Network Coding Gaps for Completion Time of Multiple Unicasts

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Abstract

Arguably the most common network communication problem is multiple-unicasts: Distinct packets at different nodes in a network need to be delivered to a destination specific to each packet as fast as possible.

The famous multiple-unicast conjecture posits that, for this natural problem, there is no performance gap between routing and network coding, at least in terms of throughput.

We study the same network coding gap but in terms of completion time. While throughput corresponds to the completion time for asymptotically-large transmissions, we look at completion times of multiple unicasts for arbitrary amounts of data. We develop nearly-matching upper and lower bounds. In particular, we prove that the network coding gap for the completion time of $k$ unicasts is at most polylogarithmic in $k$, and there exist instances of $k$ unicasts for which this coding gap is polylogarithmic in $k$.

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1 Introduction

The multiple-unicast problem studied in this paper is the natural mathematical abstraction of what is arguably the most common network communication problem: Distinct packets of different size are at different nodes in a network and each packet needs to be delivered to a specific destination as fast as possible.

All known multiple-unicast solutions employ (fractional) routing, i.e., network nodes potentially subdivide packets and route (sub-)packets to their destination via store and forward operations. Routing paths are typically selected to minimize (i) dilation, i.e., path length, and (ii) link congestion, i.e., the number of (sub-)packets going over the same network edge. Indeed, the $O(\text{congestion} + \text{dilation})$ transmission schedule of Leighton, Maggs, and Rao [27] is, up to constants, a provably-optimal routing protocol for such paths. This also implies that, up to constants, any routing protocol is equivalent to a certain length-restricted multi-commodity flow. For this, one treats each packet as a different physical commodity/liquid which flows through the network, subject to link capacities. More precisely, an LP that captures routing protocols completing in time $\Theta(T)$ assigns each link a capacity equal to the amount of information that can be transmitted over it within time $T$. Furthermore, the flow for any commodity in this LP is required to decompose into paths of length at most $T$ (see Section 2.1 for the \textsc{ConcurrentFlow}_M(T) LP described here). As such, routing protocols for the multiple-unicast problem are quite well understood.

It seems obvious at first that routing is the only way to solve network communication problems like multiple unicasts. Surprisingly, however, results discovered in the 2000s [3] prove that information need not flow through a network like a physical commodity. For example, nodes might not just forward information, but instead send out XORs of received packets. Multiple such XORs or linear combinations can then be recombined at destinations to reconstruct any desired packets. An instructive example is to look at the XOR $C \oplus M$ of two $s$-bit packets $C, M$. While it is also $s$ bits long, one can use it reconstruct either all $s$ bits of $C$ or all $s$ bits from $M$, as long as the other packet is given. Such network coding operations are tremendously useful, particularly for broadcasts, but they do not have a physical equivalent. Indeed the $C \oplus M$ packet would correspond to some $s$ ounces of a magic “café latte” liquid with the property that one can extract either $s$ ounces of milk or $s$ ounces of coffee from it, as long as one has enough of the other liquid already. Over the last two decades thousands of results demonstrating gaps between the power of network coding and routing have been published, and attempts to build a comprehensive theory explaining what is or is not achievable by going beyond routing have given rise to an entire research area called network information theory.

The question asked in this paper is if there exists such a network coding gap for multiple unicasts; i.e., “How much faster than routing can an unrestricted (i.e., network coding) protocol for a given instance of the multiple-unicast problem be?” Given how natural this question is, we are obviously not the first to ask it. Indeed, in 2004 Li and Li [29] and Harvey et al. [18] independently put forward the multiple-unicast conjecture, which posits that the answer to the above questions is “Not one bit!”, at least as far as the throughput of multiple unicasts in undirected networks is concerned. Here throughput of a multiple-unicast instance is defined as $\lim_{t \to \infty} C(t)/t$, where $C(t)$ is the completion time of the fastest protocol for the instance after increasing all packet sizes by a factor of $t$. That is, throughput exactly captures the completion time for the special case of asymptotically large packet sizes.

What makes the throughput setting easier than the general completion times we study in this paper, is that it forces the completion time $T$ to be asymptotically large. Formally, large $T$ make the length restriction of the above described multi-commodity flow LP vacuously true, which means that the routing throughput is perfectly characterized by the standard multi-commodity flow LP. On an intuitive level this stems from the fact that an asymptotically large time horizon makes the number of packets that can be sent along a path of length $L$ become essentially independent of $L$. In particular, one can completely stop worrying about path length and solely maximize flows subject to capacity constraints.
This significantly simplifies things. In particular, since the sparsest cut is an obvious upper bound on the performance of any (network coding) protocol, the multiple-unicast conjecture automatically holds for any network with no multi-commodity max-flow min-cut gap. This includes for example any instance with two unicasts [19] and planar networks in which packet sources and destinations lie on one face [32]. Similarly, the long line of work on flow-cut gaps (e.g., [5, 13, 21, 22, 28, 30]), starting with the celebrated result of Leighton and Rao [28] directly implies the throughput network coding gap for \( k \) unicasts is at most \( O(\log k) \) [5, 30] and lower for special families of graphs [8, 24, 26].

