Computational semantics of Cubical Inductive Types

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with Bob Harper

Carnegie Mellon University

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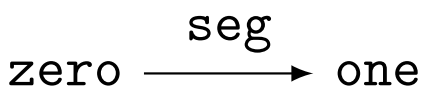
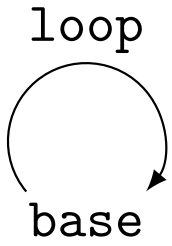
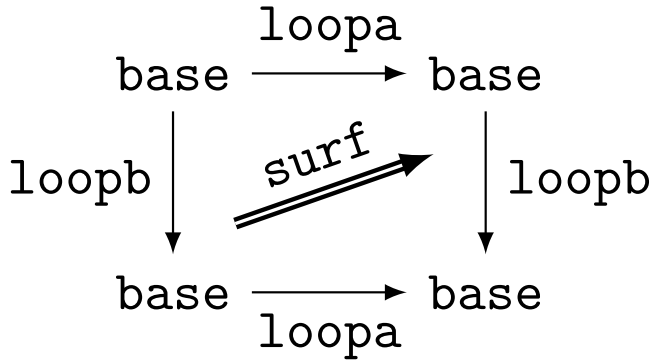
Higher Inductive Types



(a) Interval



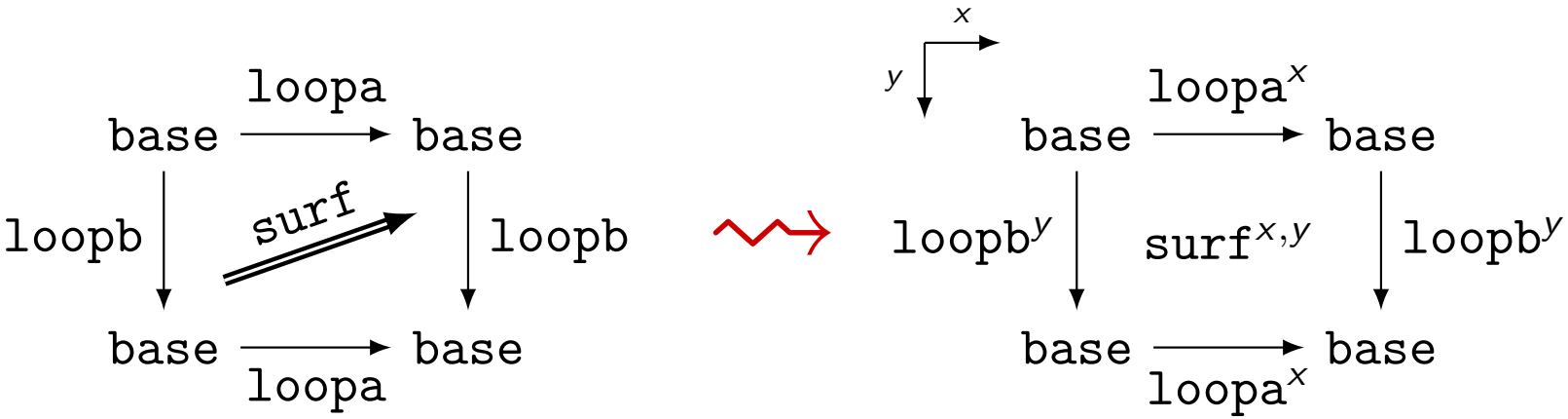
(b) Circle



(c) Torus

