Relative Kan completion in a nominal sets model

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MURI 2017

This work arose as an aspect of the implementation of inductive constructions in computational higher type theory (CHTT) [3, 2, 1]. Specifically, it pertains to the implementation of inductive families: families of types defined as generated by some list of constructors. A popular example is the family of identity types: the family $\Gamma, x_1{:}X, x_2{:}X \vdash \mathsf{Id}_X(x_1, x_2)$ generated by the single constructor $\Gamma, x{:}X \vdash \mathsf{refl}_X(x) : \mathsf{Id}_X(x, x)$. In HoTT as defined in e.g. [11], $\mathsf{Id}_X(M_1, M_2)$ is regarded as the "type of paths from M_1 to M_2," and is the means by which one accesses higher-dimensional structure.

Existing cubical type theories [3, 6], in pursuit of a computational interpretation, instead represent higher dimensional structure by allowing terms to depend on a context of dimension variables, which are thought of as varying from 0 to 1. Using this structure, it is possible to define a type $\mathsf{Path}_X(M_0, M_1)$ whose elements are terms $\langle a\rangle P$ varying over an abstracted dimension name a which satisfy $P(0/a) = M_0$ and $P(1/a) = M_1$. There is then a family $\Gamma, x_1{:}X, x_2{:}X \vdash \mathsf{Path}_X(x_1, x_2)$, but in existing theories this family is *not* "generated by reflexivity." By this we mean that, although one has a "reflexivity" constructor $\Gamma, x{:}X \vdash \langle a\rangle x : \mathsf{Path}_X(x, x)$, one cannot (to our knowledge) define an eliminator which satisfies the appropriate β-rule.[1]

Swan has shown that it nonetheless possible in the cubical setting to define a family generated by reflexivity in the above sense [10]. Here "the cubical setting" is Pitts' category **01Sub**, a category of nominal sets which is equivalent to the category of cubical sets on on the cube category with face maps, degeneracies, and permutations.[2] Swan describes his construction as *labelled name abstraction*: intuitively, one is modifying the type $\mathsf{Path}_X(M_0, M_1)$ so that degenerate paths can be "labelled," so that an eliminator can behave correctly on such paths. This intuition is adapted to type theory in [6, §9.1], though using a somewhat different definition.

Here we will approach the problem of identity types from a different perspective, that of Kan completion, an operation described for a particular case in [5] and expounded upon in [8]. The idea here is to take an arbitrary morphism and construct a free fibration from it by adding formal filler elements to satisfy the

[1] This is related to the failure of the *regularity condition* for certain types; see [8, §3.4].

[2] This is the cube category used in [5]. For comparison, the cube category used by the same group in the later [6] also includes diagonals and connections, while CHTT adds only diagonals.

Kan condition. Swan uses this construction to define his algebraic weak factorization system on **01Sub**. However, since the factorization is not stable under pullback in the sense required by [4], the factorization of the diagonal cannot be used to model identity types, so he turns to labelled path abstraction. We will instead seek to define a "relative Kan completion," which will factorize a morphism between *fibrations over some context*, in such a way as to be stable under pullback along morphisms between contexts.

Our aim in doing this is to give a construction of identity types in a way which will more easily generalize to arbitrary inductive families in the future. Our relative Kan completion will serve as an interpretation of the *family of homotopy fibers* of a morphism $\Gamma \vdash f : X \to Y$, the family $\Gamma, y{:}Y \vdash \mathsf{Fib}_f(y)$ generated by the constructor $\Gamma, x{:}X \vdash \mathsf{r}(x) : \mathsf{Fib}_f(f(x))$.[3] This corresponds to factorization of f in the same sense that identity types correspond to factorization of the diagonal map (as described in [4]): we can define the latter in terms of the former as $\mathsf{Id}_X(M_1, M_2) \triangleq \mathsf{Fib}_{\lambda x.\langle x,x\rangle}(\langle M_1, M_2\rangle)$. (It is also possible to define the former using the latter: $\mathsf{Fib}_f(N) \triangleq \Sigma_{x\in X} \mathsf{Id}_Y(f(x), N)$.) By phrasing this construction in terms of a completion process, we hope it will be easier to generalize to inductive families with recursive constructors, where one wants to interleave the process of Kan completion with constructor completion.

After recalling the definition of the category **01Sub** from [9], as well as the category **01NomSub** corresponding to cubical sets with diagonals (appearing in [9] under the name $\mathbf{Set}^{\mathbf{Sb}}_{\mathrm{fs}}$), we will briefly describe the algebraic weak factorization system on **01Sub** given by Swan. We will then give a construction of an algebraic weak factorization system on **01NomSub**, and show how the definition of the factorization can be restricted to obtain a relative Kan completion. Finally, we will discuss the question of adapting this construction to **01Sub**.

1 Nominal sets models

We work over a distinguished set $\mathbb{A}$ of *atoms*.

Definition 1. The *category of* $\mathbb{A}$*-sets*, written $\mathbb{A}$-**Set**, is the category where

- objects are sets X with an action $\cdot : \mathrm{Perm}(\mathbb{A}) \times X \to X$, where $\mathrm{Perm}(\mathbb{A})$ is the set of finitely supported permutations of $\mathbb{A}$,
- morphisms $X \to Y$ are set functions $f : X \to Y$ which are equivariant: $f(\pi \cdot x) = \pi \cdot f(x)$ for $\pi \in \mathrm{Perm}(\mathbb{A})$ and $x \in X$.

Definition 2. For $X \in \mathbb{A}$-**Set**, we say that $A \supseteq \mathbb{A}$ is a *support* for $x \in X$ if for any $\pi \in \mathrm{Perm}(A)$, if $\pi a = a$ for all $a \in A$ then $\pi \cdot x = x$.

Definition 3. For $X \in \mathbb{A}$-**Set**, we say that $a \in \mathbb{A}$ is *fresh for* $x \in X$, written $a \# x$, if x has a support A such that $a \notin A$.

[3]The idea of taking the family of homotopy fibers as a primitive type former was introduced to me by Steve Awodey, in a set of notes from a Summer 2012 seminar at CMU and in personal communication.

Definition 4. The *category of nominal sets*, written **Nom**, is the subcategory of the category of $\mathbb{A}$-sets consisting of those objects X for which every $x \in X$ has a finite support.

