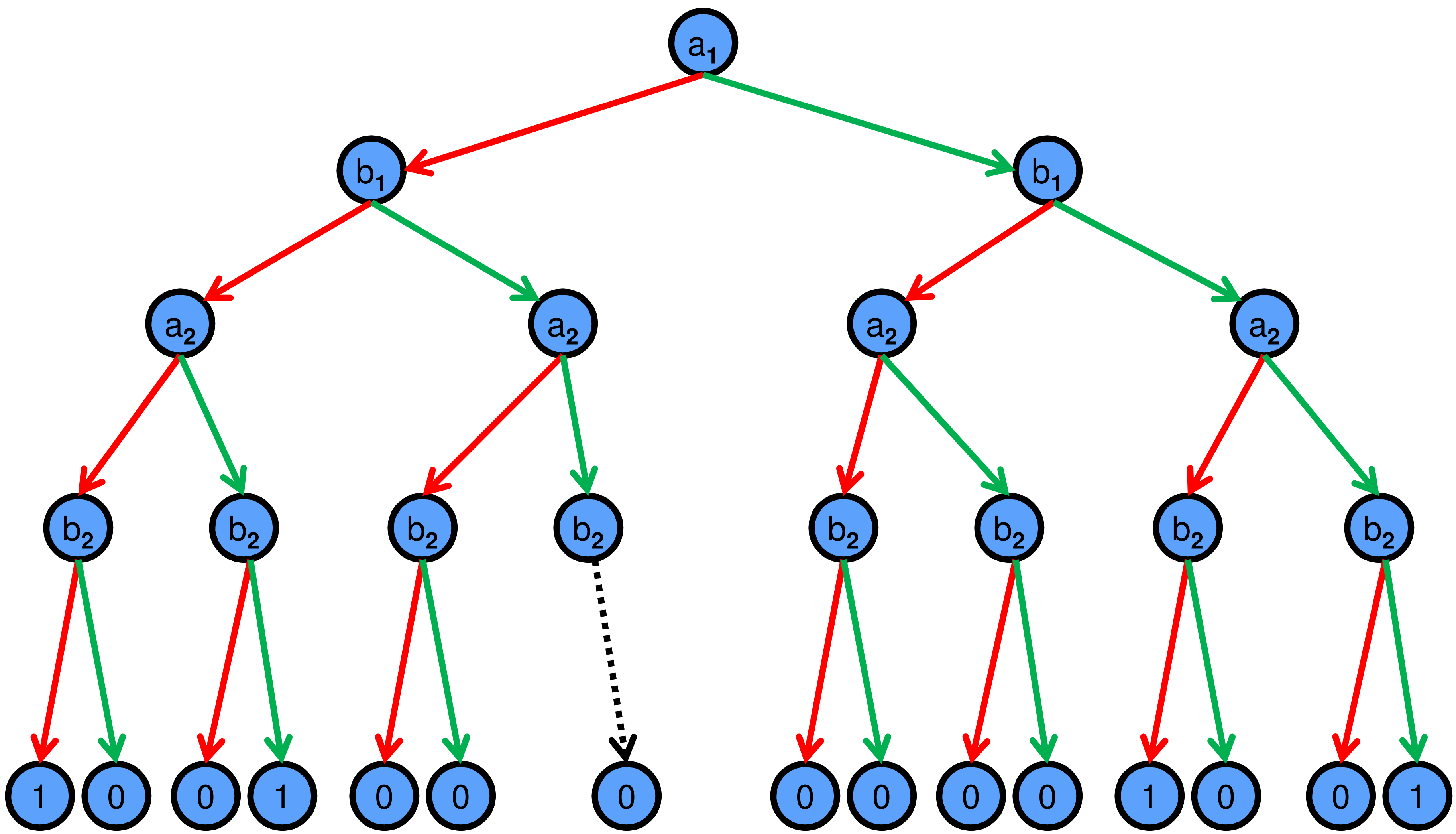
OBDT to ROBDD



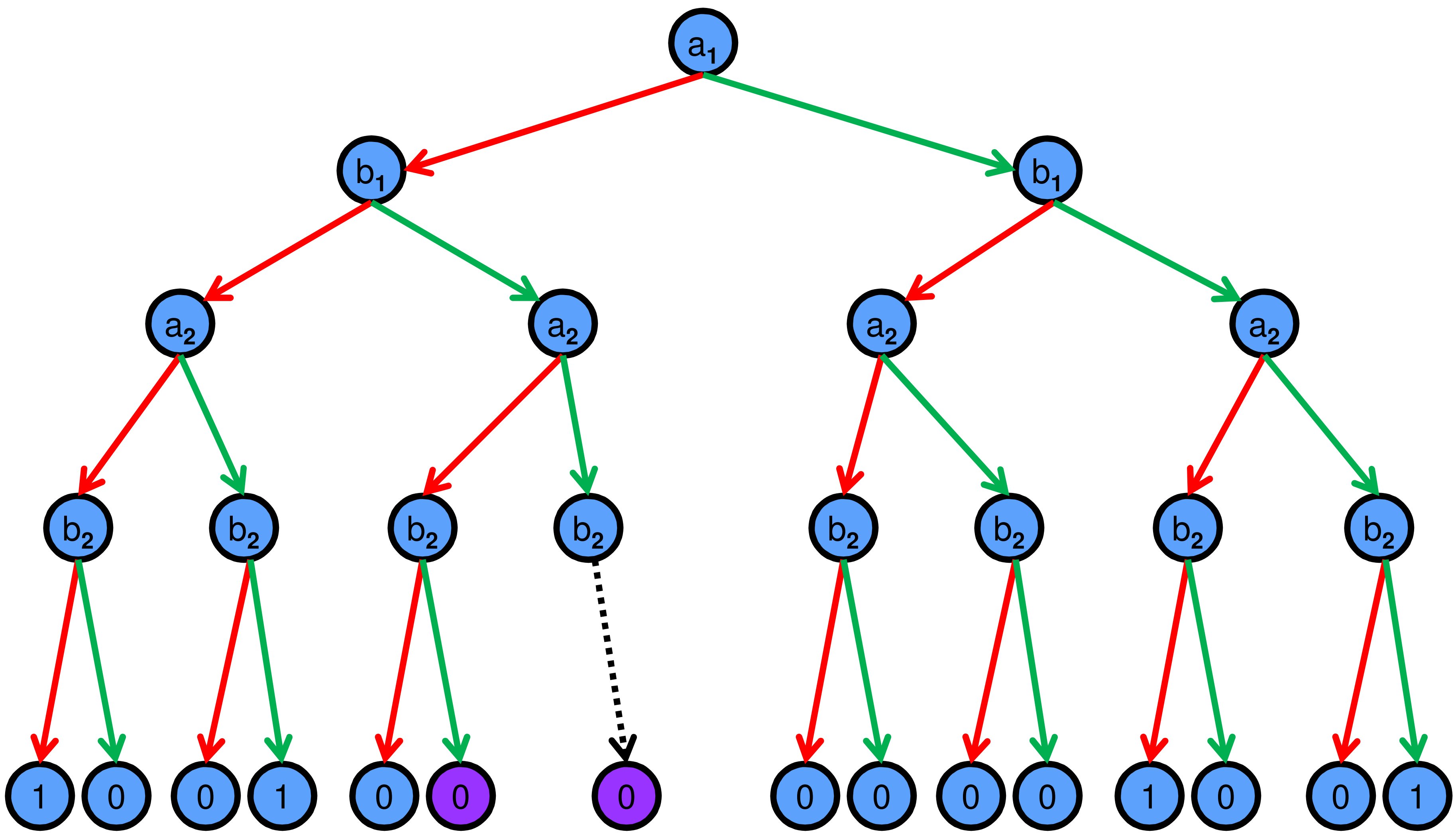
OBDT to ROBDD



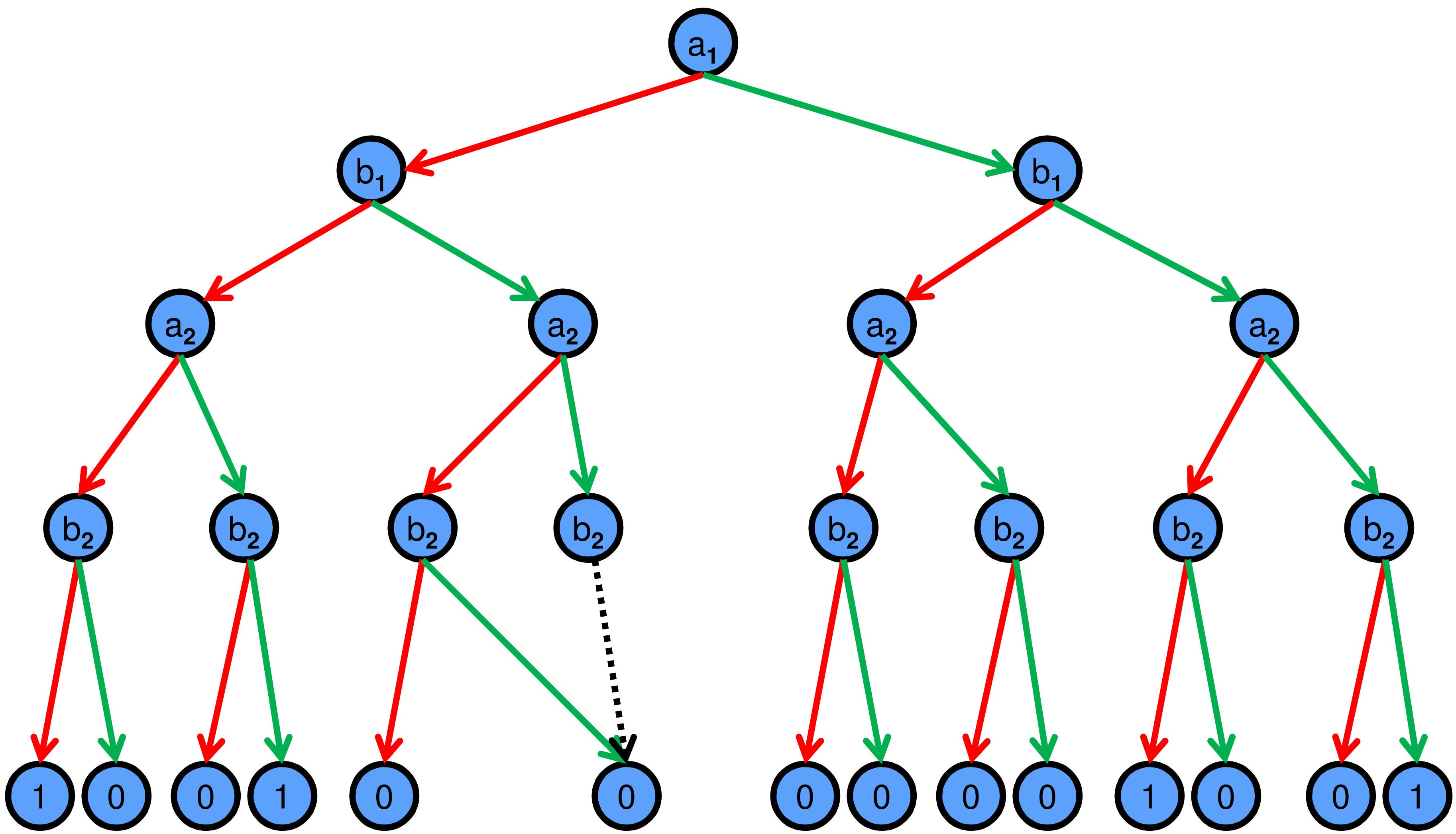
OBDT to ROBDD



OBDT to ROBDD



OBDT to ROBDD



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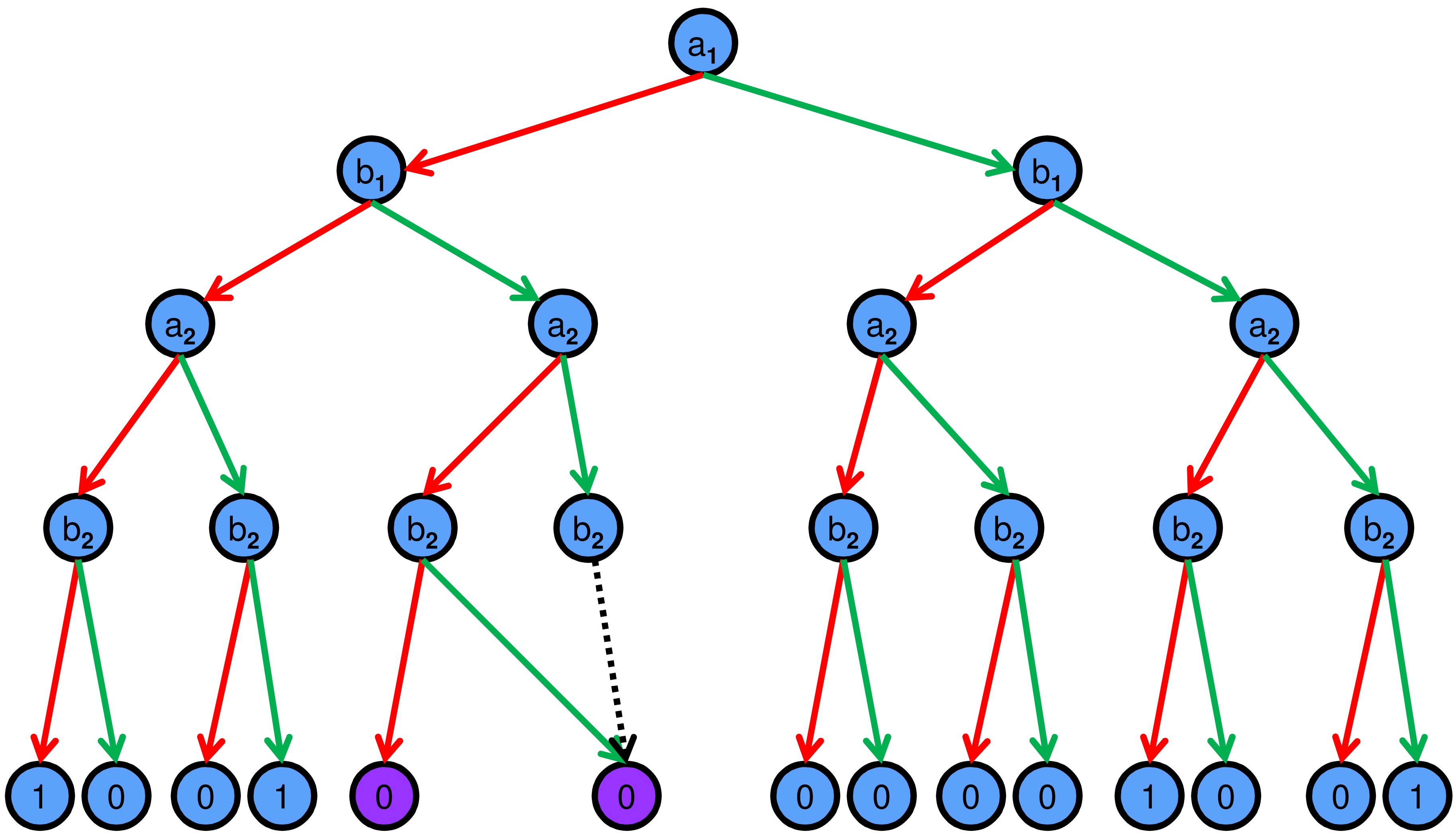