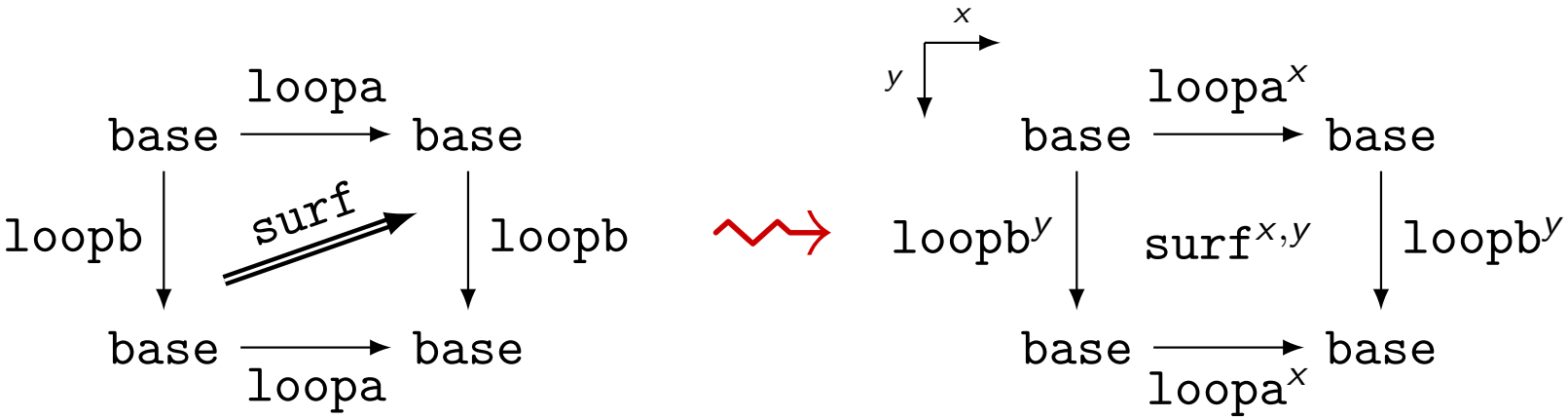
- Types generated by specified points and paths
  - Homotopy-initial algebras [AGS17]
  - Weakly stable typal initial algebras [LS17]
- Can we regard these as computational objects?

Cubical Inductive Types



1. Choose a format for organizing higher structure

Cubical Inductive Types



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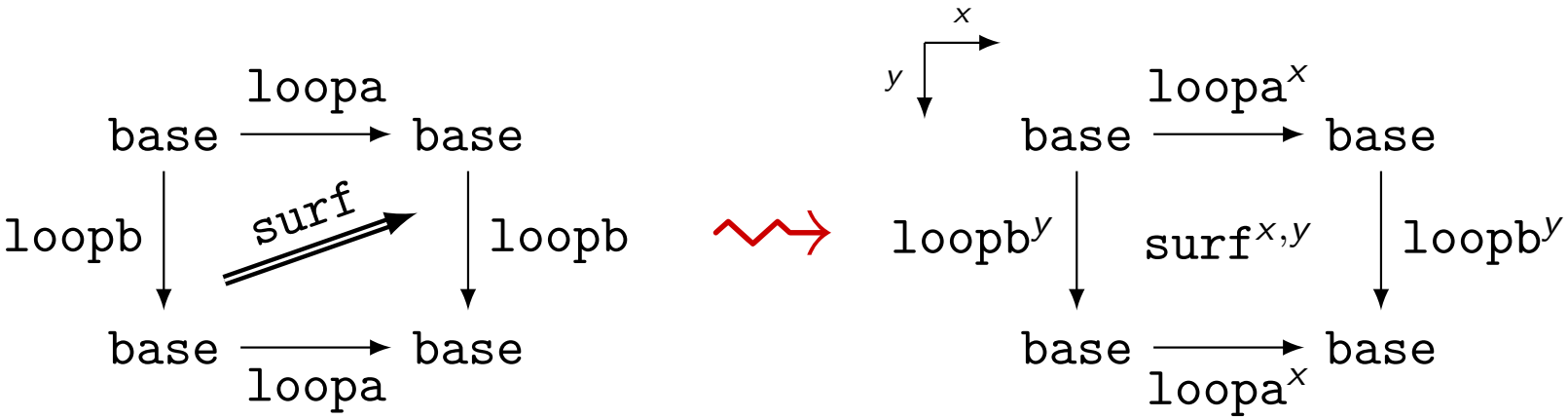