Definition 5. The *category of* 01*-substitution sets*, written **01Sub**, is the category where

- objects are nominal sets X equipped with operations $-(0/a)$ and $-(1/a)$ for each $a \in \mathbb{A}$ such that for all $x \in X$ and $(a, i), (a', i') \in \mathbb{A} \times 2$,
  1. $a \# x(i/a)$,
  2. if $a \# x$, then $x(i/a) = x$,
  3. if $a \neq a'$, then $x(i/a)(i'/a') = x(i'/a')(i/a)$,
- morphisms $f : X \to Y$ are equivariant functions such that $f(x(i/a)) = f(x)(i/a)$ for all $(a, i) \in A \times 2$.

Definition 6. The *category of* 01$\mathbb{A}$*-substitution sets*, written **01NomSub**, is the category where

- objects are nominal sets X equipped with operations $-(r/a)$ for each $a \in \mathbb{A}$ and $r \in \mathbb{A} \amalg 2$ such that for all $x \in X$ and $(a, r), (a', r') \in \mathbb{A} \times (\mathbb{A} \amalg 2)$,
  1. $a \# x(r/a)$,
  2. if $a \# x$, then $x(r/a) = x$,
  3. if $a \neq a'$, then $x(r/a)(r'/a') = x(r'/a')(r/a)$,
- morphisms $f : X \to Y$ are equivariant functions such that $f(x(r/a)) = f(x)(r/a)$ for all $(a, r) \in A \times (A \amalg 2)$.

Notation 7. Given $k \in 2$, we write $\overline{k} \triangleq 1 - k$.

Each of **Nom**, **01Sub**, and **01NomSub** can be more compactly described as a subpresheaf of the presheaf category on the appropriate monoid of substitutions.

Given $X \in$ **Nom**, we can define the nominal set $[\mathbb{A}]X$ of *name abstractions* of elements of X, whose elements are equivalence classes of pairs $(a, x) \in \mathbb{A} \times X$ quotiented by the relation of α-equivalence: $(a, x) \approx_\alpha (a', x')$ if there exists $a'' \in \mathbb{A}$ with $a'' \# a, a', x, x'$ and $(a a'') \cdot x = (a' a'') \cdot x$. We write $\langle a \rangle x$ for the equivalence class of (a, x). The action by permutations is defined by $\pi \cdot \langle a \rangle x = \langle \pi a \rangle (\pi \cdot x)$. In **01Sub** and **01NomSub**, we define substitution in $[\mathbb{A}]X$ by $(\langle a \rangle x)(r/a') = \langle a \rangle (x(r/a'))$, where we are able to assume that $a \# a'$ by choosing an appropriate element of the equivalence class.

2 Algebraic weak factorization systems

2.1 Swan's awfs on 01Sub

First, we need a notion of an open box over a map $f : X \to Y$, of a filler for an open box over f, and of a filling operator for f.

Definition 8 (Swan)**.** In **01Sub**, for any map $f : X \to Y$, a *k-open (A, a)-box over f* is a pair (u, y) where

1. $u : A \times 2 \setminus (a, k) \to X$ is such that for any $(a', i'), (a'', i'') \in A \times 2 \setminus (a, k)$,
   (a) if $a \neq a'$ then $u(a', i')(i''/a'') = u(a'', i'')(i'/a')$,
   (b) $a' \# u(a', i')$,
2. $y \in Y$ is such that $f(u(a', i')) = y(i'/a')$ for all $(a', i') \in A \times 2 \setminus (a, k)$.

Definition 9 (Swan)**.** In **01Sub**, a *filler* for a k-open (A, a)-box (u, y) over f is an element $x \in X$ such that $x(i/a') = u(a', i)$ for every $(a', i) \in A \times 2 \setminus (a, k)$ and $f(x) = y$.

Definition 10 (Swan)**.** In **01Sub**, a *filling operator* for $f : X \to Y$ consists of a choice of filler $f \uparrow (u, y)$ for every open box (u, y) over f, satisfying the *uniformity conditions*:

1. for any $\pi \in \mathrm{Perm}(A)$, $f \uparrow (\pi(u, y)) = \pi(f \uparrow (u, y))$,
2. for (u, y) a k-open (A, a)-box, $c \# A$ and $i \in 2$, we have $(f \uparrow (u, y))(i/c) = f \uparrow ((u, y)(i/c))$.

Definition 11 (Swan)**.** In **01Sub**, a map $f : X \to Y$ is a *Kan fibration* if it admits a filling operator.

Given a map $f : X \to Y$, we define its factorization $X \xrightarrow{\lambda_f} Kf \xrightarrow{\rho_f} Y$ via an inductive construction, defining Kf and ρ_f simultaneously. (This informal presentation is made precise in [10] as a sequential colimit.)

Definition 12 (Swan)**.** The object Kf is generated by elements

- $x \in Kf$ for $x \in X$. We set $\rho_f(x) = f(x)$.
- $(u, y) \in Kf$ for (u, y) a k-open (A, a)-box over ρ_f. We set $\rho_f((u, y)) = y$.
- $\langle a \rangle(u, y) \in Kf$ for (u, y) a k-open (A, a)-box over ρ_f. We set $\rho_f(\langle a \rangle(u, y)) = y(k/a)$.

The map $\lambda_f : X \to Kf$ is the obvious inclusion.

The second and third clauses introduce formal fillers and composites, respectively: 01-substitution in Kf is defined so that $(u, y)(k/a) = \langle a \rangle(u, y)$.

Theorem 13 (Swan)**.** *This factorization operation can be extended to an algebraic weak factorization system.*

Theorem 14 (Swan)**.** *The right maps of the awfs so defined are exactly the Kan fibrations.*

2.2 An awfs on 01NomSub

In **01NomSub**, an adjustment is necessary, as the open boxes described above are not well-behaved under arbitary substitutions: for $a \neq b$, we have

$$(u, y)(a/b)(k/a) = \langle a \rangle (u(a/b), y(a/b))$$

and

$$(u, y)(k/a)(k/b) = \langle a \rangle (u(k/b), y(k/b)),$$

but these two substitutions should give the same result. We solve this problem by keeping the open boxes under an abstraction. For the purposes of defining our relative factorization, it also appears necessary to require filling between any two "dimension terms" $r, r' \in \mathbb{A} \amalg 2$. (This matches the requirement for typehood in CHTT.)