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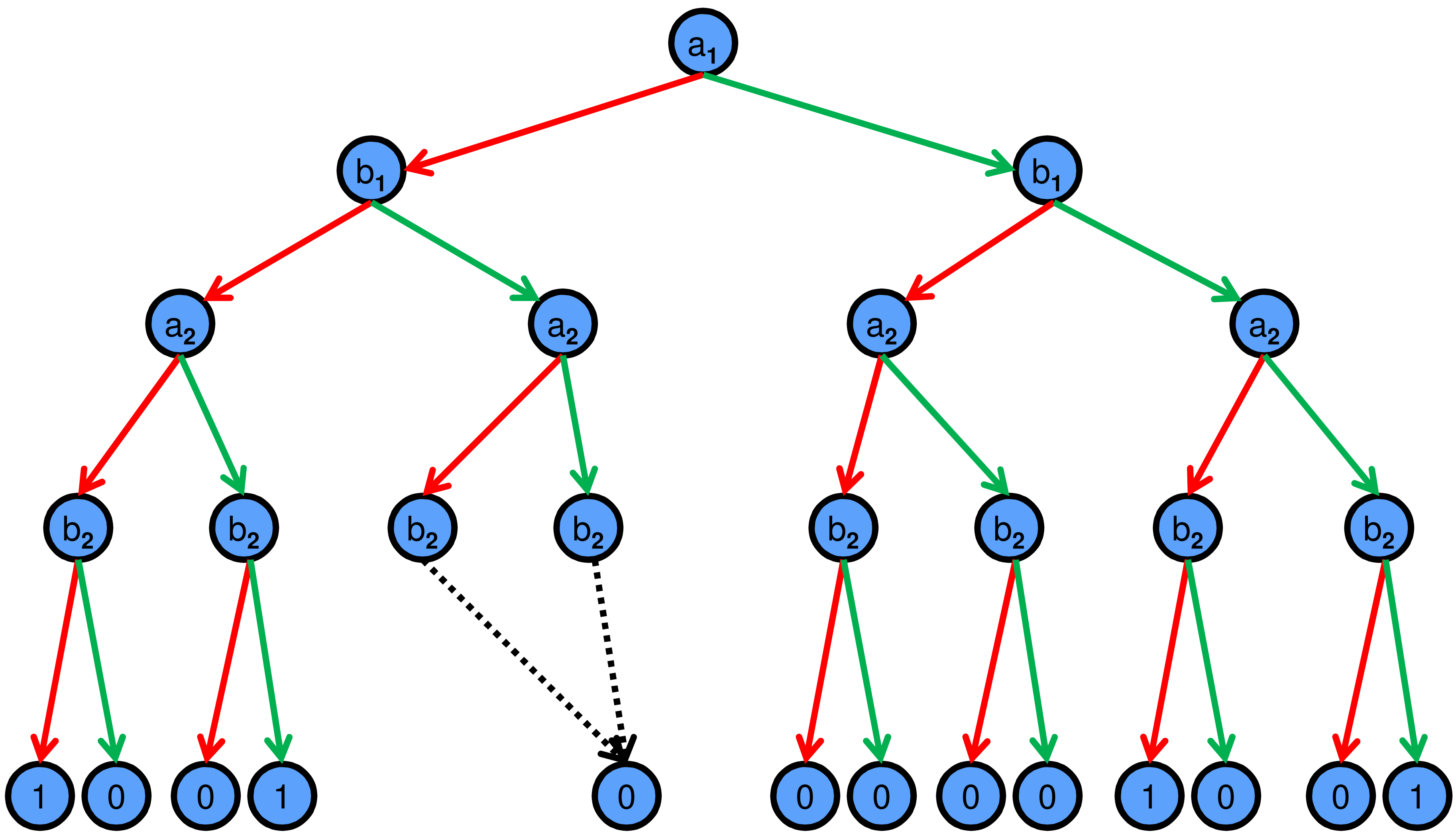

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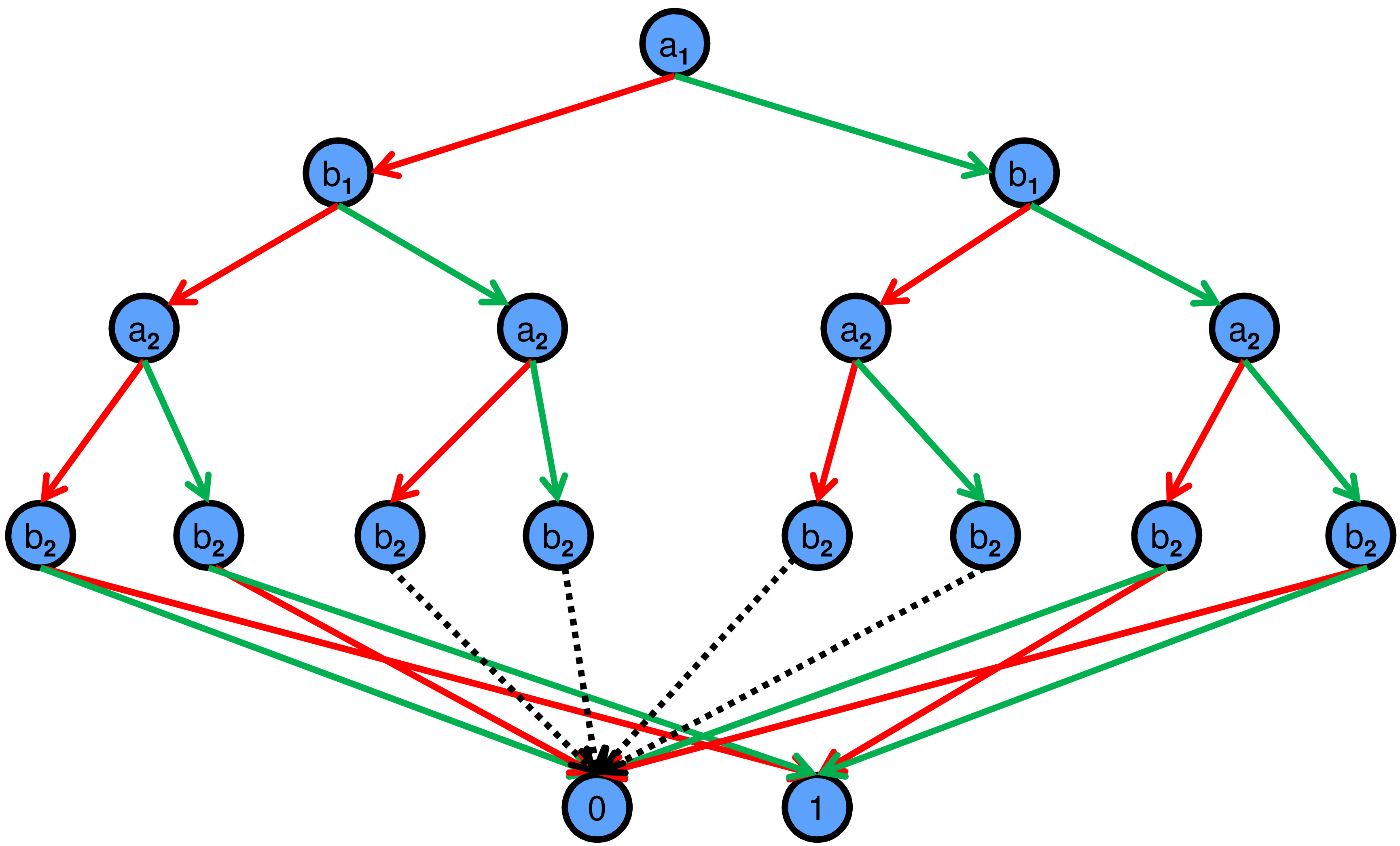
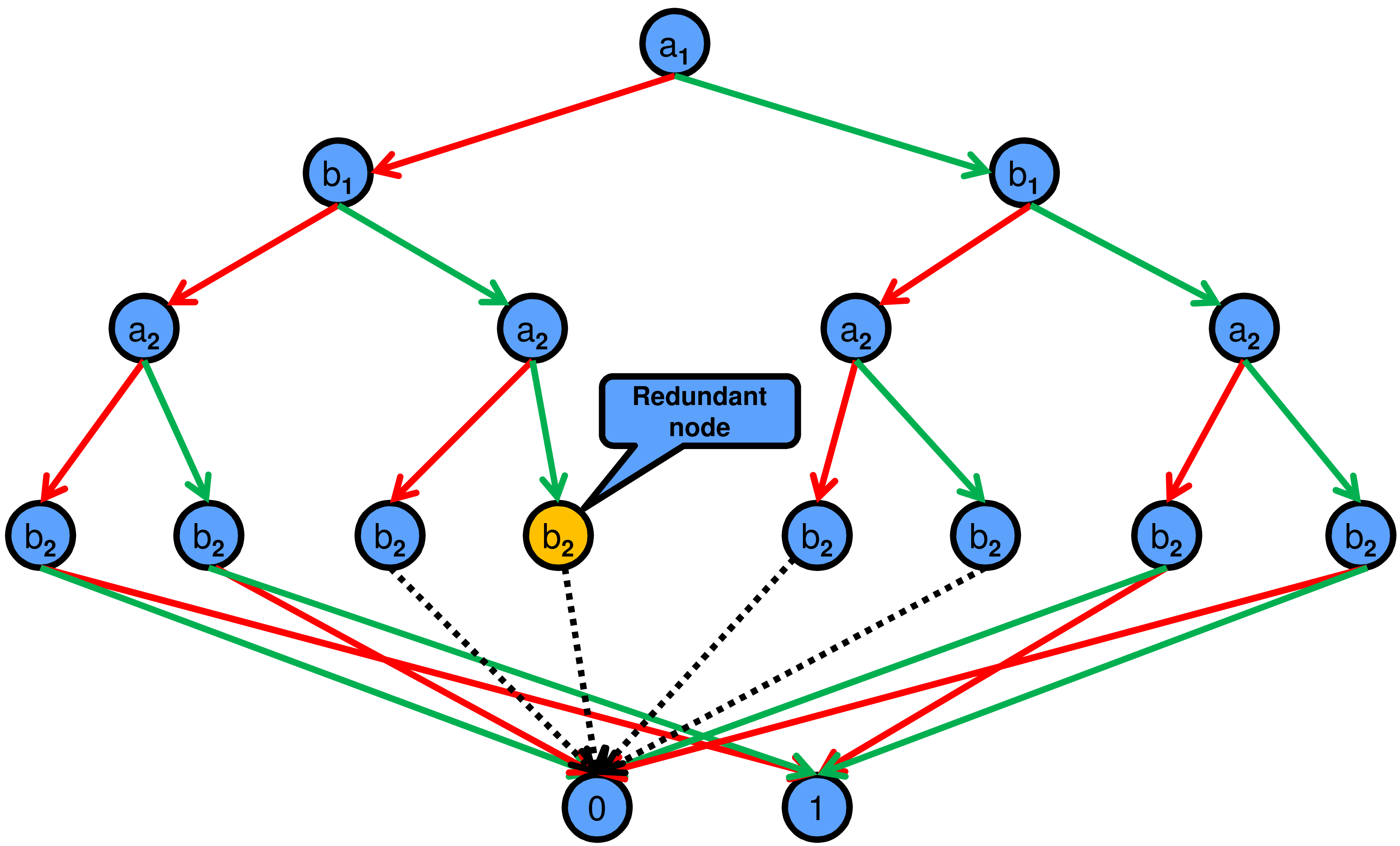
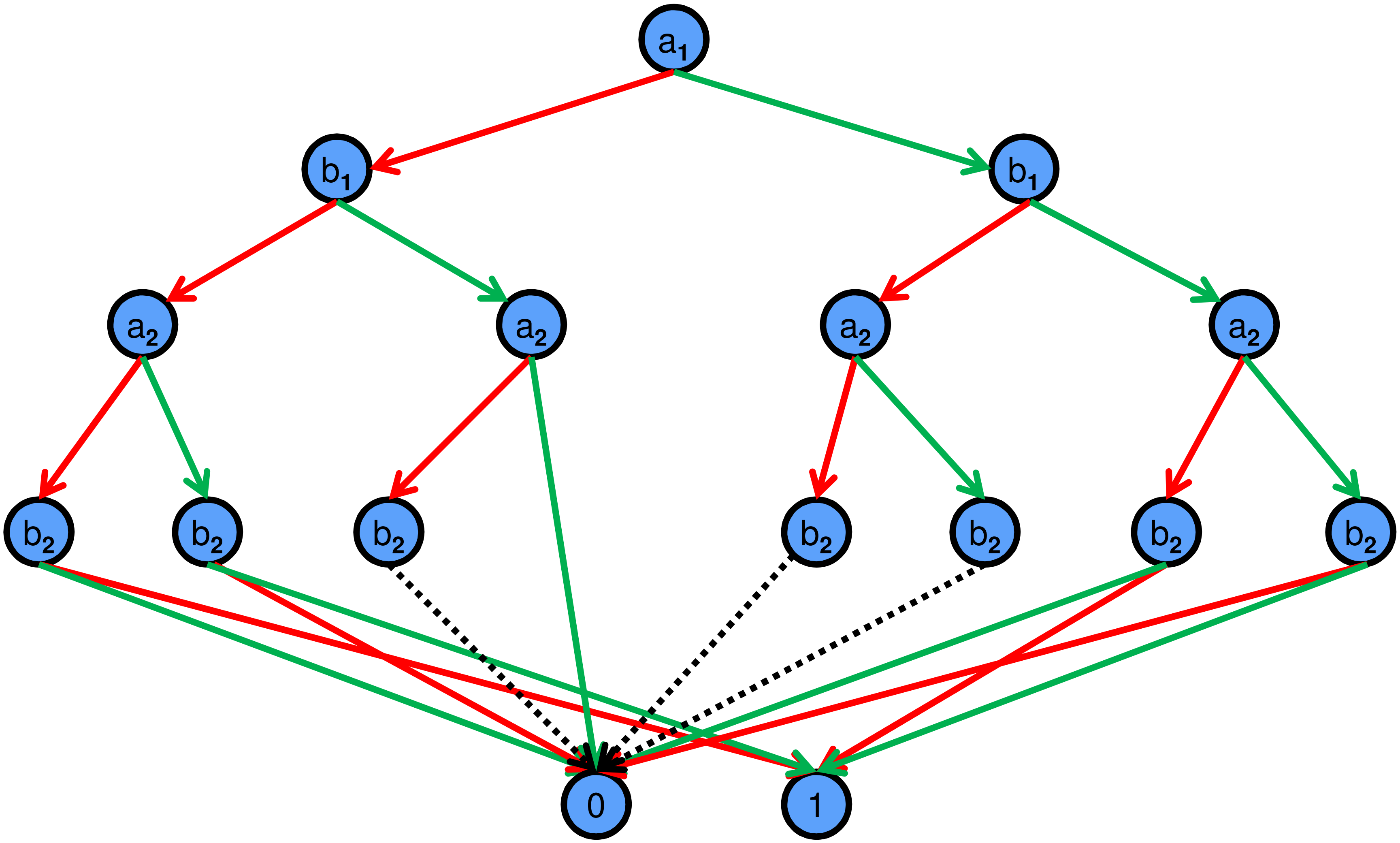