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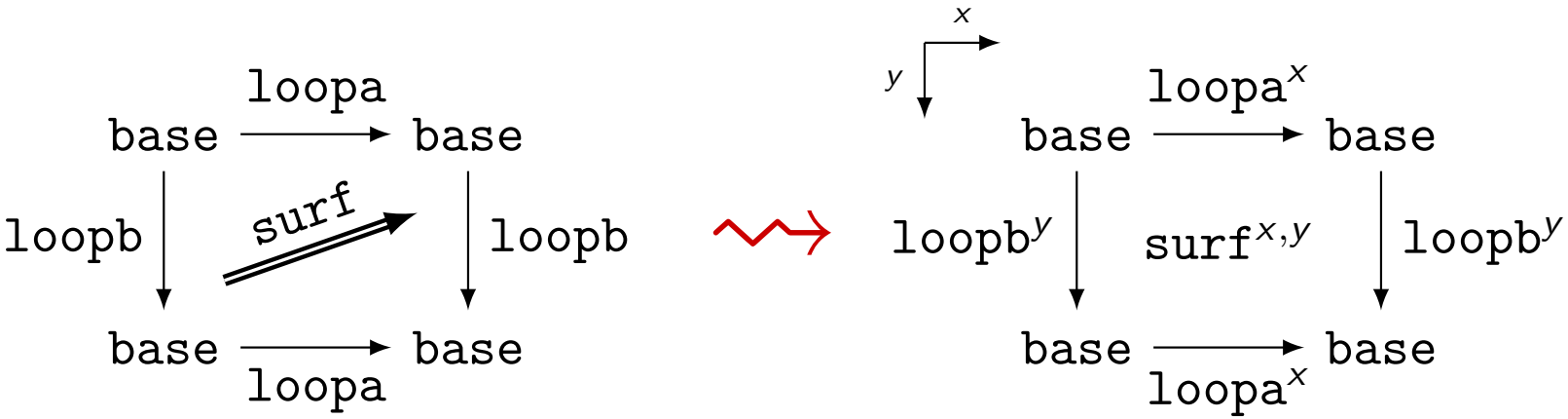

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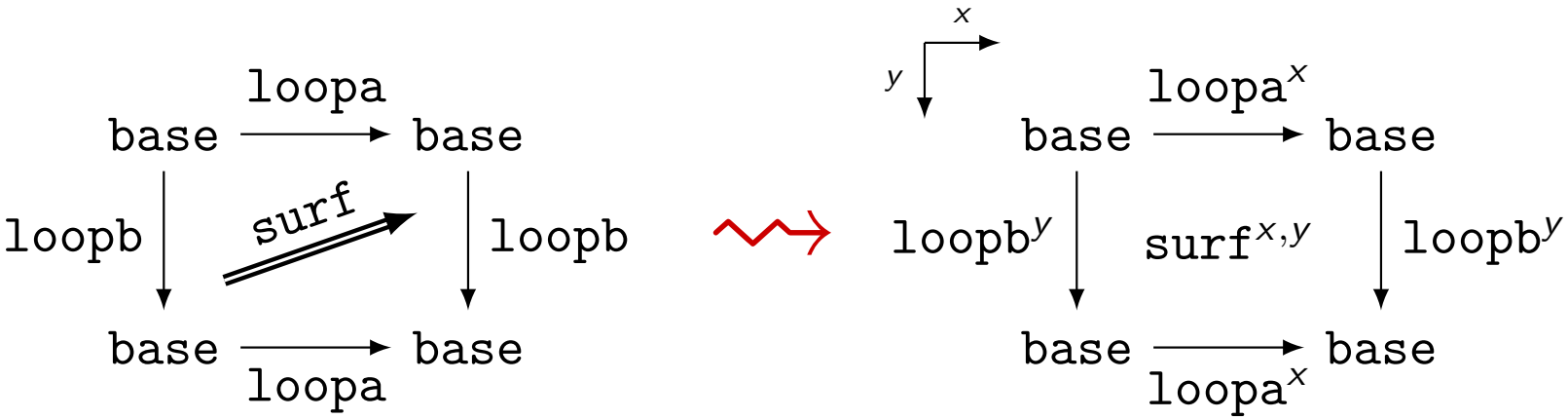

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Feature Requests for Cubical Inductive Types

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- ...or recursive arguments.

$$\frac{M \in \|A\| \qquad N \in \|A\|}{M \xrightarrow{\texttt{path}^x(M;N)} N \;\; \in \|A\|}$$

Schemata for Inductive Types

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- W-types [ML82]:

$$\frac{A\ \mathtt{type} \qquad a : A \gg B\ \mathtt{type}}{\mathtt{W}(A; a.B)\ \mathtt{type}}$$

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$$\begin{array}{lcl} \mathtt{nat} & := & \mathtt{W}(\mathtt{bool}; b.\mathtt{if}(b; \mathtt{void}, \mathtt{unit})) \\ \mathtt{zero} & := & \mathtt{sup}(\mathtt{true}; \lambda e.\mathtt{void\text{-}elim}(e)) \\ \mathtt{s}(M) & := & \mathtt{sup}(\mathtt{false}; \lambda\_.M) \end{array}$$