Definition 15. In **01NomSub**, for any map $f : X \to Y$, a *abstracted r-capped A-box over f* is a term $\langle a \rangle (u, y)$ where

1. $u : (A \times 2) \cup \{(a, r)\} \to X$ is such that for any $(a', r'), (a'', r'') \in (A \times 2) \cup \{(a, r)\}$,
   (a) if $a \neq a'$ then $u(a', r')(r''/a'') = u(a'', r'')(r'/a')$,
   (b) $u(a', r') = u(a', r')(r'/a')$,
2. $y \in Y$ is such that $f(u(a', r')) = y(r'/a')$ for all $(a', r') \in (A \times 2) \cup \{(a, k)\}$.

Definition 16. In **01NomSub**, a *filler at $r' \in \mathbb{A} \amalg 2$* for an abstracted r-capped A-box $\langle a \rangle (u, y)$ over f is an element $x \in X$ such that

1. $x(i/a') = u(a', i)(r'/a)$ for $(a', i) \in A \times 2$,
2. if $r = r'$ then $x = u(a, r)$,
3. $f(x) = y(r'/a)$.

(Note: we do not require above that $x(r/r') = u(a, r)$ when $r' \in \mathbb{A}$; this condition is enforced for filling operators by the uniformity condition below.)

Definition 17. In **01NomSub**, a *filling operator* for $f : X \to Y$ consists of a choice of filler $f \uparrow_r^{r'} \langle a \rangle (u, y)$ at r' for every abstracted r-capped A-box $\langle a \rangle (u, y)$ over f, satisfying the *uniformity condition*:

1. for $(b, s) \in (\mathbb{A} \times (\mathbb{A} \amalg 2)) \setminus A \times 2$, we have

$$(f \uparrow_r^{r'} (\langle a \rangle (u, y)))(s/b) = f \uparrow_{r(s/b)}^{r'(s/b)} ((\langle a \rangle (u, y))(s/b)).$$

Here, $(\langle a \rangle (u, y))(s/b)$ for $(b, s) \notin A \times 2$ is the $A(s/b)$-box $\langle a \rangle (u', y(s/b))$ defined by

$$u'(a', i) \triangleq \begin{cases} u(b, i) & \text{if } a' = s \\ u(a', i)(s/b) & \text{otherwise.} \end{cases}$$

Definition 18. In **01NomSub**, a map $f : X \to Y$ is a *Kan fibration* if it admits a filling operator.

Just as in **01Sub**, we can define a factorization $X \xrightarrow{\lambda_f} Kf \xrightarrow{\rho_f} Y$ in **01NomSub** where Kf is obtained by adding formal filler elements to X. Here, we define Kf and ρ_f as generated by elements

- $x \in Kf$ for $x \in X$. We set $\rho_f(x) = f(x)$.
- $(\langle a\rangle(u, y), r, r') \in Kf$ for (u, y) an abstracted r-capped A-box over ρ_f and $r' \in \mathbb{A} \amalg 2 \setminus r$. We set $\rho_f((\langle a\rangle(u, y), r, r')) = y(r'/a)$.

Substitution in Kf is defined on elements of X by substitution in X, and on r-capped A-boxes by

$$(\langle a\rangle(u, y), r, r')(s/b) \triangleq \begin{cases} u(a, r), & \text{if } r(s/b) = r'(s/b) \\ u(b, s)(r'/a), & \text{if } (b, s) \in A \times 2 \\ ((\langle a\rangle(u, y))(s/b), r(s/b), r'(s/b)), & \text{otherwise} \end{cases}$$

Again, $\lambda_f : X \to Kf$ is the inclusion. For convenience we sometimes write $(\langle a\rangle(u, y), r, r)$ to mean $u(a, r)$.

Given a commutative square

$$\begin{array}{ccc} X & \xrightarrow{h} & U \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle g} \\ Y & \xrightarrow{k} & V, \end{array}$$

we define the functorial action $K(h, k) : Kf \to Kg$ so that

$$\begin{array}{ccc} X & \xrightarrow{h} & U \\ {\scriptstyle \lambda_f}\downarrow & & \downarrow{\scriptstyle \lambda_g} \\ Kf & \xrightarrow{K(h,k)} & Kg \\ {\scriptstyle \rho_f}\downarrow & & \downarrow{\scriptstyle \rho_g} \\ Y & \xrightarrow{k} & V \end{array}$$

commutes. We simultaneously define $K(h, k)(t)$ and verify that the equation $\rho_g(K(h, k)(t)) = k(\rho_f(t))$ holds by induction.

- Case 1: Suppose $t = x \in X$. Then we define $K(h, k)(t) = h(x)$. We have $\rho_g(K(h, k)(t)) = \rho_g(h(x)) = g(h(x)) = k(f(x)) = k(\rho_f(t))$.
- Case 2: Suppose $t = (\langle a\rangle(u, y), r, r')$ where $\langle a\rangle(u, y)$ is an r-capped A-box over ρ_f. By induction hypothesis, $\langle a\rangle(K(h, k) \circ u, k(y))$ is an r-capped A-box over ρ_g. We thus define $K(h, k)(t) = (\langle a\rangle(K(h, k) \circ u, k(y)), r, r')$. We have $\rho_g(K(h, k)(t)) = \rho_g((\langle a\rangle(K(h, k) \circ u, k(y)), r, r')) = k(y)(r'/a) = k((\langle a\rangle(u, y), r, r'))$.

We now have a functorial factorization K, λ, ρ. In order to define an algebraic weak factorization system, we need to give a comonad $\mathsf{L} = (L, \Phi, \Sigma)$ and monad $\mathsf{R} = (R, \Lambda, \Pi)$ where L, R are the endofunctors on $(\mathbf{Set}_{\mathrm{fs}}^{\mathbf{Sb}})^{\mathbf{2}}$ defined by $Lf = \lambda_f$ and $Rf = \rho_f$ [7]. Modulo the necessary adjustment of definitions, the construction and proofs are essentially identical to those of Swan.