OBDT to ROBDD



OBDT to ROBDD



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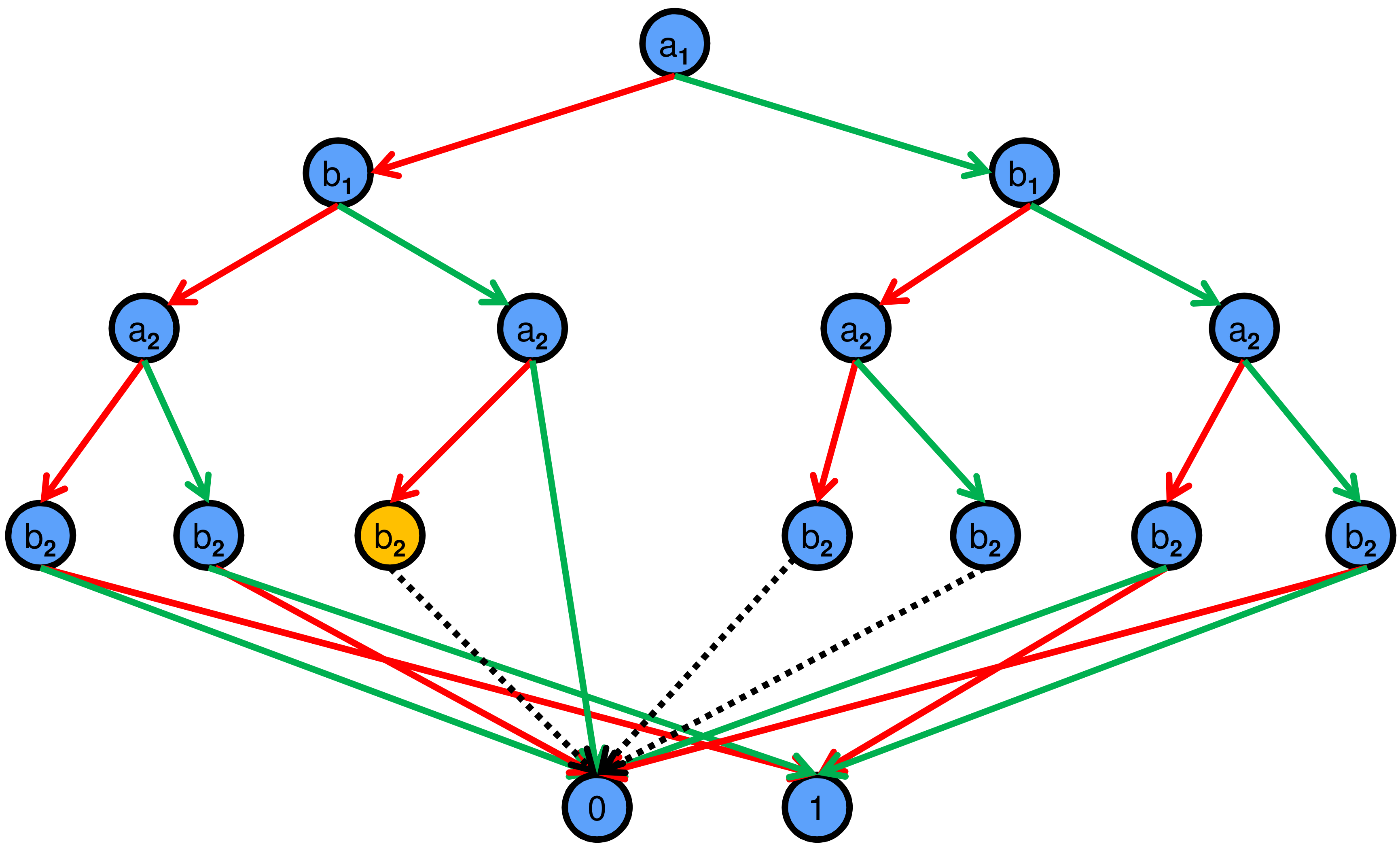

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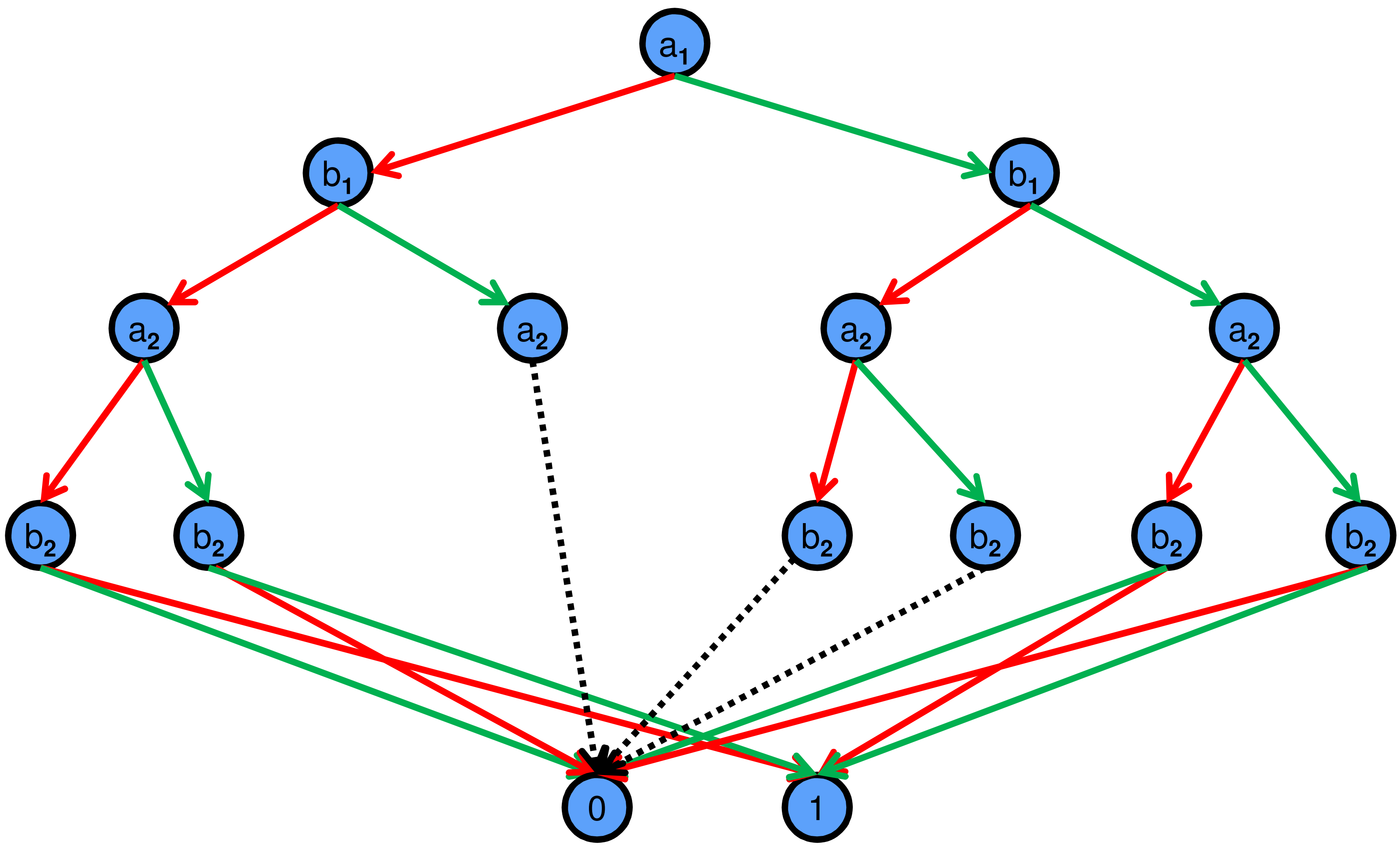
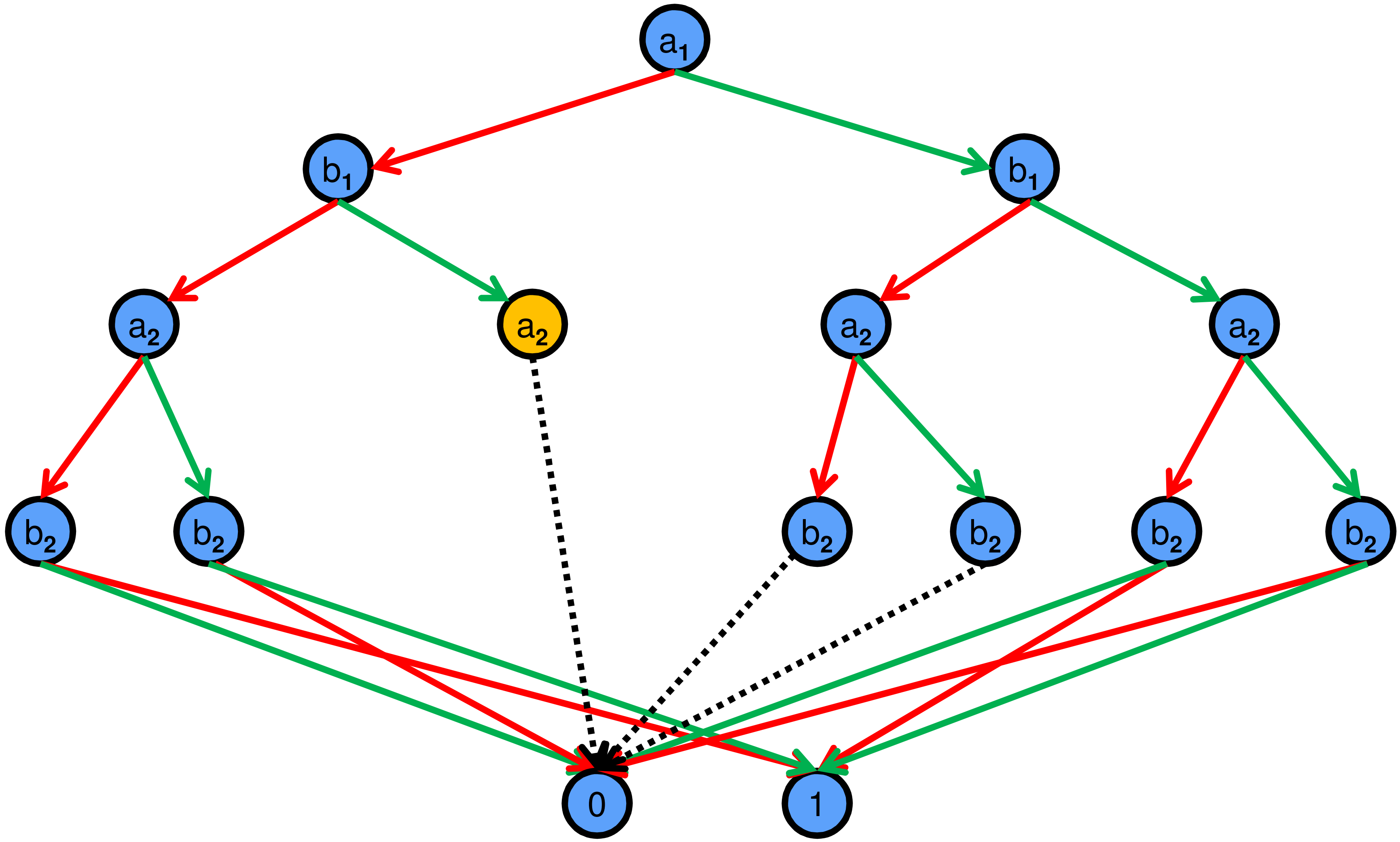
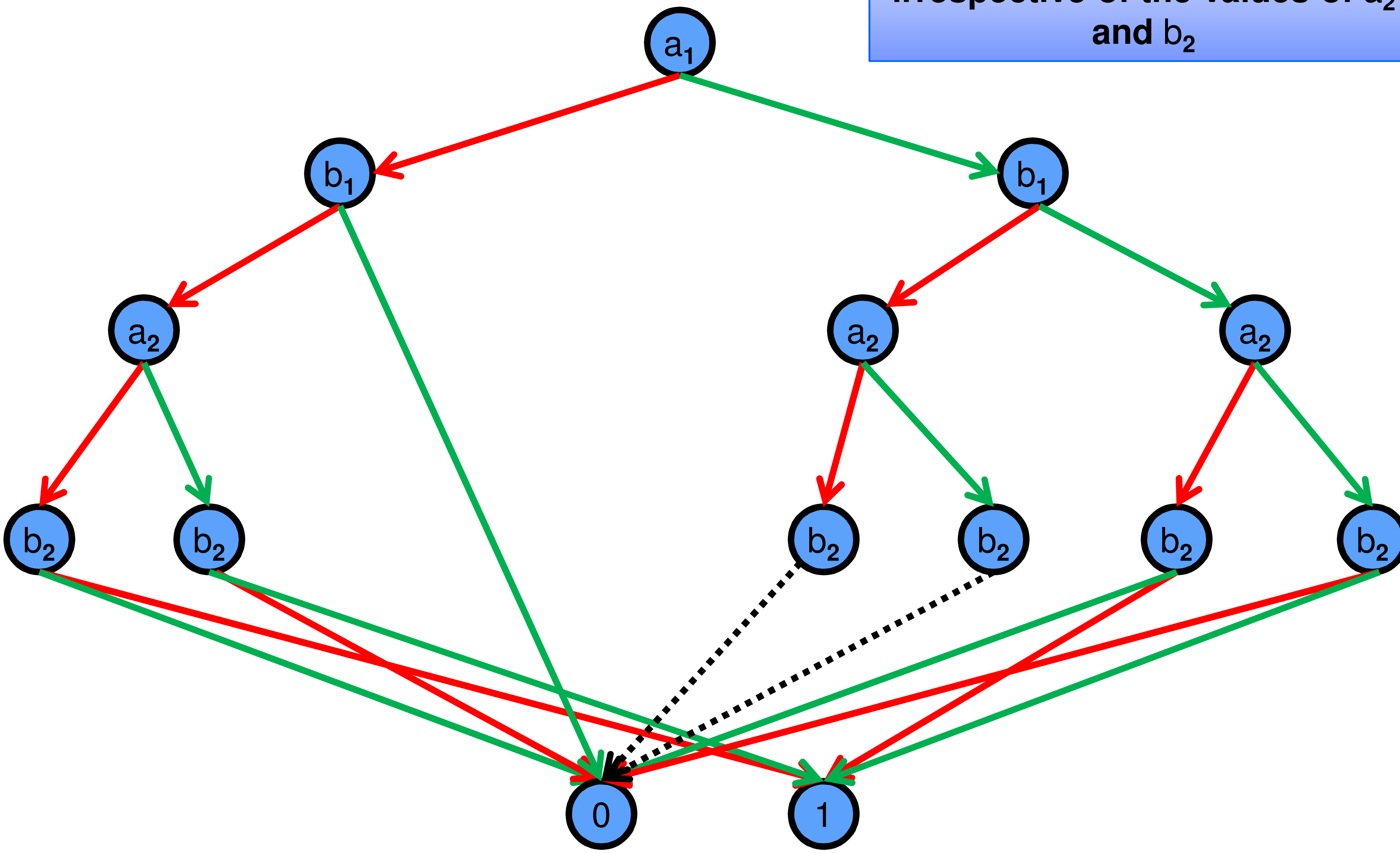