Beyond settings with no flow-cut gap, the multiple-unicast conjecture has been proven for several other restricted settings including the Okamura-Seymour instance [20, 23] and certain bipartite networks [1, 20]. Still, despite attempts by many prominent researchers [4, 17, 20, 25, 29, 36, 37] the conjecture remains open and has established itself as a notoriously hard open problem. In addition to being considered “arguably the most important open problem in the field of network coding” [1], the conjecture has also been connected to other seemingly unrelated areas of theoretical computer science. For example, a positive resolution of this conjecture has been shown to imply (1) an answer to a longstanding open question in external memory algorithm complexity [1, 12], (2) improved lower bounds for computation in the cell-probe model [1], and (3) (very recently) an \( \Omega(n\log n) \) circuit size lower bound for multiplication of \( n \)-bit integers [2] (matching an even more recent breakthrough algorithmic result for this fundamental problem [16]).

In this paper we study the completion-time network coding gap for arbitrary packet sizes (and not just arbitrarily large, which correspond to throughput coding gaps). Whether completion-time coding gaps have as far-reaching implications as throughput coding gaps discussed above is yet to be determined. However, we argue that, while potentially more technically challenging, completion-time coding gaps are arguably at least as natural and important to study in their own right as throughput ones, if not more. In many practical settings packet sizes are not large enough compared to the network size to allow one to ignore the length of paths along which packets are routed, or otherwise communicated.

Before formally describing the multiple-unicast instance problem and its completion time and stating our results, we mention two points regarding this work which we find particularly appealing. The first is that this work has found a connection between this problem and other fields of TCS. Specifically, a crucial ingredient in our proofs, which we refer to as moving cut, was inspired by instances that have been fundamental in allowing to apply communication complexity to distributed computing [9, 10, 33], allowing us to apply the former to network coding, where to the best of our knowledge communication complexity had not been used before. Even more pleasing, while the proofs considered in the distributed computing literature were in some sense tailored to this instance, our proof allows us to obtain optimal (up to polylog terms) bounds, for any instance, underlying our upper bounds on completion-time coding gaps.

We proceed to formally describe the problem studied in this paper before presenting our results.

### 1.1 Preliminaries

We start by formally defining the problem studied in this paper and families of protocols solving it. A multiple-unicast instance \( \mathcal{M} = (G, S) \) is defined over a communication network, represented by an undirected graph \( G = (V, E) \) with capacity \( c_e \in \mathbb{Z}_{\geq 1} \) for each edge \( e \). The \( k \geq |S| \) sessions of \( \mathcal{M} \) are denoted by \( S = \{(s_i, t_i, q_i)\}_{i=1}^k \). Each session consists of source node \( s_i \), which wants to transmit a packet to the sink \( t_i \), consisting of \( q_i \in \mathbb{Z}_{\geq 1} \) sub-packets. Without loss of generality we assume that a uniform sub-packetization is used; i.e., all sub-packets have the same size (think of sub-packets as the underlying data type, e.g., field elements or bits). For brevity, we refer to an instance with \( k \) sessions as a \( k \)-unicast instance.
A protocol for a multiple-unicast instance is conducted over synchronous time steps. Initially each source \( s_i \) knows its packet, consisting of \( q_i \) sub-packets. At any time step, each node \( v \) sends a different packet along each of its edges \( e \), consisting of at most \( c_e \) sub-packets, with each packet being some predetermined function of the time and the sub-packets received by \( v \) by that time, or originating at \( v \). A protocol for multiple-unicast instance has completion time \( T \) if after \( T \) time steps of the protocol each sink \( t_i \) can determine the \( q_i \)-sized packet of its source \( s_i \). Network coding protocols are unrestricted protocols, allowing each node to send out any function of the packets it has received so far. On the other hand, routing protocols are a restricted family of protocols where each node can only send packets comprised of forwarding predetermined sub-packets it has received so far or that originate at this node.

By definition, for any multiple-unicast instance, the fastest routing protocol is no faster than the fastest coding protocol. The completion-time coding gap characterizes how much faster the latter is.

**Definition 1.1.** (Completion-time coding gap) The completion-time coding gap for a multiple-unicast instance \( \mathcal{M} = (G, S) \) is the ratio of the smallest completion time of any routing protocol for \( \mathcal{M} \) and the smallest completion time of any network coding protocol for \( \mathcal{M} \).