We begin with the comonad structure. We define $\Phi : L \to \mathrm{id}$ by

$$\Phi\begin{pmatrix} X \\ \big\downarrow f \\ Y \end{pmatrix} = \begin{array}{ccc} X & = & X \\ {\scriptstyle \lambda_f}\big\downarrow & & \big\downarrow{\scriptstyle f} \\ Kf & \xrightarrow{\rho_f} & Y \end{array}$$

and $\Sigma : L \to L^2$ by

$$\Sigma\begin{pmatrix} X \\ \big\downarrow f \\ Y \end{pmatrix} = \begin{array}{ccc} X & = & X \\ {\scriptstyle \lambda_f}\big\downarrow & & \big\downarrow{\scriptstyle \lambda_{\lambda_f}} \\ Kf & \xrightarrow{\sigma_f} & K\lambda_f \end{array}$$

where we must define $\sigma_f : Kf \to K\lambda_f$. We define $\sigma_f(t)$ by induction on t while simultaneously checking that $\rho_{\lambda_f}(\sigma_f(t)) = t$:

- Case 1: Suppose $t = x \in X$. We set $\sigma_f(t) = x$; we have $\rho_{\lambda_f}(\sigma_f(t)) = \rho_{\lambda_f}(x) = \lambda_f(x) = x$.
- Case 2: Suppose $t = (\langle a\rangle(u, y), r, r')$ where $\langle a\rangle(u, y)$ is a r-capped A-box over ρ_f. By induction hypothesis, $\sigma_f \circ u$ is defined and we know that $\rho_{\lambda_f}((\sigma_f \circ u)(a', s)) = u(a', s)$ for $(a', s) \in (\mathbb{A} \times 2) \cup \{(a, r)\}$, which implies that $(\sigma_f \circ u, (\langle a\rangle(u, y), r, a))$ is an r-capped A-box over ρ_{λ_f}. We thus define $\sigma_f(t) = (\langle a\rangle(\sigma_f \circ u, (\langle a\rangle(u, y), r, a)), r, r')$, and we see that $\rho_{\lambda_f}(\sigma_f(t)) = (\langle a\rangle(u, y), r, a))(r'/a) = (\langle a\rangle(u, y), r, r')) = t$.

It is immediate from the first case that $\sigma_f \circ \lambda_f = \lambda_{\lambda_f}$ as required.

In order to check that (L, Φ, Σ) is a comonad, we must confirm that $\Phi_{Lf} \circ \Sigma_f = \mathrm{id}_{Lf}$, $L\Phi_f \circ \Sigma_f = \mathrm{id}_{Lf}$, and $L\Sigma_f \circ \Sigma_f = \Sigma_{Lf} \circ \Sigma_f$. We know the first of these, as it amounts to checking $\rho_{\lambda_f} \circ \sigma_f = \mathrm{id}_{Kf}$; the others follow by a straightforward induction.

We proceed to the monad structure. We define $\Lambda : \mathrm{id} \to R$ by

$$\Lambda\begin{pmatrix} X \\ \big\downarrow f \\ Y \end{pmatrix} = \begin{array}{ccc} X & \xrightarrow{\lambda_f} & Kf \\ {\scriptstyle f}\big\downarrow & & \big\downarrow{\scriptstyle \rho_f} \\ Y & = & Y \end{array}$$

and $\Pi : R^2 \to R$ by

$$\Pi\begin{pmatrix} X \\ \big\downarrow f \\ Y \end{pmatrix} = \begin{array}{ccc} K\rho_f & \xrightarrow{\pi_f} & Kf \\ {\scriptstyle \rho_{\rho_f}}\big\downarrow & & \big\downarrow{\scriptstyle \rho_f} \\ Y & = & Y \end{array}$$

where we must define $\pi_f : K\rho_f \to Kf$. Again, we define $\pi_f(t)$ by induction on t, simultaneously verifying that $\rho_f(\pi_f(t)) = \rho_{\rho_f}(t)$.

- Case 1: Suppose $t \in Kf$. Then we define $\pi_f(t) = t$; we have $\rho_f(\pi_f(t)) = \rho_f(t) = \rho_{\rho_f}(t)$.
- Case 2: Suppose $t = (\langle a\rangle(u, y), r, r')$ where $\langle a\rangle(u, y)$ is an r-capped A-box over ρ_{ρ_f}. By induction hypothesis, $\pi_f \circ u$ is defined and $\rho_f((\pi_f \circ u)(a', s)) = \rho_{\rho_f}(u(a', s)) = y(a'/s)$ for $(a', s) \in (\mathbb{A} \times 2) \cup \{(a, r)\}$, so $\langle a\rangle(\pi_f \circ u, y)$ is an r-capped A-box over ρ_f. We therefore define $\pi_f(t) = (\langle a\rangle(\pi_f \circ u, y), r, r')$. We have $\rho_f(\pi_f(t)) = y(r'/a) = \rho_{\rho_f}(t)$.

To check that (R, Λ, Π) is a monad, we must confirm that $\Pi_f \circ \Lambda_{Rf} = \mathrm{id}_{Rf}$, $\Pi_f \circ R\Lambda_f = \mathrm{id}_{Rf}$, and $\Pi_f \circ R\Pi_f = \Pi_f \circ \Pi_{Rf}$. These can be checked by straightforward induction.

2.3 Characterizing fibrations

As Swan does in **01Sub**, we can confirm that the class of right maps induced by the awfs is exactly the class of Kan fibrations. Again, these proofs are not substantially different from those in [10].

Lemma 19. *Suppose that $f : X \to Y$ has an R-algebra structure where we regard R as a pointed endofunctor. Then we can define a canonical filling operator for f.*

Proof. Suppose f is as above; the R-algebra structure amounts to a map $g : Kf \to X$ satisfying

$$\begin{array}{ccc} Kf & \xrightarrow{g} & X \\ {\scriptstyle \rho_f}\downarrow & & \downarrow{\scriptstyle f} \\ Y & = & Y \end{array} \qquad\qquad \begin{array}{ccc} X & \xrightarrow{\lambda_f} & Kf \\ & \diagdown\!\!\diagdown & \downarrow{\scriptstyle g} \\ & & X \end{array}$$

Given a r-capped A-box $\langle a\rangle(u, y)$ over f, the second diagram implies that $\langle a\rangle(\lambda_f \circ u, y)$ is an r-capped A-box over ρ_f; we thus define its filler at r' by $f \uparrow_r^{r'} \langle a\rangle(u, y) \triangleq g((\langle a\rangle(\lambda_f \circ u, y), r, r'))$. The definition of substitution in Kf and the equivariance of g ensure that this definition is well-behaved with respect to substitution, and we have $f(f \uparrow_r^{r'} \langle a\rangle(u, y)) = y(r'/a)$ by the first diagram above. □

Lemma 20. *Suppose that $f : X \to Y$ supports a filling operator. Then there is a canonical R-algebra structure on f.*

Proof. Suppose we have f as above. We define a map $g : Kf \to X$. We define $g(t)$ by induction on t while simultaneously proving that $f(g(t)) = \rho_f(t)$.