OBDT to ROBDD

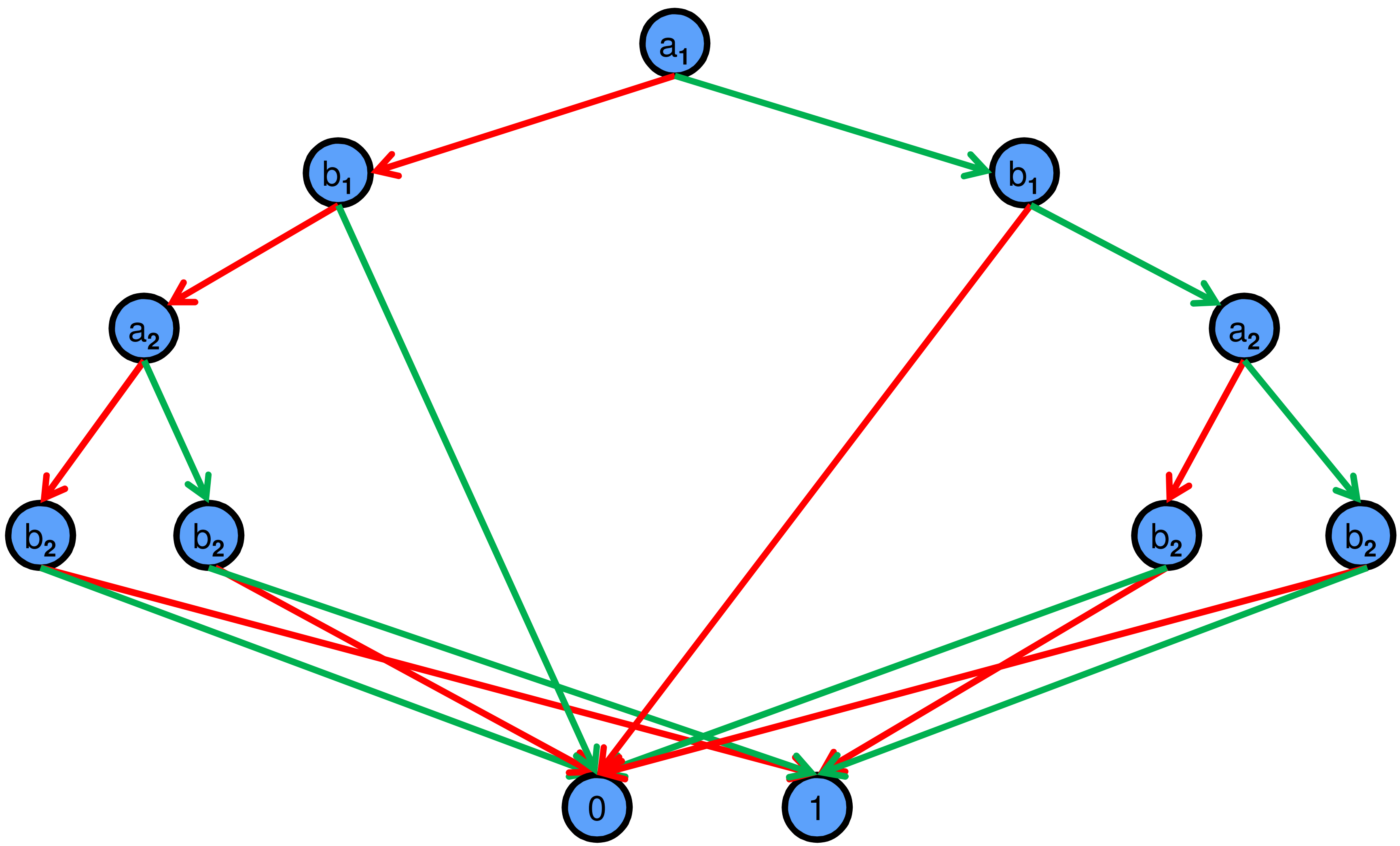


OBDT to ROBDD

If a_1 = 0 and b_1 = 1 then f = 0 irrespective of the values of a_2 and b_2



OBDT to ROBDD



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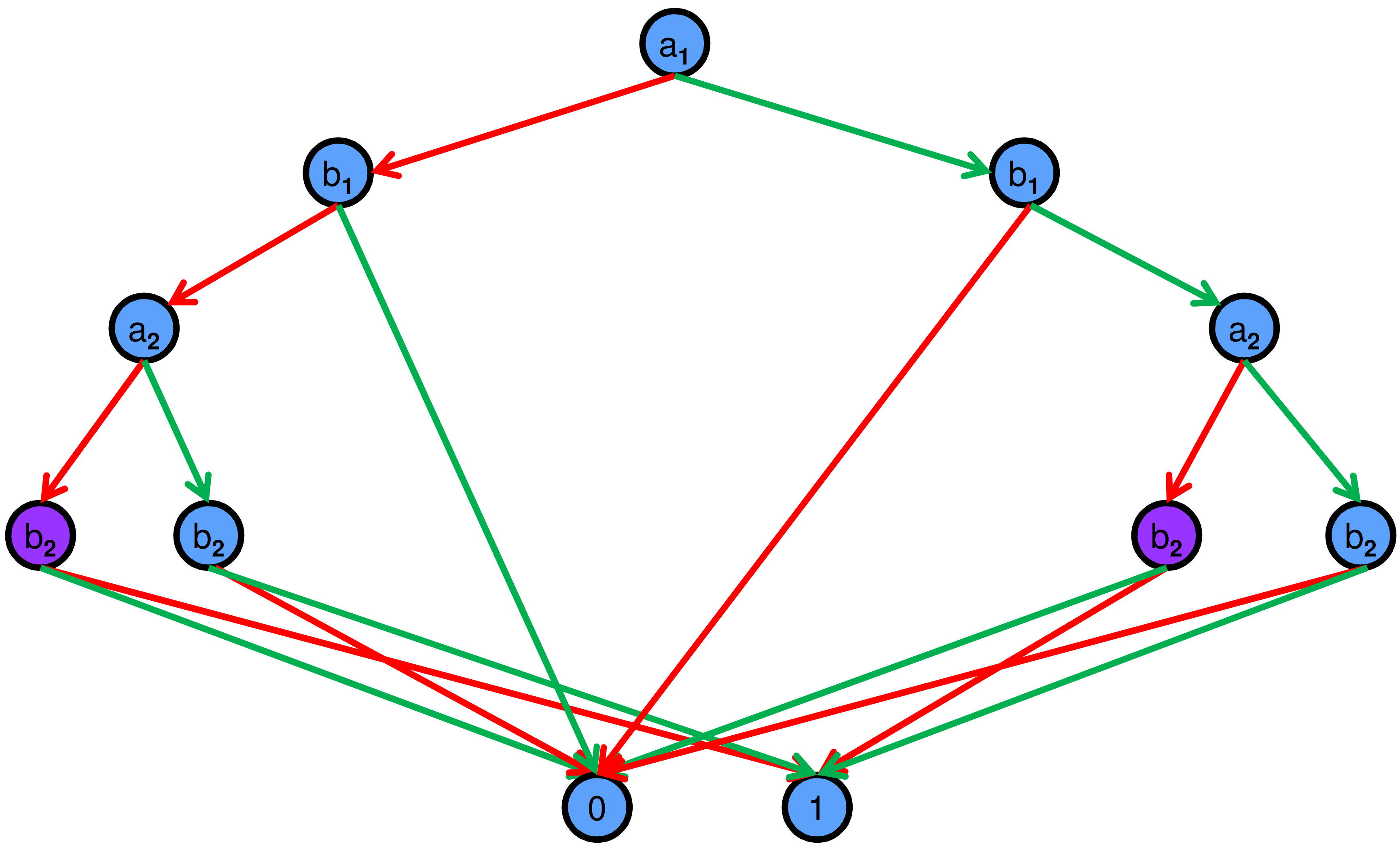

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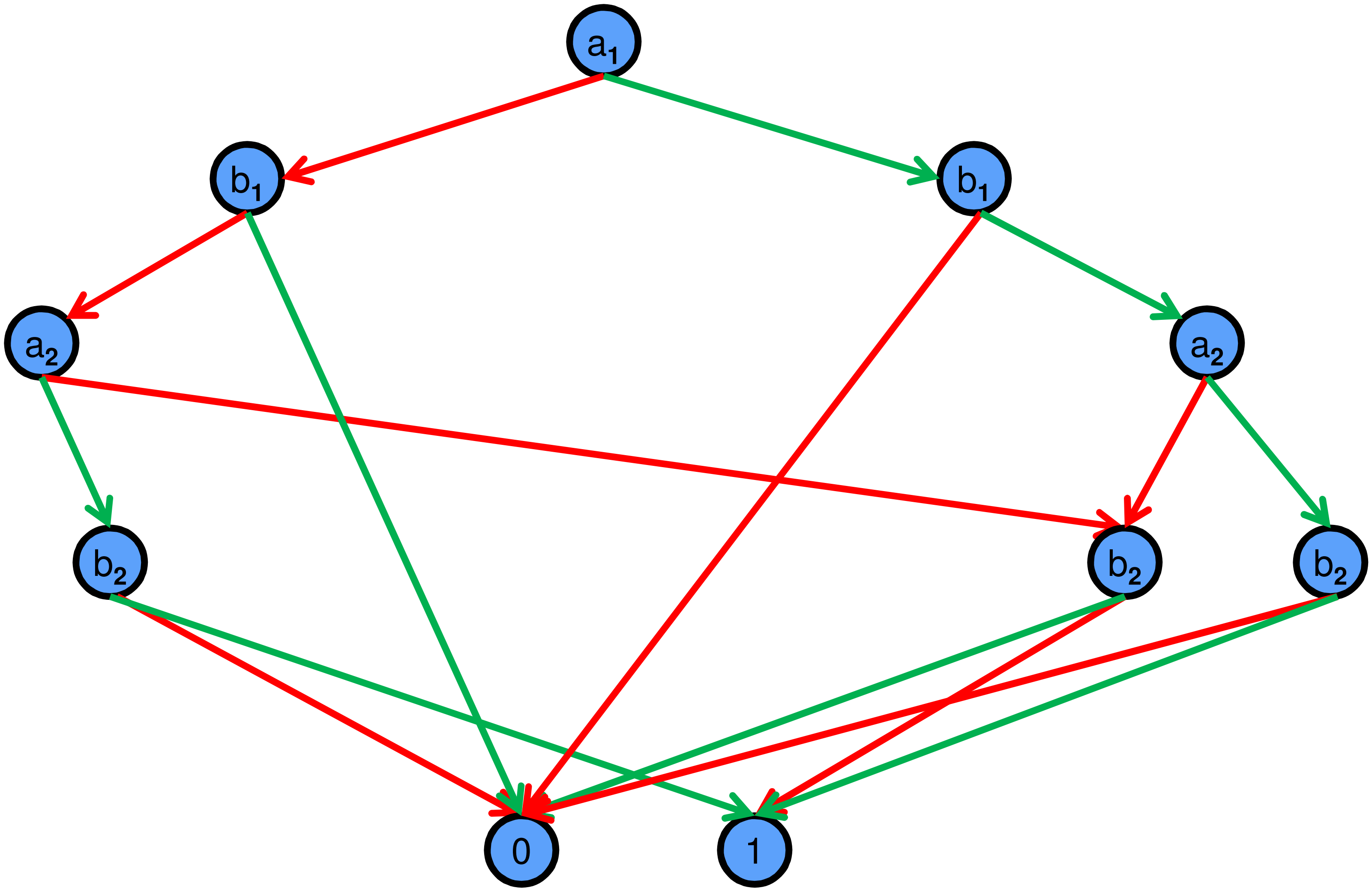

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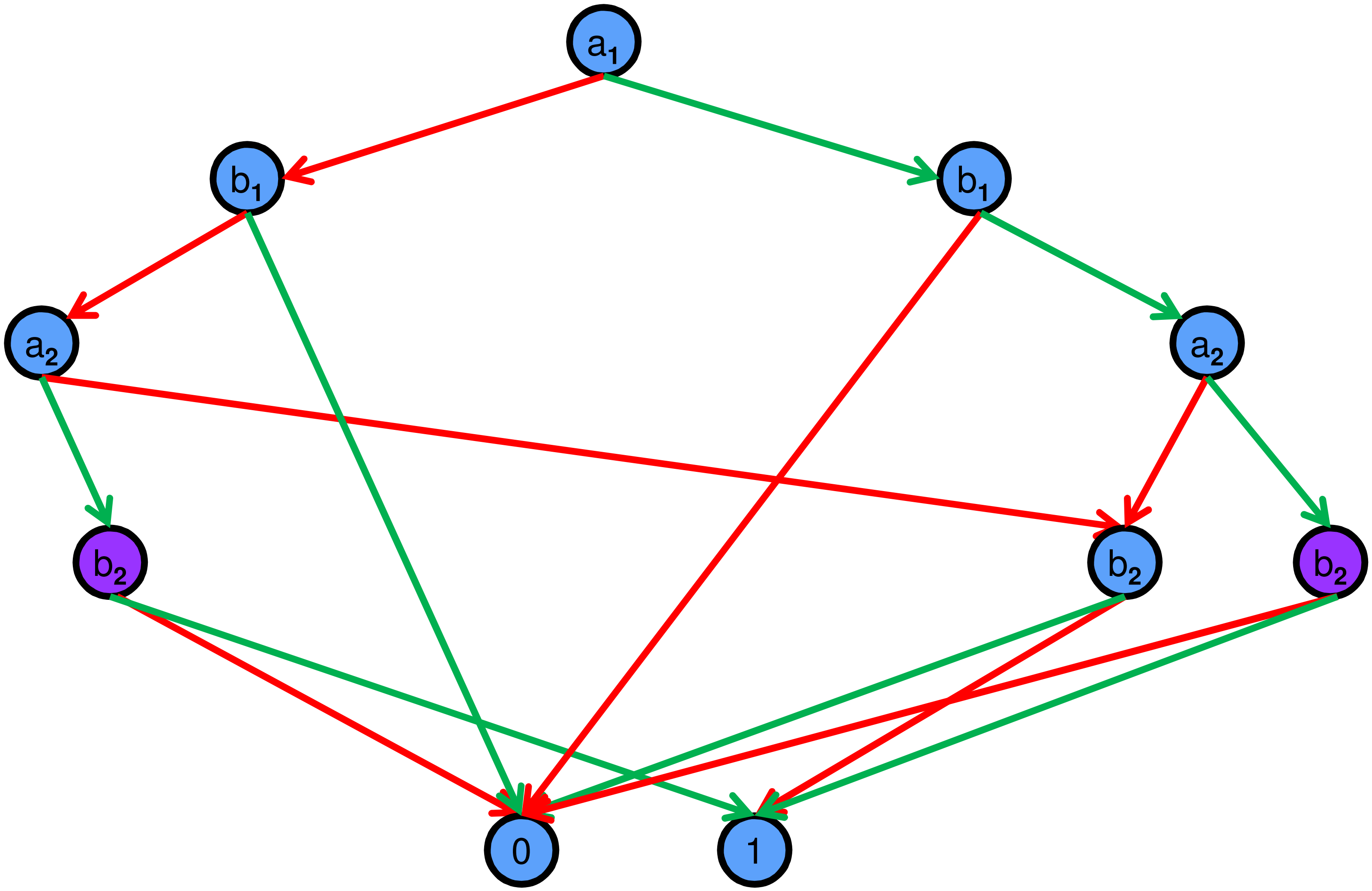