Schemata for Inductive Types

- More explicit schemata [CP88; Dyb94]:

$$\begin{array}{l}\texttt{data X where}\\ \quad \texttt{intro}_1 \in (a_1 : A_1) \cdots (a_n : A_n) \to (B_1 \to \texttt{X}) \to \cdots \to (B_k \to \texttt{X}) \to \texttt{X}\\ \quad \vdots\\ \quad \texttt{intro}_m \in \cdots\end{array}$$

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Adding Dimensions and Boundaries

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\xi & ::= & r = r' \\
\mathrm{M} & ::= & \texttt{intro}_i(\cdots) \mid \texttt{fcom}(\cdots) \mid \lambda a.\mathrm{M} \mid \texttt{app}(\mathrm{M}, M) \\
 & & (+ \text{ typing rules for } \mathrm{M} : \mathrm{C})
\end{array}
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Formulating elimination

$$\texttt{data}\ A \sqcup_C^{F,G} B\ \texttt{where}$$

$$\texttt{left} \in (a : A) \to A \sqcup_C^{F,G} B$$

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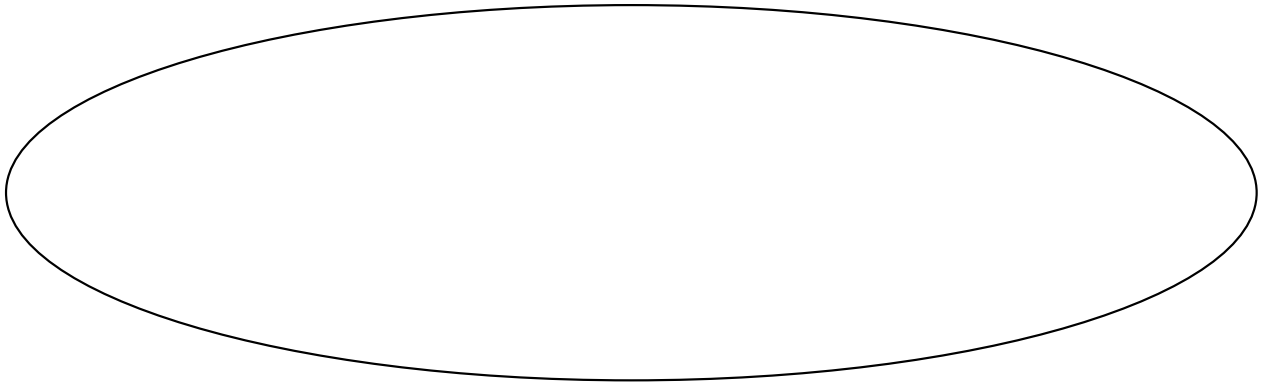