- Case 1: Suppose $t = x \in X$. We define $g(x) = x$. We have $f(g(t)) = f(x) = \rho_f(t)$.

- Case 2: Suppose $t = (\langle a \rangle(u, y), r, r')$ where (u, y) is an r-capped A-box over ρ_f. By induction hypothesis, $g \circ u$ is defined and we have $f((g \circ u)(a', i)) = \rho_f(u(a', i)) = y$, so $(g \circ u, y)$ is an r-capped A-box over f. We thus define $g(t) = f \uparrow_r^{r'} \langle a \rangle(g \circ u, y)$. We have $f(g(t)) = y(r/a) = \rho_f(t)$.

To see that g defines a pointed endofunctor R-algebra structure, we need the diagrams

$$\begin{array}{ccc} Kf & \xrightarrow{g} & X \\ {\scriptstyle \rho_f}\downarrow & & \downarrow{\scriptstyle f} \\ Y & = & Y \end{array} \qquad \begin{array}{ccc} X & \xrightarrow{\lambda_f} & Kf \\ & \searrow\!\!\!\!= & \downarrow{\scriptstyle g} \\ & & X \end{array}$$

to commute; we have already shown the first and the second is immediate. $\square$

The constructions in Lemmas 19 and 20 are clearly inverse; thus we have an exact correspondence between Kan fibrations and pointed endofunctor R-algebras on f.

3 Relative factorization

3.1 Construction

We are now ready to define a relative factorization operator in **01NomSub**. Our desiderata are the following:

1. For any triangle

$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ & {\scriptstyle g}\searrow \quad \swarrow{\scriptstyle h} & \\ & Z & \end{array}$$

   in **01NomSub** with g, h fibrations, there is a factorization

$$\begin{array}{ccc} & K^{g,h}(f) & \\ {\scriptstyle \tilde{\lambda}_f}\nearrow & & \searrow{\scriptstyle \tilde{\rho}_f} \\ X & \xrightarrow{f} & Y \\ {\scriptstyle g}\searrow & & \swarrow{\scriptstyle h} \\ & Z & \end{array}$$

   with $\tilde{\lambda}_f$ a left map and $\tilde{\rho}_f$ a fibration.

2. For any map

$$\begin{tikzcd} X' \arrow[r, dashed, "k_X"] \arrow[d, "f'"'] \arrow[dd, bend right=60, "g'"'] & X \arrow[d, "f"] \arrow[dd, bend left=60, "g"] \\ Y' \arrow[r, dashed, "k_Y"] \arrow[d, "h'"'] & Y \arrow[d, "h"] \\ Z' \arrow[r, dashed, "k_Z"] & Z \end{tikzcd}$$

of such triangles (with g, g', h, h' fibrations), there is an induced map $K(k_X, k_Y, k_Z) : K^{g',h'}(f') \to K^{g,h}(f)$ making the diagram

$$\begin{tikzcd} X' \arrow[r, "k_X"] \arrow[d, "\tilde{\lambda}_{f'}"'] & X \arrow[d, "\tilde{\lambda}_f"] \\ K^{g',h'}(f') \arrow[r, "K(k_X, k_Y, k_Z)"] \arrow[d, "\tilde{\rho}_{f'}"'] & K^{g,h}(f) \arrow[d, "\tilde{\rho}_f"] \\ Y' \arrow[r, "k_Y"] & Y \end{tikzcd}$$

commute.

3. In the above situation, if the two squares in the assumed diagram are pullbacks, then the square

$$\begin{tikzcd} K^{g',h'}(f') \arrow[r, "K(k_X, k_Y, k_Z)"] \arrow[d, "h' \circ \tilde{\rho}_{f'}"'] & K^{g,h}(f) \arrow[d, "h \circ \tilde{\rho}_f"] \\ Z' \arrow[r, "k_Z"'] & Z \end{tikzcd}$$

is also a pullback.

In type-theoretic terms, the fibration $\tilde{\rho}_f : K^{g,h}(f) \to Y$ will be the interpretation of the family of homotopy fibers $z{:}Z, y{:}Y \vdash \mathsf{Fib}_f(y)$ of the map $z{:}Z \vdash f : X \to Y$, and $\tilde{\lambda}_f : X \to K^{g,h}(f)$ will be the interpretation of the introduction form $z{:}Z, x{:}X \vdash \mathsf{r}_f(x) : \mathsf{Fib}_f(f(x))$. The fact that $\tilde{\lambda}_f$ is a left map gives rise to the dependent eliminator. Naturally, the interpretation must be stable under substitution for $z : Z$; this translates into the pullback stability condition above.

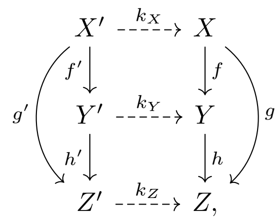
Definition 21. Suppose we have a diagram

$$\begin{tikzcd} X \arrow[rr, "f"] \arrow[dr, "g"'] & & Y \arrow[dl, "h"] \\ & Z & \end{tikzcd}$$

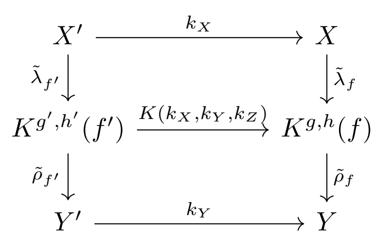
We define a subset $K^{g,h}(f) \subseteq Kf$ as follows.

- If $x \in X$, then $x \in K^{g,h}(f)$.
- If $\langle a\rangle(u,y)$ is an r-capped A-box over ρ_f with $u(a',i) \in K^{g,h}(f)$ for all $(a',i) \in A \times 2 \setminus (a,k)$, $r' \neq r$, and $a \# h(y)$, then $(\langle a\rangle(u,y), r, r') \in K^{g,h}(f)$.