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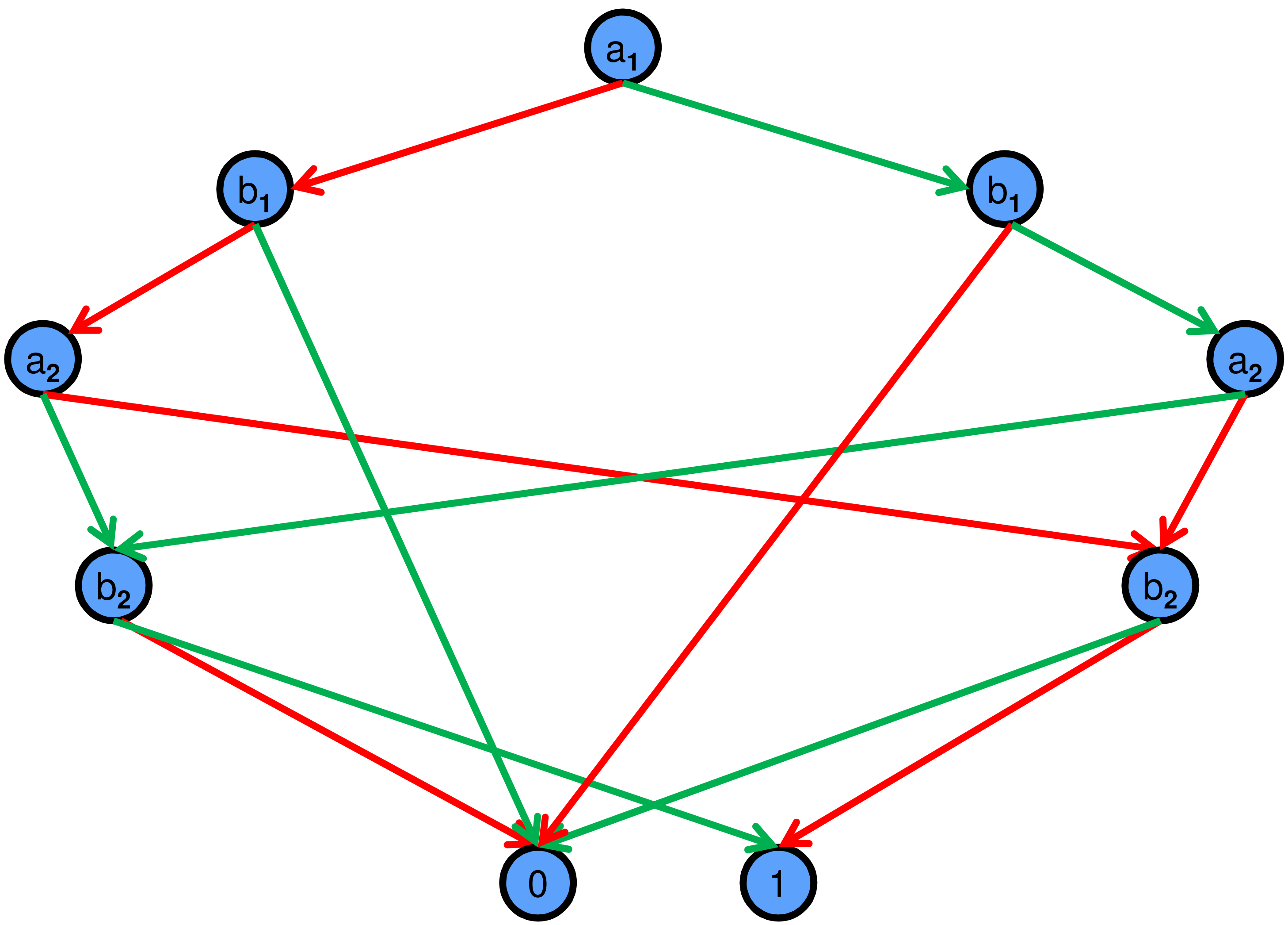

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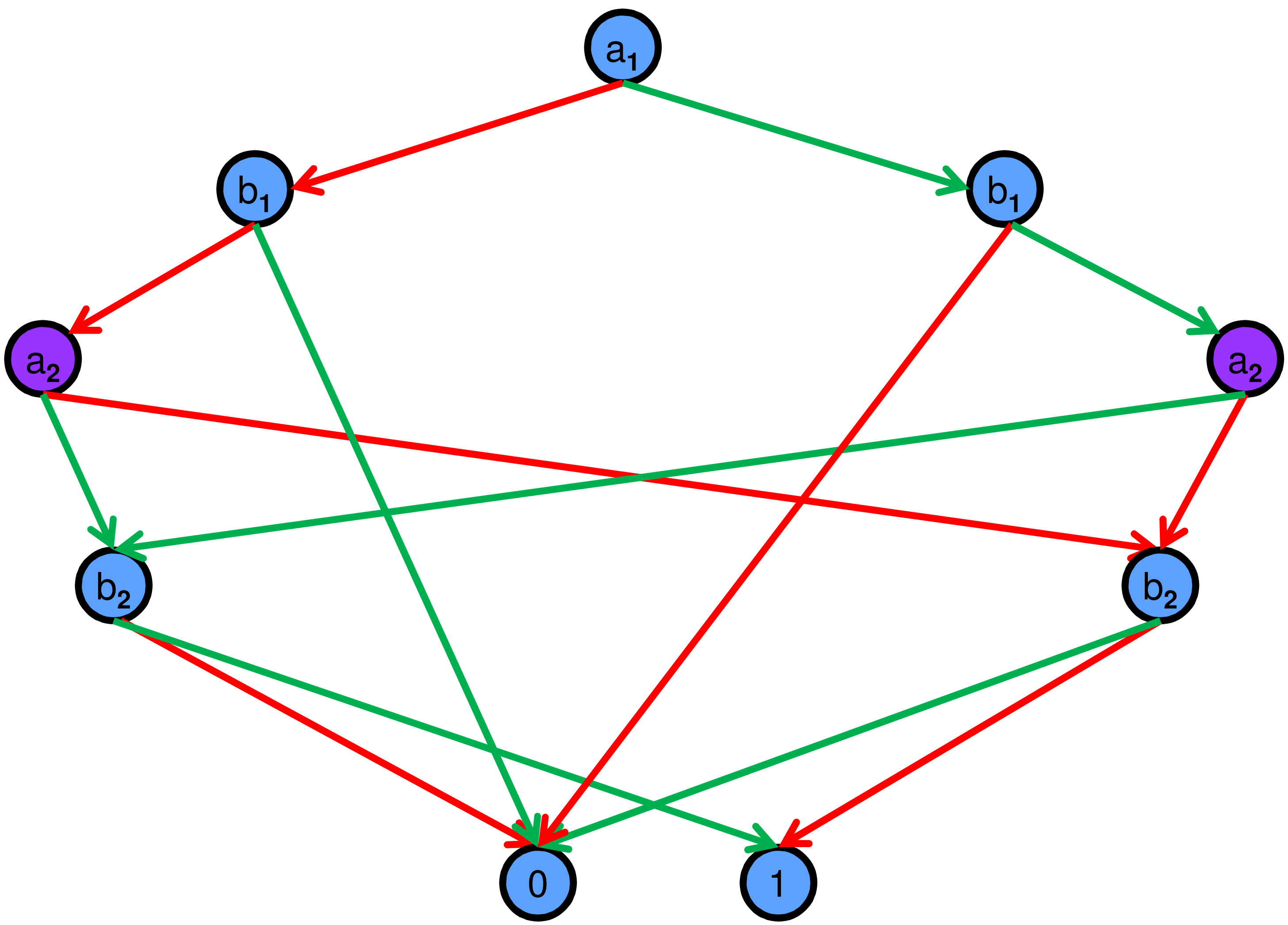
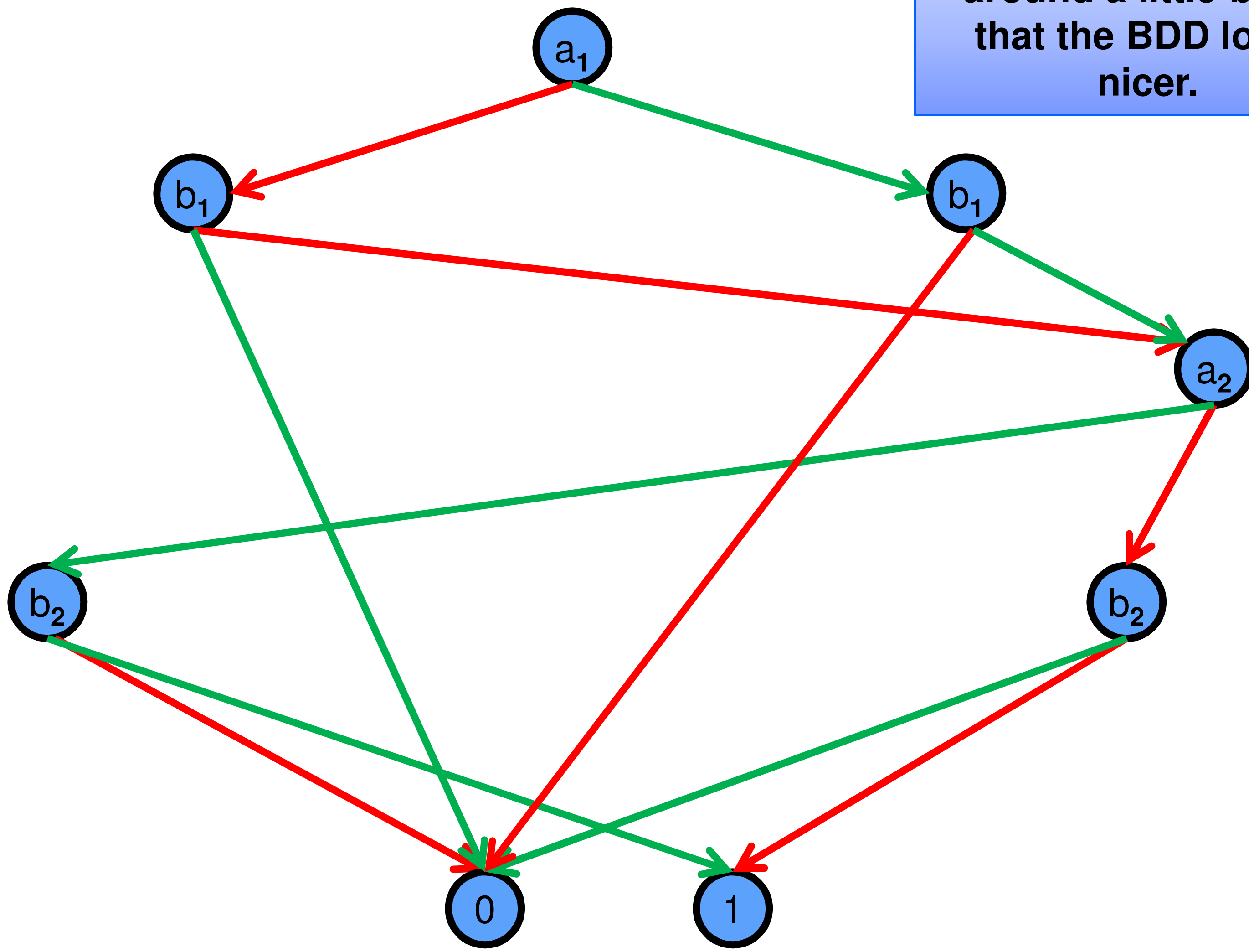
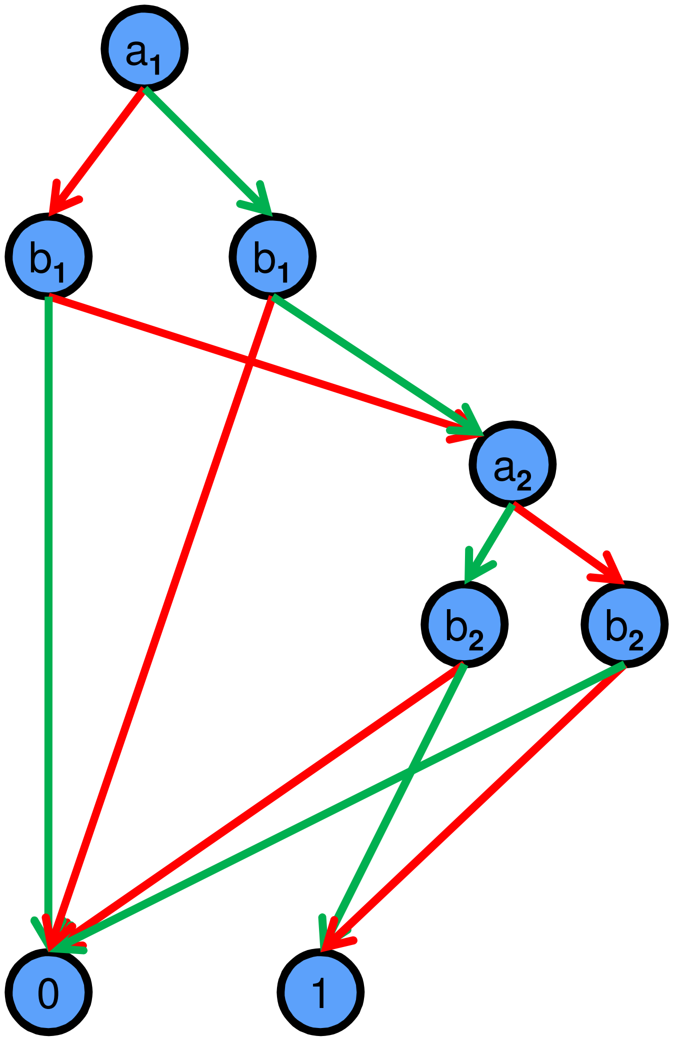


OBDT to ROBDD

Let's move things around a little bit so that the BDD looks nicer.