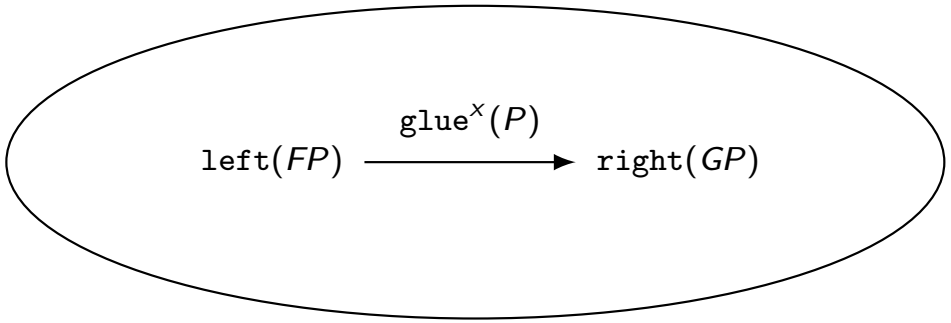

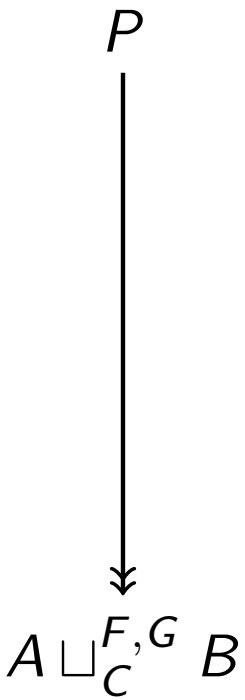

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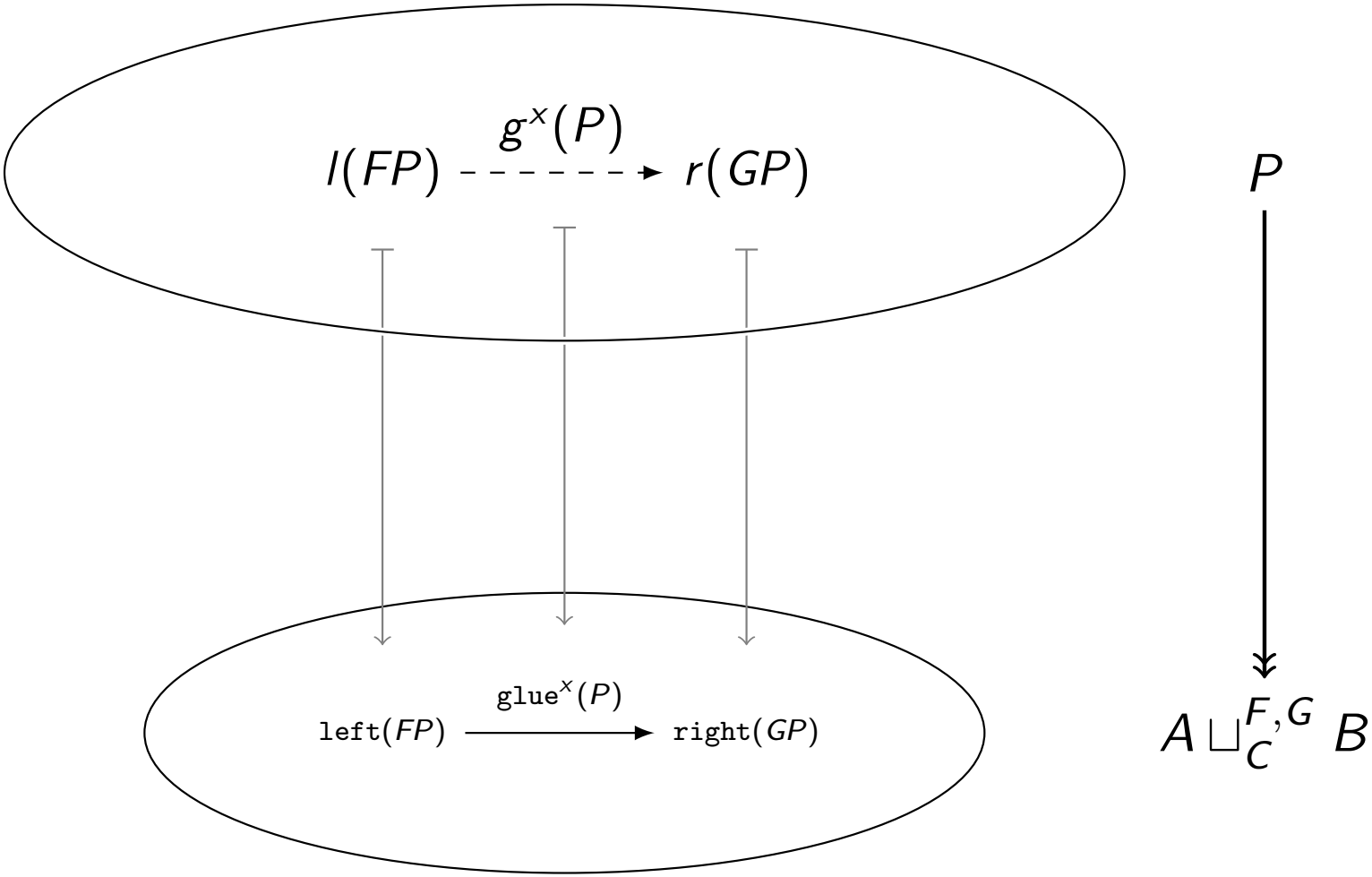