We write $\tilde{\lambda}_f : X \to K^{g,h}(f)$ for the inclusion of X and $\tilde{\rho}_f : K^{g,h}(f) \to Y$ for the restriction of ρ_f to $K^{g,h}(f)$. Given



the map $K(k_X, k_Y) : Kf' \to Kf$ restricts to a map $K(k_X, k_Y, k_Z) : K^{g',h'}(f') \to K^{g,h}(f)$ making the diagram

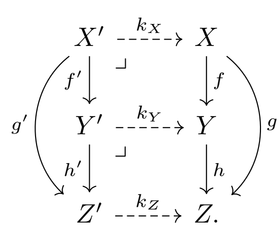


commute.

Remark 22. We do not need g and h to be fibrations for this definition. This assumption is required to show that $\tilde{\rho}_f$ is a fibration, but not needed to check that the construction is stable under pullback. Note that for any $X, Y \in \mathbf{01NomSub}$ (possibly non-fibrant) and $f : X \to Y$, we have $Kf = K^{!_X,!_Y}(f)$ for the unique maps $!_X : X \to 1$ and $!_Y : Y \to 1$.

The sole restriction on elements of $K^{g,h}(f)$ relative to Kf is the requirement that $a \# h(y)$ in the case of a formal filler $(\langle a\rangle(u,y), r, r')$. This ensures that a filler lives over the same element of Z as its cap, making the construction "fiberwise" with respect to Z and therefore stable under pullback along maps into Z.

Theorem 23. *Suppose we have a map*



Then

$$\begin{array}{ccc} K^{g',h'}(f') & \xrightarrow{K(k_X,k_Y,k_Z)} & K^{g,h}(f) \\ {\scriptstyle h'\circ\tilde{\rho}_{f'}}\downarrow & & \downarrow{\scriptstyle h\circ\tilde{\rho}_f} \\ Z' & \xrightarrow[k_Z]{} & Z \end{array} \tag{1}$$

is a pullback square.

Proof. We abbreviate $k_K \triangleq K(k_X, k_Y, k_Z)$. We will show that the map $\langle h' \circ \tilde{\rho}_{f'}, k_K\rangle : K^{g',h'}(f') \to Z' \times_Z K^{g,h}(f)$ is an isomorphism by exhibiting an inverse $\ell : Z' \times_Z K^{g,h}(f) \to K^{g',h'}(f')$. We simultaneously define $\ell(z', t)$ and prove that $\langle h' \circ \tilde{\rho}_{f'}, k_K\rangle(\ell(z', t)) = (z', t)$ by induction on t.

- Case 1: Suppose $t = x \in X$. We take $\ell(z', t)$ to be the element of X' corresponding to $(z', x) \in Z' \times_Z X$. As such we have $\langle h' \circ \tilde{\rho}_{f'}, k_K\rangle(\ell(z', t)) = (z', x) = (z', t)$.

- Case 2: Suppose $t = (\langle a\rangle(u, y), r, r')$ where $\langle a\rangle(u, y)$ is an r-capped A-box over $\tilde{\rho}_f$ with $a \# h(y)$. We may assume that $a \# z'$. We have

$$\begin{aligned} k_Z(z') &= (h \circ \tilde{\rho}_f)(\langle a\rangle(u, y), r, r') \\ &= h(y(r'/a)) \\ &= h(y). \end{aligned}$$

  because $a \# h(y)$. Since $Y' \cong Z' \times_Z Y$, we have a unique $y' \in Y'$ corresponding to the pair $(z', y) \in Z' \times_Z Y$.

  For each $(a', s) \in (A \times 2) \cup \{(a, r)\}$, we may define

$$u'(a', s) = \ell(z'(s/a'), u(a', s)).$$

  By induction hypothesis we know that $h'(\tilde{\rho}_{f'}(u'(a', s)) = z'(s/a')$ and $k_K(u'(a', s)) = u(a', s)$. From the second it follows that

$$\begin{aligned} k_Y(\tilde{\rho}_{f'}(u'(a', s))) &= \tilde{\rho}_f(k_K(u'(a', s))) \\ &= \tilde{\rho}_f(u(a', s)) \\ &= y(s/a'). \end{aligned}$$

  Then $\tilde{\rho}_{f'}(u'(a', s))$ is the unique element of Y' corresponding to the pair $(z'(s/a'), y(s/a')) \in Z' \times_Z Y$, implying that $\tilde{\rho}_{f'}(u'(a', s)) = y'(s/a')$. Hence $\langle a\rangle(u', y')$ is a r-capped A-box over $\tilde{\rho}_{f'}$. Moreover, we have $h'(y') = z'$, so $a \# h'(y')$. Thus $(\langle a\rangle(u', y'), r, r') \in K^{g',h'}(f')$, and we define $\ell(z', t) = (\langle a\rangle(u', y'), r, r')$. We have $h'(\tilde{\rho}_{f'}(\ell(z', t))) = h'(y') = z'$ and $k_K(\ell(z', t)) = (\langle a\rangle(k_K \circ u', k_Y(y')), r, r') = (\langle a\rangle(u, y), r, r') = t$ as needed.

This completes the definition of ℓ; it remains to check that $\ell \circ \langle h' \circ \tilde{\rho}_{f'}, k_K\rangle = \mathrm{id}$. We show that $\ell(h'(\tilde{\rho}_{f'}(t)), k_K(t)) = t$ by induction on t. For $t = x \in X$ this is clear. Suppose $t = (\langle a\rangle(u', y'), r, r')$ where $\langle a\rangle(u', y')$ is an r-capped A-box over

$\tilde{\rho}_{f'}$ with $a \# h'(y')$. In this case, $\ell(h'(\tilde{\rho}_{f'}(t)), k_K(t)) = (\langle a\rangle(\overline{u}', \overline{y}'), r, r')$ where $\overline{u}'(a', s) = \ell(h'(\tilde{\rho}_{f'}(t))(a/s), k_K(u'(a', s))$ and $\overline{y}'$ is the unique element of Y' corresponding to $\langle h'(\tilde{\rho}_{f'}(t), k_K(t)\rangle \in Z' \times_Z Y$. By induction hypothesis applied at each face of u' we have $\overline{u}' = u'$. Also, $h'(y') = h'(\tilde{\rho}_{f'}(t))$ and $k_Y(y') = k_K(t)$, so by uniqueness $\overline{y}' = y'$. □

In order for $\tilde{\rho}_f$ to be a fibration, we must define a filling operator $\tilde{\rho}_f\!\uparrow$. As $K^{g,h}(f)$ only contains formal fillers for boxes which do not move between Z-indices, the remaining fillers must be implemented. This can be accomplished when g and h are fibrations. Intuitively, the filling operations for g and h make it possible to move between Z-indices, so that formal fillers are only necessary within each fiber. The following lemma, which is used to show that $\tilde{\rho}_f$ is a fibration, implements coercion between Z-indices.