OBDT to ROBDD



Bryant gave a linear-time algorithm (called Reduce) to convert OBDT to ROBDD.

In practice, BDD packages don't use Reduce directly. They apply the two reductions on-the-fly as new BDDs are constructed from existing ones. Why?

ROBDD (a.k.a. BDD) Summary

BDDs are canonical representations of Boolean formulas

- $f_1 = f_2 \Leftrightarrow$?

ROBDD (a.k.a. BDD) Summary

BDDs are canonical representations of Boolean formulas

- $f_1 = f_2 \Leftrightarrow$ BDD(f_1) and BDD(f_2) are isomorphic
- f is unsatisfiable $\Leftrightarrow$?

ROBDD (a.k.a. BDD) Summary

BDDs are canonical representations of Boolean formulas

- $f_1 = f_2 \Leftrightarrow$ BDD(f_1) and BDD(f_2) are isomorphic
- f is unsatisfiable $\Leftrightarrow$ BDD(f) is the leaf node "0"
- f is valid $\Leftrightarrow$?

ROBDD (a.k.a. BDD) Summary

BDDs are canonical representations of Boolean formulas

- $f_1 = f_2 \Leftrightarrow$ BDD(f_1) and BDD(f_2) are isomorphic
- f is unsatisfiable $\Leftrightarrow$ BDD(f) is the leaf node “0”
- f is valid $\Leftrightarrow$ BDD(f) is the leaf node “1”
- BDD packages do these operations in constant time

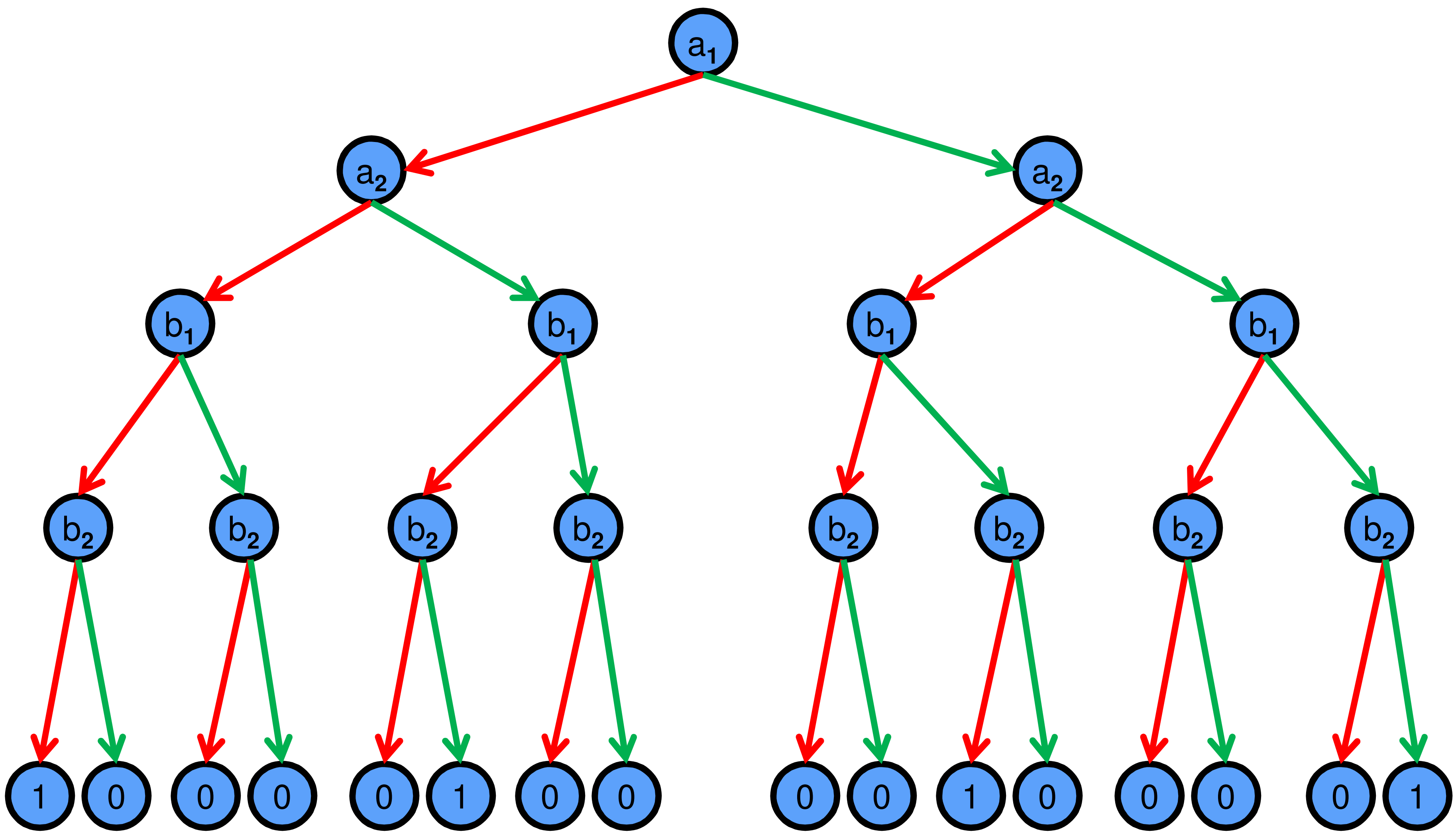
Logical operations can be performed efficiently on BDDs

- Polynomial in argument size
- More details in next lecture

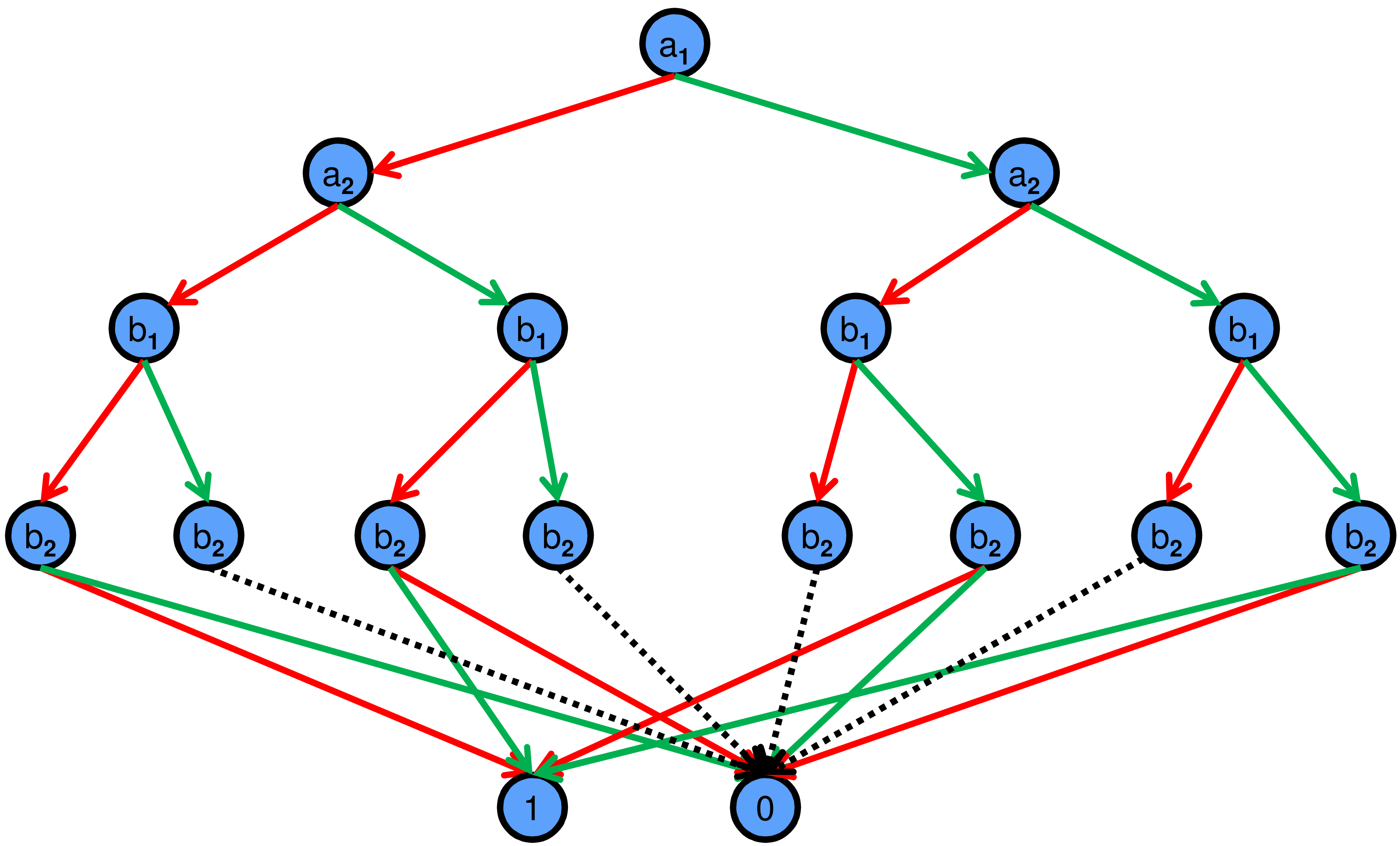
BDD size depends critically on the variable ordering

- Some formulas have exponentially large sizes for all ordering
- Others are polynomial for some ordering and exponential for others

ROBDD and variable ordering



ROBDD and variable ordering



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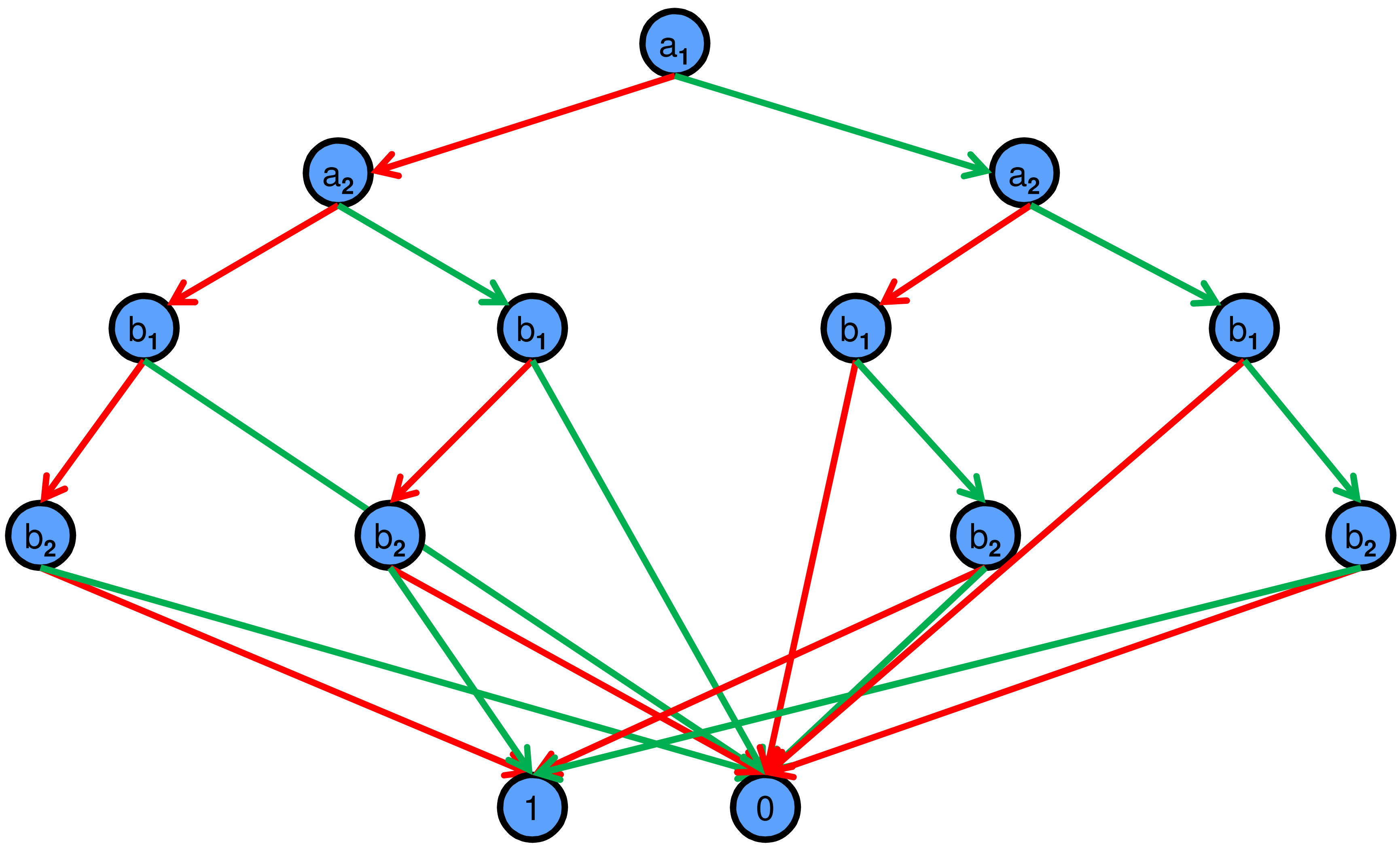