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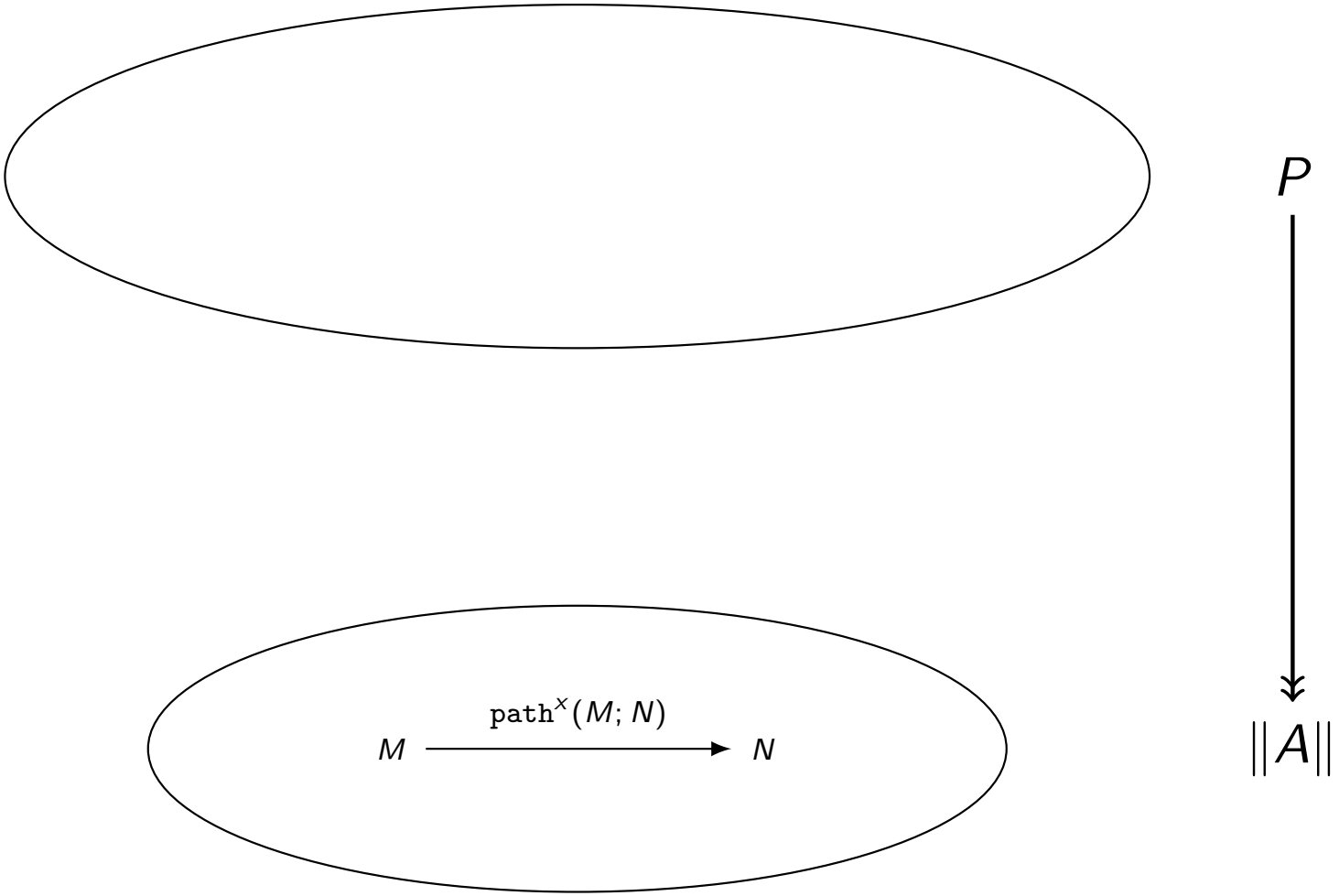