Lemma 24. *Assume g and h are fibrations.*

Let $\langle a\rangle(y, z)$ be an r-capped $\emptyset$-box over h. (We will treat y as an element of Y.) Since h is a fibration, we have $h \uparrow^b_r \langle a\rangle(y, z) \in Y$ for any $b \in \mathbb{A}$. Let $\langle a\rangle(t, h \uparrow^a_r \langle a\rangle(y, z))$ be an r-capped $\emptyset$-box over $\tilde{\rho}_f$. For any $r' \in \mathbb{A} \amalg 2$, there is a filler at r' for $\langle a\rangle(t, h \uparrow^a_r \langle a\rangle(y, z))$, for which we will write $\tilde{\rho} \upuparrows^{r'}_r \langle a\rangle(t, y, z)$. Pictorially, we have

$$\begin{array}{cccccc}
\xrightarrow{a} & t & \overset{\tilde{\rho}\upuparrows^a_r\langle a\rangle(t,y,z)}{\dashrightarrow} & \in & K^{g,h}(f) & \\
 & & & & \downarrow \tilde{\rho}_f & \\
 & y & \xrightarrow{h\uparrow^a_r\langle a\rangle(y,z)} & \in & Y & \\
 & & & & \downarrow h & \\
 & z(r/a) & \xrightarrow{z} & \in & Z &
\end{array}$$

This choice of fillers is moreover uniform, in the sense that

1. *$\tilde{\rho}_f \upuparrows^r_r \langle a\rangle(t, y, z) = t$,*
2. *$(\tilde{\rho}_f \upuparrows^{r'}_r \langle a\rangle(t, y, z))(s/b) = \tilde{\rho}_f \upuparrows^{r'(s/b)}_{r(s/b)} (\langle a\rangle(t, y, z))(s/b)$.*

Proof. By induction on t.

- Case 1: Suppose $t = x \in X$. As g is a fibration and $g(t) = h(f(t)) = h(y) = z(r/y)$, we have a filler $g \uparrow^{r'}_r \langle a\rangle(t, z) \in X$. As desired, this is an element which is mapped to z by g, but it may not be mapped to $h \uparrow^{r'}_r \langle a\rangle(y, z)$ by f. This we fix with a formal filler.

  Choose $b \in \mathbb{A}$ fresh. Observe that $f(g \uparrow^b_r \langle a\rangle(t, z)) \in Y$ is mapped by h to $z(b/a)$. Thus, $\langle a\rangle(f(g \uparrow^b_r \langle a\rangle(t, z)), z)$ is a b-capped $\emptyset$-box over h, so has a filler $y' \triangleq h \uparrow^{r'}_b \langle a\rangle(f(g \uparrow^b_r \langle a\rangle(t, z)), z)$. By uniformity of filling operators, we have

$$\begin{aligned} y'(r/b) &= h \uparrow_r^{r'} \langle a\rangle(f(g \uparrow_r^{r} \langle a\rangle(t,z)),z) \\ &= h \uparrow_r^{r'} \langle a\rangle(f(t),z) \\ &= h \uparrow_r^{r'} \langle a\rangle(y,z) \end{aligned}$$

and

$$\begin{aligned} y'(r'/b) &= h \uparrow_{r'}^{r'} \langle a\rangle(f(g \uparrow_r^{r'} \langle a\rangle(t,z)),z) \\ &= f(g \uparrow_r^{r'} \langle a\rangle(t,z)). \end{aligned}$$

The latter equation implies that $\langle b\rangle(g \uparrow_r^{r'} \langle a\rangle(t,z),y')$ is an r'-capped $\emptyset$-box over $\tilde{\rho}_f$. Thus, we have $t' \triangleq (\langle b\rangle(g \uparrow_r^{r'} \langle a\rangle(t,z),y'),r',r) \in K^{g,h}(f)$. We set $\tilde{\rho}_f \upuparrows_r^{r'} (\langle a\rangle(t,y,z)) \triangleq t'$. As required, we have $\tilde{\rho}_f(t') = y'(r/b) = h \uparrow_r^{r'} \langle a\rangle(y,z)$. When $r = r'$, we get $t' = g \uparrow_r^{r'} \langle a\rangle(t,z) = t$.

- Case 2: Suppose $t = (\langle b\rangle(u,y),s,s')$ where $\langle b\rangle(u,y)$ is an r-capped B-box over $\tilde{\rho}_f$. For $(b',s') \in (B \times 2) \cup \{(b,s)\}$, we have $\tilde{\rho}_f(u(b',s')) = y(s'/b')$ By induction hypothesis, $\tilde{\rho}_f \upuparrows_r^{r'} (\langle a\rangle(u(b',s'),y(s'/b'),z(s'/b')))$ is defined. Define $u' : (B \times 2) \cup \{(b,s)\} \to K^{g,h}(f)$ by

  $$u'(b',s') \quad = \quad \tilde{\rho}_f \upuparrows_r^{r'} (\langle a\rangle(u(b',s'),y(s'/b'),z(s'/b')))$$

  The boundary conditions are satisfied by uniformity of $\tilde{\rho}_f \upuparrows$, so $\langle b\rangle(u',h\uparrow_r^{r'} \langle a\rangle(y,z))$ is an s-capped B-box over $\tilde{\rho}_f$. Moreover, we may assume that $b \# z$. We thus have $t' \triangleq (\langle b\rangle(u',h \uparrow_r^{r'} \langle a\rangle(y,z)),s,s') \in K^{g,h}(f)$. We set $\tilde{\rho}_f \upuparrows_r^{r'} (\langle a\rangle(t,y,z)) \triangleq t'$. As required, we have $\tilde{\rho}_f(t') = h\uparrow_r^{r'} \langle a\rangle(y,z)$. When $r = r'$, we have $u = u'$ and so $t' = t$. □