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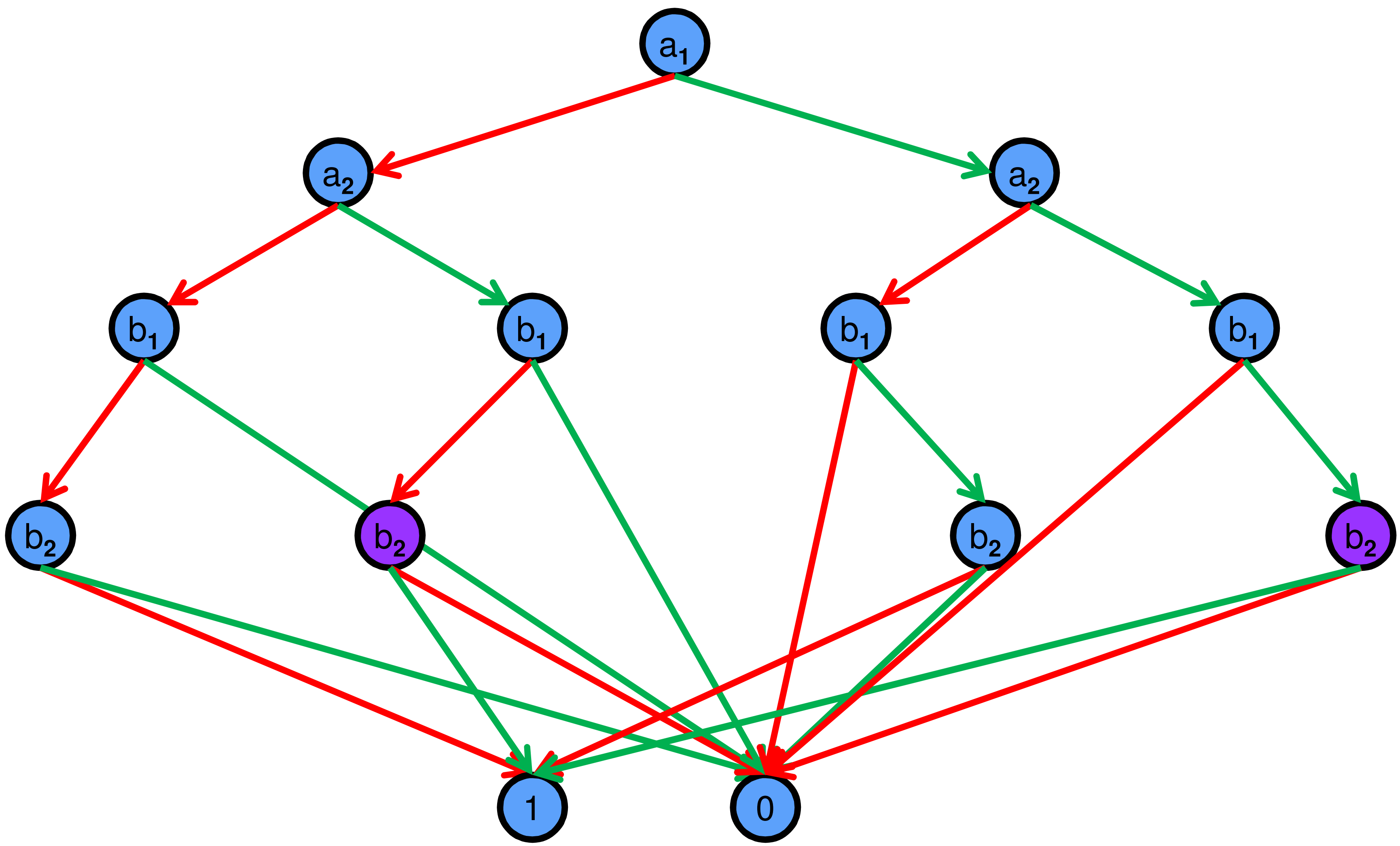

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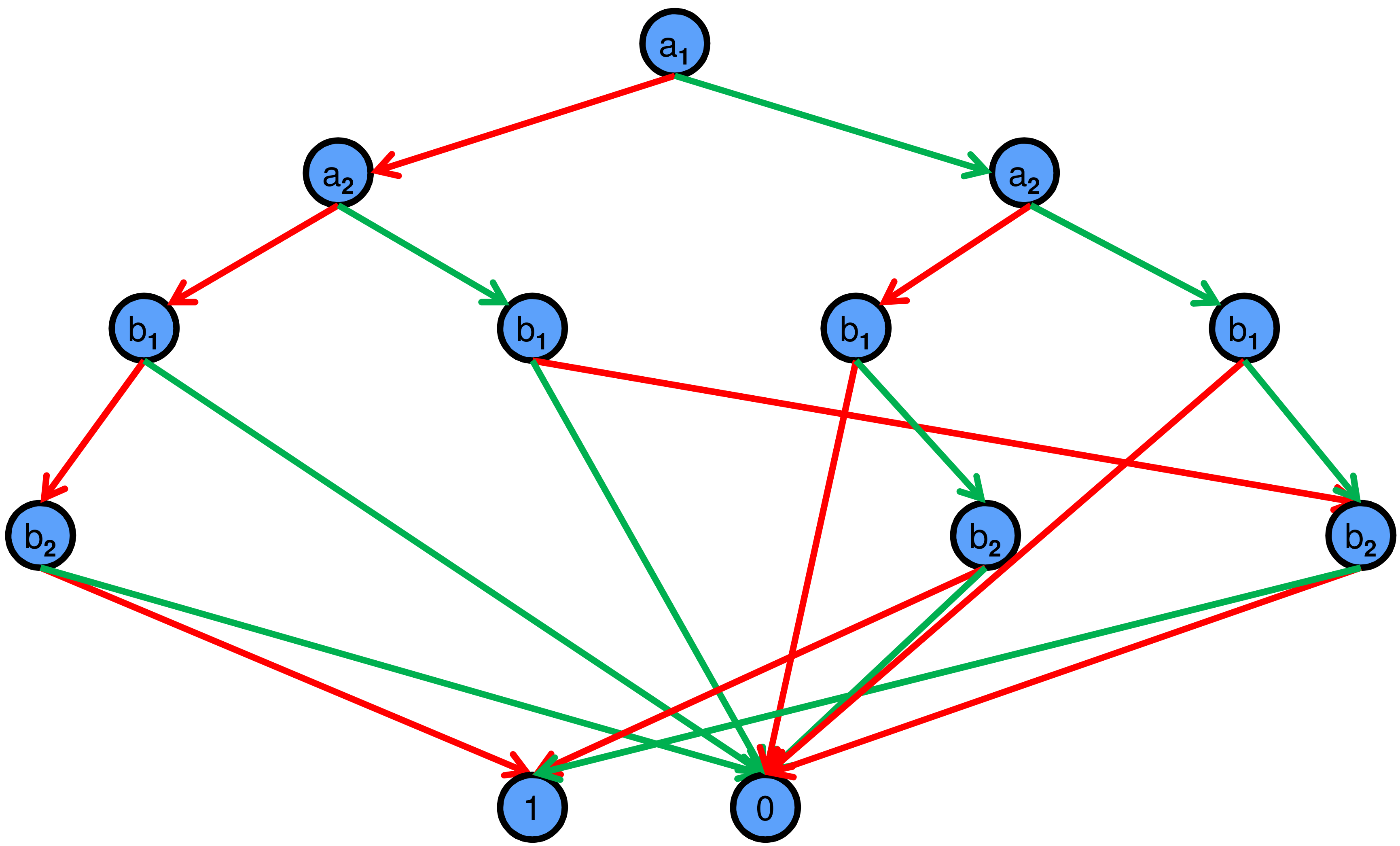

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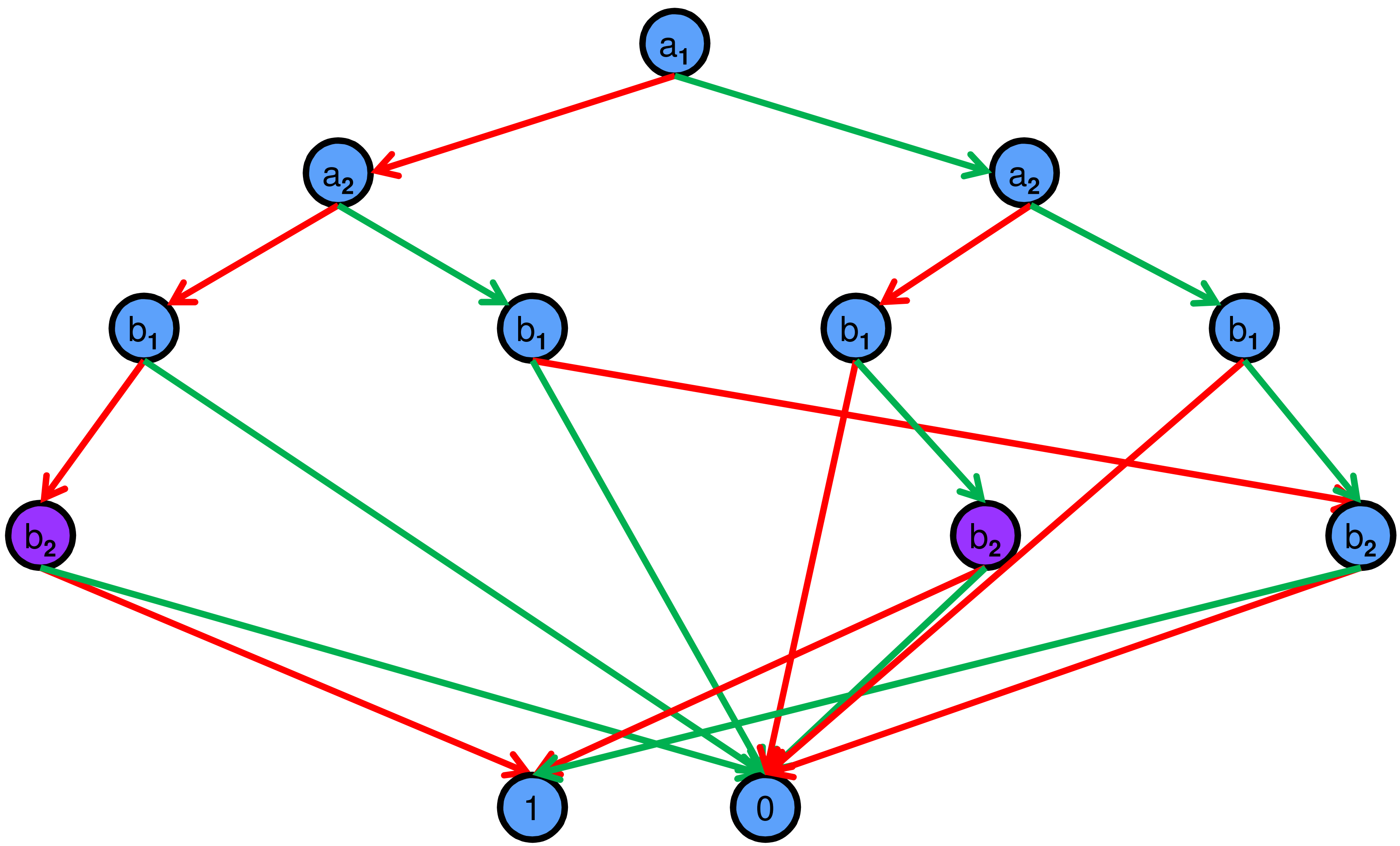
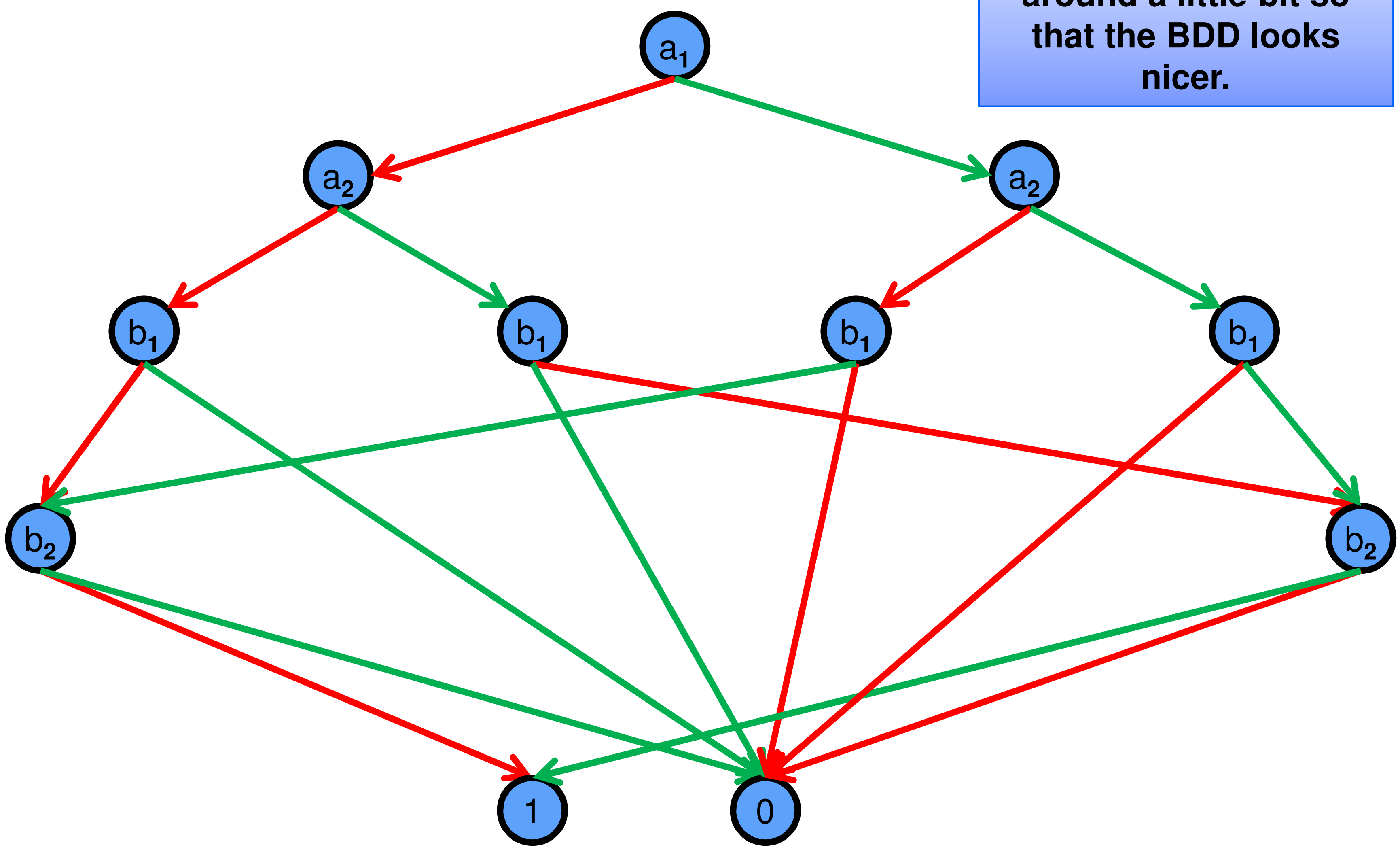
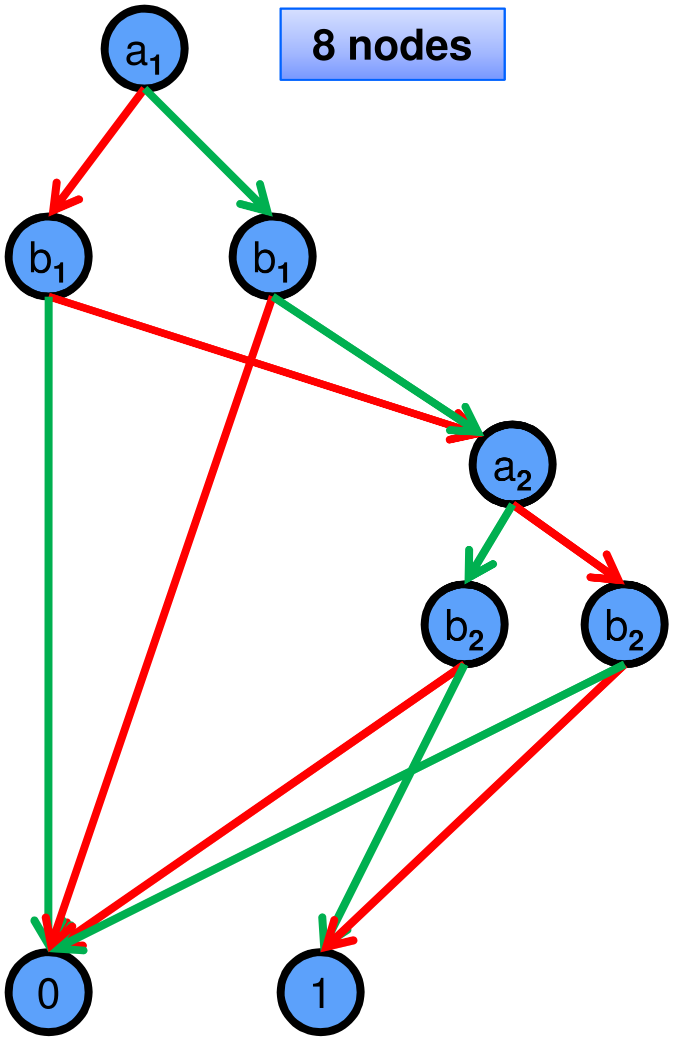