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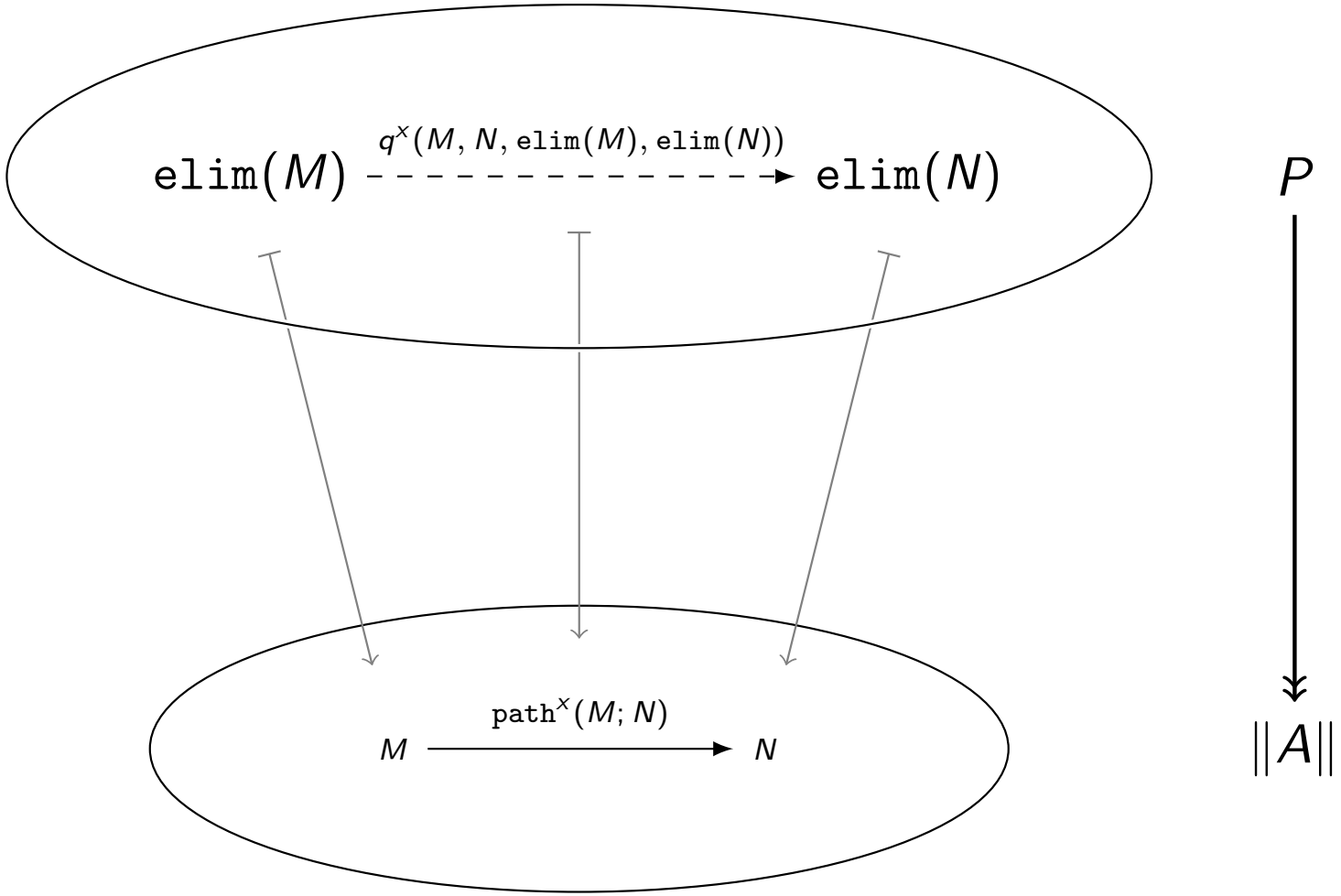

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\quad \texttt{intro}_{\texttt{m}}{}^{\cdots} \in \cdots \\
\qquad [\cdots]
\end{array}
$$

To define a function $(u : \texttt{X}) \to P(u)$, provide:

- $r_1^{x_1,\ldots,x_l} : (a_1, \ldots, a_n)(f_1, \ldots, f_k)(f_1^*, \ldots, f_k^*) \to P(\texttt{intro}_1(\ldots))$, such that
  - $r_1^{\cdots}(\cdots) \doteq \mathrm{M}_i{}^*(\cdots) \in P(\texttt{intro}_1(\ldots))$ within ξ_i, for each i

$\vdots$

- $r_m^{\cdots} \cdots$

Examples

- Circle
- Pushouts
- (-1)-Truncations (and n-truncations, via hub-and-spokes)
- Localizations
- ~~Lumsdaine-Shulman §9~~