Theorem 25. *If g and h are fibrations, then $\tilde{\rho}_f$ is a fibration.*

Proof. Let $\langle a\rangle(u,y)$ be an r-capped A-box over $\tilde{\rho}_f$ and let $r' \in \mathbb{A} \amalg 2$. Choose $b \in \mathbb{A}$ fresh. We define an r-capped A-open box $\langle b\rangle(u',h \uparrow_b^{r'} (y(b/a),h(y)))$ over $\tilde{\rho}_f$ by

$$\begin{aligned} u'(r,b) \quad &= \quad \tilde{\rho}_f \upuparrows_r^{r'} \langle a\rangle(u(a,r),y(r/a),h(y)) \\ u'(a',i) \quad &= \quad \tilde{\rho}_f \upuparrows_b^{r'} \langle a\rangle(u(a',i)(b/a),y(i/a')(b/a),h(y)(i/a')) \end{aligned}$$

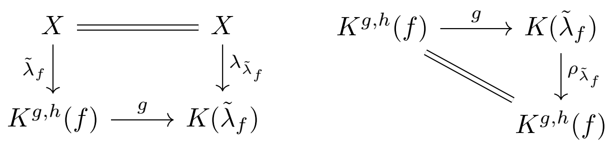
The edge conditions are satisfied by uniformity of $\tilde{\rho}_f \upuparrows$. We know that $b \# h(y)(r'/a) = h(h \uparrow_b^{r'} \langle a\rangle(y(b/a),h(y)))$ because b is fresh, so we can define $t \triangleq (\langle b\rangle(u',h \uparrow_b^{r'} (y(b/a),h(y))),r,r') \in K^{g,h}(f)$. We set $\tilde{\rho}_f \uparrow_r^{r'} \langle a\rangle(u,y) \triangleq t$. For any $(a',i) \in A \times 2$, we have

$$t(a'/i) = (\tilde{\rho}_f \upuparrows_b^{r'} \langle a\rangle(u(a',i)(b/a),y(i/a')(b/a),h(y)(i/a')))(r'/b) = u(a',i)(r'/a)$$

and when $r = r'$ we have $t = u'(r,b) = u(a,r)$. □

Theorem 26. *$\tilde{\lambda}_f$ is a left map.*

Proof. We have to give a pointed endofunctor L-coalgebra structure on $\tilde{\lambda}_f$. This amounts to defining $g : K^{g,h}(f) \to K(\tilde{\lambda}_f)$ so that the diagrams



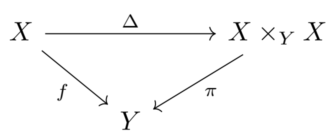
commute. We define $g(t)$ by induction on $t \in K^{g,h}(f)$ while simultaneously verifying that $\rho_{\tilde{\lambda}_f}(g(t)) = t$.

- Case 1: Suppose $t = x \in X$. Then define $g(t) = x$. We have $\rho_{\tilde{\lambda}_f}(g(t)) = \rho_{\tilde{\lambda}_f}(x) = \tilde{\lambda}_f(x) = x$.
- Case 2: Suppose $t = (\langle a\rangle(u, y), r, r')$ where $\langle a\rangle(u, y)$ is an r-capped A-box over $\tilde{\rho}_f$. By induction hypothesis, $\langle a\rangle(g \circ u, (\langle a\rangle(u, y), r, a))$ is an r-capped A-box over $\rho_{\tilde{\lambda}_f}$. We thus define $g(t) = (\langle a\rangle(g \circ u, (\langle a\rangle(u, y), r, a)), r, r')$. We have $\rho_{\tilde{\lambda}_f}(g(t)) = (\langle a\rangle(u, y), r, a)(r'/a) = t$.

That the first diagram commutes is immediate, and we have just proven that the second commutes. $\square$

3.2 Comparison with labelled path abstraction

For a fibration $f : X \to Y$, we have a diagonal map



where $\pi \triangleq f \circ \pi_1 = f \circ \pi_2$. Swan gives a factorization $X \to P_Y X \to X \times_Y X$ of maps of this form in **01Sub**; this provides an interpretation of identity types. We reproduce the definition below, which relies on an auxiliary notion of pre-normal form in a given direction.

Definition 27 (Swan)**.** The subset $P^a_Y X \subseteq K\Delta$ of *pre-normal forms in a direction* $a \in \mathbb{A}$ is generated by

1. $x \in P^a_Y X$ if $x \in X$ and $a \# x$,
2. $(u, (x_1, x_2)) \in P^a_Y X$ if $(u, (x_1, x_2))$ is a 1-open A, a-box over ρ_f, $u(a, 0) = x_1$, and $u(a', i) \in P^a_Y X$ for every $(a', i) \in (A \setminus a) \times 2$.

Definition 28 (Swan)**.** The subset $P_Y X \subseteq K\Delta$ of *normal forms* consists of those $t \in K\Delta$ such that $t = p(1/a)$ for some $a \in \mathbb{A}$ and $p \in P^a_Y X$.

By inspection of the definition of substitution in $K\Delta$, we can give the following alternative characterization of normal forms.

Proposition 29. *An element $t \in K\Delta$ is a normal form iff*

1. $t = x \in X$, *or*

2. $t = \langle a\rangle(u, (x_1, x_2))$ *where* $(u, (x_1, x_2))$ *is a* 1*-open* A, a*-box over* ρ_f, $u(a, 0) = x_1$, *and* $u(a', i) \in P^a_Y X$ *for every* $(a', i) \in (A \setminus a) \times 2$.

Note that, for a pre-normal form which is a formal filler $(u, (x_1, x_2)) \in P^a_Y X$, we have $\pi(x_1, x_2) = x_1 \# a$; this matches our restriction to fiberwise fillers and (as Swan observes) makes this definition stable under pullback.

It does not appear possible to naively adapt our definition to **01Sub** by restricting Kf to fiberwise fillers. Recall that in **01Sub**, for lack of a general substitution operation, formal filler elements of Kf are non-abstracted A, a-open boxes (u, y). As a is not bound, we cannot make the crucial assumption that a is fresh (for z') in the proof of Theorem 23 pullback stability. In Swan's definition, this problem is evaded because all the fillers in a normal form $p(1/a)$ are in a direction a which is made abstract at the "outermost layer." As we have mentioned, stable factorization of diagonal maps $\Delta : X \to X \times_Y X$ suffices to implement stable factorization of all morphisms. It is not clear to us, however, how to directly give a natural definition of $K^{g,h}(f)$ in **01Sub**.

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