ROBDD and variable ordering

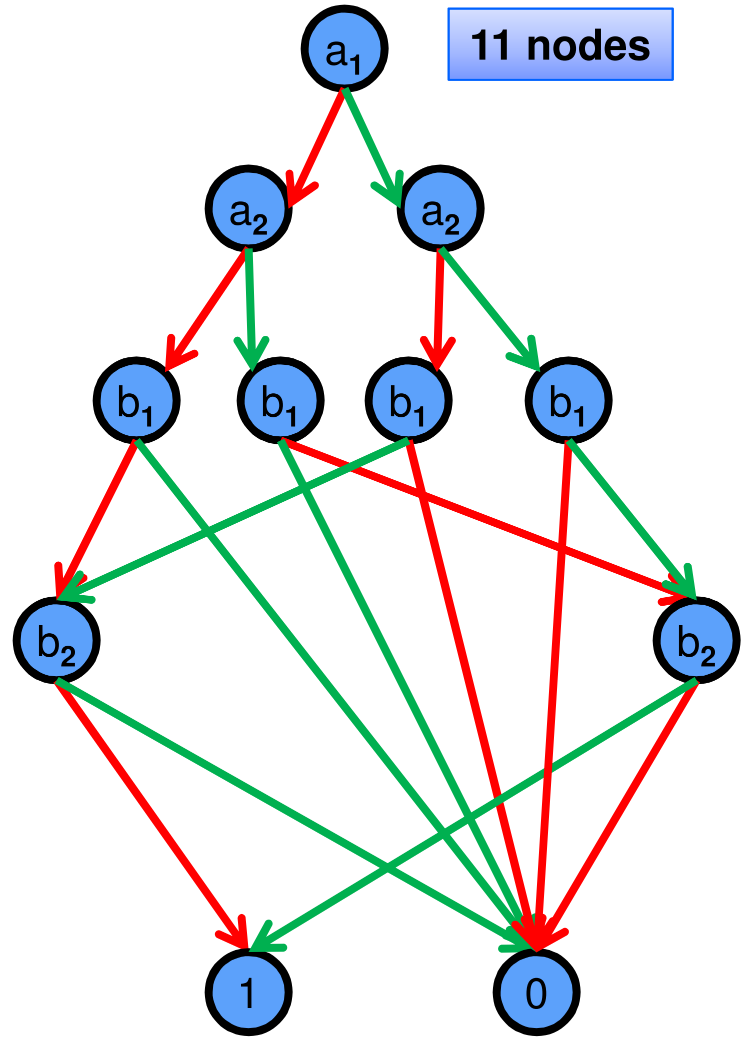
Let's move things around a little bit so that the BDD looks nicer.



ROBDD and variable ordering



$a_1 < b_1 < a_2 < b_2$



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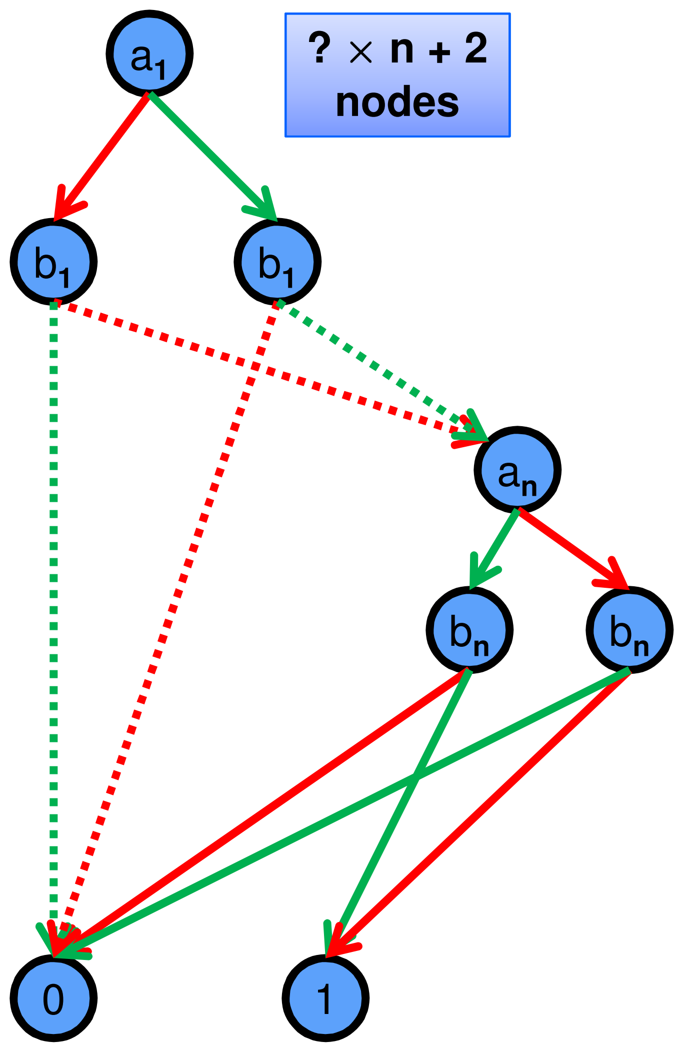
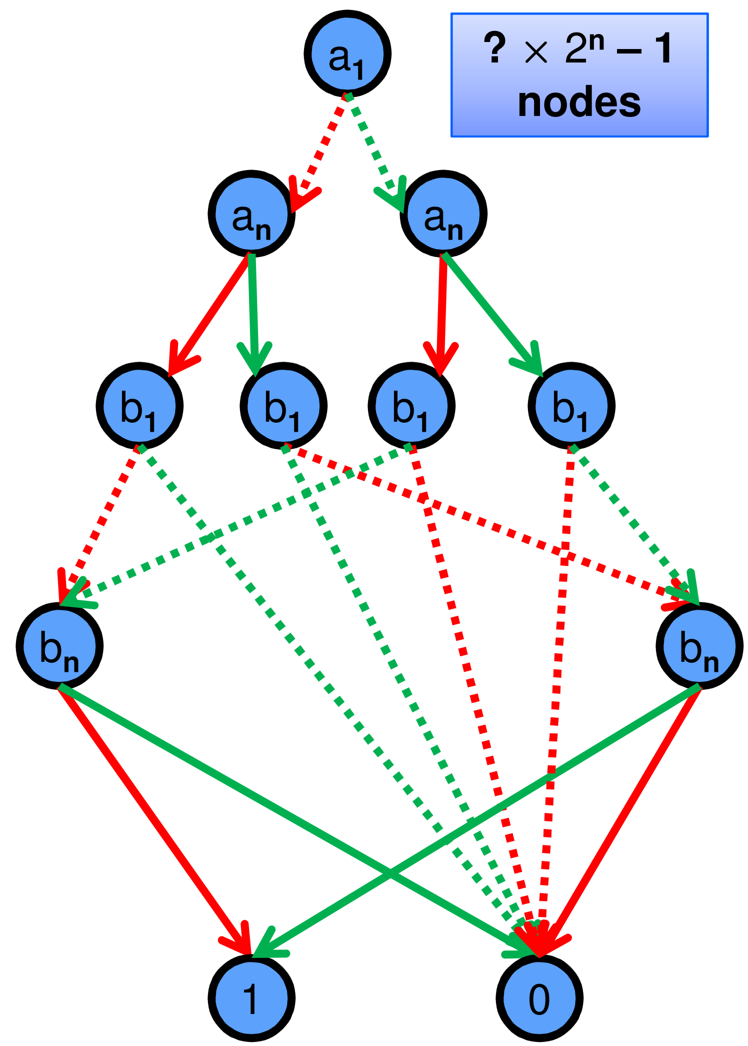
ROBDD and variable ordering



$a_1 < b_1 < \ldots < a_n < b_n$



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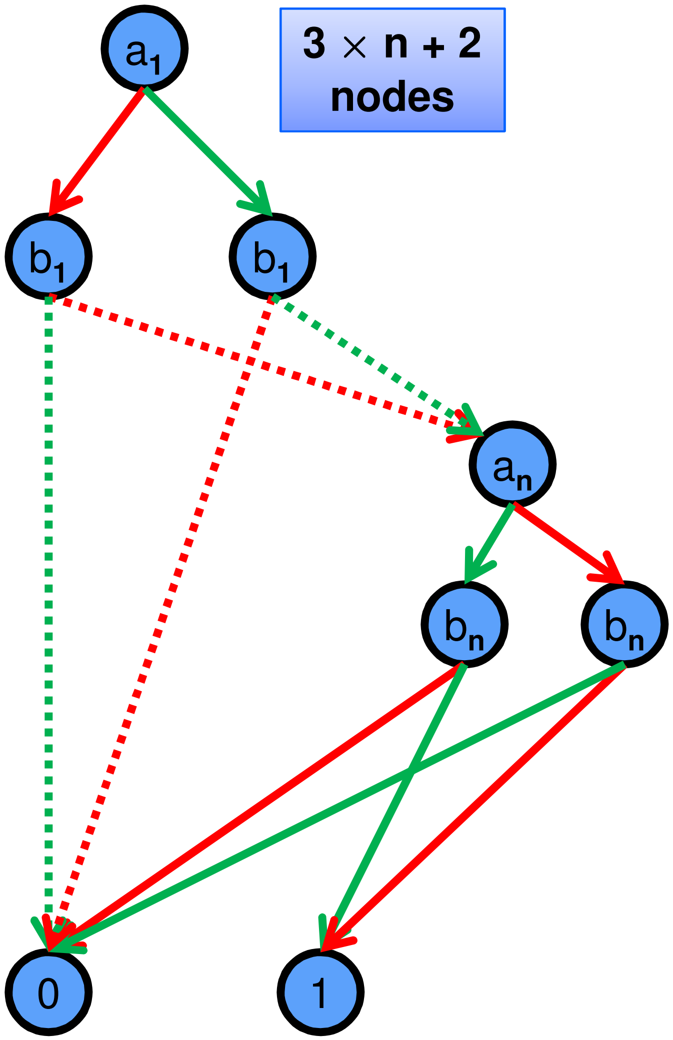
ROBDD and variable ordering



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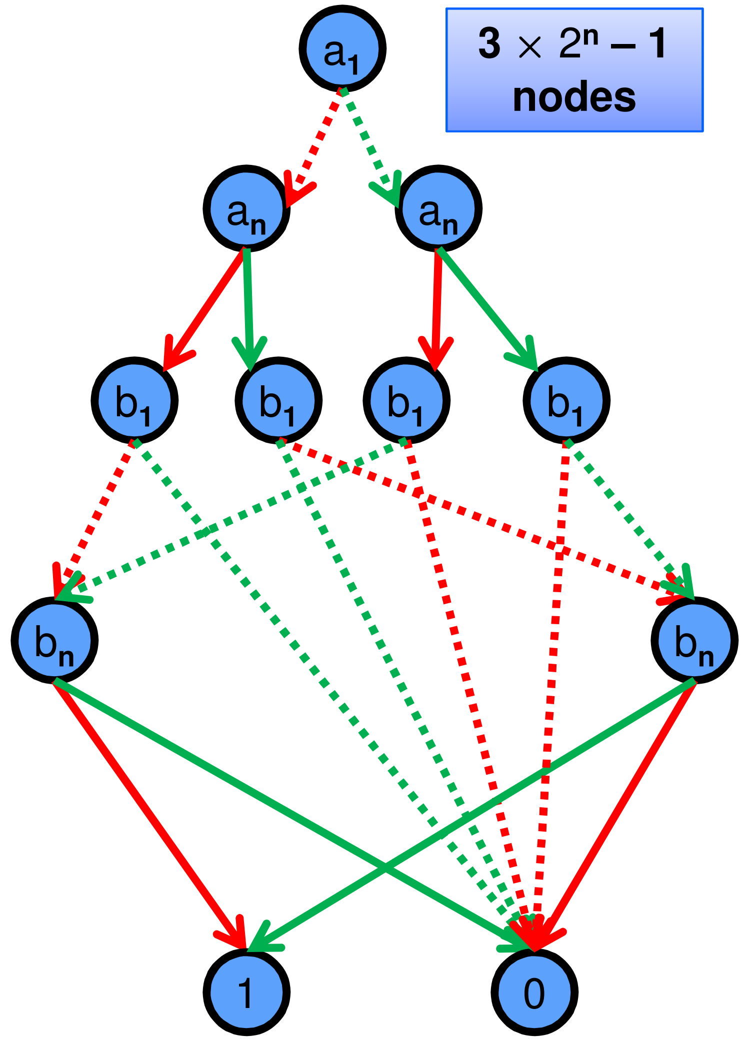

$a_1 < \ldots < a_n < b_1 < \ldots < b_n$

Next Class

BDD recap

BDD operations

BDD applications

Next homework

See you then …

Questions?

Sagar Chaki

Senior Member of Technical Staff

RTSS Program

Telephone: +1 412-268-1436

Email: chaki@sei.cmu.edu

U.S. Mail

Software Engineering Institute

Customer Relations

4500 Fifth Avenue

Pittsburgh, PA 15213-2612

USA

Web

www.sei.cmu.edu/staff/chaki

Customer Relations

Email: info@sei.cmu.edu

Telephone: +1 412-268-5800

SEI Phone: +1 412-268-5800

SEI Fax: +1 412-268-6257