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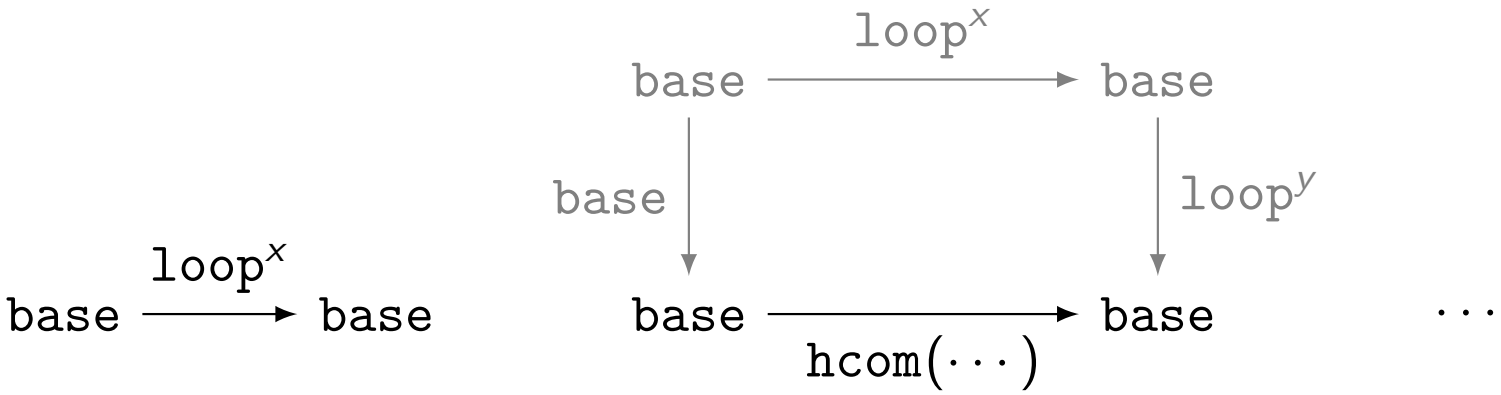
Elements of the circle $\mathbb{S}^1$:



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Define the meaning of $\mathbb{S}^1$ as the least family $(\sigma_\Psi)_\Psi$ of value PERs such that

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Kan conditions are achieved by a fiberwise fibrant replacement

- The construction of $\mathtt{ind}(\mathcal{K})$ from $\mathcal{K}$ is parametric in the ambient dimension context Ψ
- Add solutions to boxes in each fiber, and get the rest from the Kan structure of the component types

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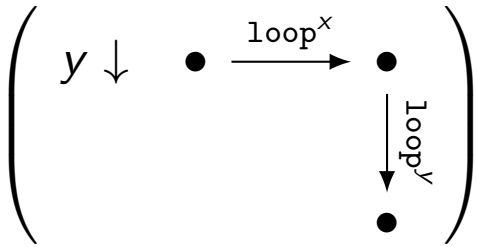
BAD: $\texttt{fcom}^{0 \rightsquigarrow 1}$ 

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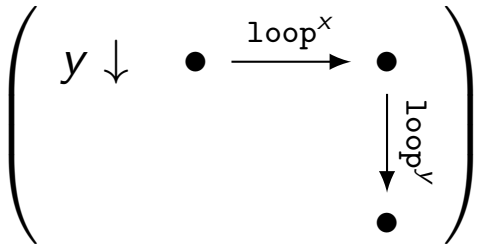
BAD: $\texttt{fcom}^{0 \rightsquigarrow 1}$ 

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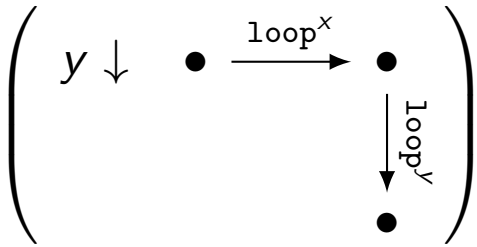
BAD: $\texttt{fcom}^{0 \rightsquigarrow 1}$ 

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   - Implement `hcom` directly for 0-dimensional inductive types.

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3. We have parameterized inductives "for free"

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- Even with 0-constructors, need new elements to satisfy the Kan condition

$$\texttt{coe}^{0 \rightsquigarrow y}_{x.\texttt{Id}_{\mathbb{S}1}(\texttt{base},\texttt{loop}^x)}(\texttt{refl}(\texttt{base})) \longmapsto ???$$

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- To eliminate into a Kan type, need only supply the `refl` case
- Exact β-rule for `refl`! (see also [Swa14; Coh+15])

The future

- Indexed inductive types in generality
- Inductive-inductive types
- Implementation in RedPRL

References

Evan Cavallo and Robert Harper. *Computational Higher Type Theory IV: Inductive Types*. arXiv:1801.01568.

Evan Cavallo and Robert Harper. *Higher Inductive Types in Cubical Computational Type Theory*. Preprint.

THANKS