We note that the multiple-unicast instance problem can be further generalized, so that each edge has both capacity and delay, corresponding to the amount of time needed to traverse the edge. This more general problem can be captured by replacing each edge \( e \) with a path with unit delays of total length proportional to \( e \)'s delay. As we show, despite path length being crucially important in characterizing completion times for multiple-unicast instances, this transformation does not affect the worst-case coding gaps, which are independent of the network size (including after this transformation). We therefore consider only unit-time delays in this paper, without loss of generality.

### 1.2 Our Contributions

In this work we show that completion-time coding gaps of multiple unicast are vastly different from their throughput counterparts, which are conjectured to be trivial (i.e., equal one). For example, while the throughput coding gap is always one for instances with \( k = 2 \) sessions [19], for completion-time coding gaps it is easy to derive an instance with \( k = 2 \) sessions and coding gap of \( 4/3 \) (based on the butterfly network). Having observed that completion-time coding gap can in fact be non trivial, we proceed to study the potential asymptotic growth of such coding gaps as the network parameters grow.

In this work we show that completion-time coding gap of multiple unicast with \( k \) sessions and packet sizes \( \{q_i\}_{i \in [k]} \) is polylogarithmic in the problem parameters, \( k \) and \( \sum_i q_i / \min_i q_i \), but independent of the network size, \( n \). The positive part of this result is given by the following theorem.

**Theorem 1.2.** The network coding gap for completion-time of any \( k \)-unicast instance is at most

\[
O \left( \log(k) \cdot \log \left( \sum_i q_i / \min_i q_i \right) \right).
\]

Note that for similarly-sized packets, this bound simplifies to \( O(\log^2 k) \). For different-sized packets, our proofs and ideas in [34] imply a coding gap of \( O(\log k \log(nk)) \). We note moreover that our proof of theorem 1.2 is constructive, yielding for any \( k \)-unicast instance \( \mathcal{M} \) a routing protocol which is no more than \( O(\log(k) \cdot \log(\sum_i q_i / \min_i q_i)) \) times slower than the fastest protocol (of any kind) for \( \mathcal{M} \).

On the other hand, we prove that such a polylogarithmic gap as in Theorem 1.2 is inherent to the problem, by providing an infinite family of multiple-unicast instances with unit-sized packets (\( q_i = 1 \) for all \( i \in [k] \)) exhibiting a polylogarithmic coding gap for completion time. (Contrast this with the throughput coding gaps, whose existence is still wide open.)

**Theorem 1.3.** There exists an absolute constant \( c > 0 \) and an infinite family of \( k \)-unicast instances for \( k \) sufficiently large whose completion-time coding gap is at least \( \Omega(\log^c k) \).
1.3 Techniques

Here we outline the challenges faced and key ideas needed to obtain our results.

1.3.1 Completion Time Impossibility Results

As we want to bound the ratio between the best completion time of any routing protocol and any coding protocol, our proofs will need both upper and lower bounds for these best completion times. As it turns out, obtaining upper bounds (possibility results) for these completion times is somewhat easier. The major technical challenge and our main contribution is in deriving lower bounds on the optimal completion time of any given multiple-unicast instance, which we outline in this section. Most notably, we formalize a technique we refer to as the moving cut. Essentially the same technique was used to prove that distributed verification is hard on one particular graph that was designed specifically with this technique in mind [9, 11, 33]. Strikingly, we show that the moving cut technique gives an almost-tight characterization (up to polylogarithmic factors) of the coding completion time for every multiple-unicast instance (i.e., gives universally optimal bounds).

We start by considering several prospective techniques that one can use to prove that no protocol can solve an instance in fewer than \( T \) rounds and build our way up to the moving cut. For any multiple-unicast instance, \( \max_{i \in [k]} d(s_i, t_i) \), the maximum distance between any source-sink pair, clearly lower bounds the coding completion time. However, this lower bound can be arbitrarily bad since it does not take edge congestion into account; for example, if all source-sink paths pass through one common edge. Similarly, any approach that looks at sparsest cuts in a graph is also bound to fail since it does not take the source-sink distances into account.

Attempting to interpolate between both bounds, one can try to extend this idea by noting that a graph that is “close” (in the sense of few deleted edges) to another graph with large source-sink distances must have large completion time for routing protocols. For simplicity, we focus on instances where all capacities and demands are one, i.e., \( c_e = 1 \) for every edge \( e \) and \( q_i = 1 \) for all \( i \), which we refer to as simple instances. The following lemma illustrates such an approach.

**Lemma 1.4.** Let \( M = (G, S) \) be a simple \( k \)-unicast instance. Suppose that after deleting some edges \( F \subseteq E \), any sink is at distance at least \( T \) from its source; i.e., \( \forall i \in [k] \) we have \( d_{G \setminus F}(s_i, t_i) \geq T \). Then any routing protocol for \( M \) has completion time at least \( \min \left\{ T, \frac{k}{|F|} \right\} \).

The lemma is nearly immediate. For any sets of flow paths between all sinks and source, either (1) all source-sink flow paths contain at least one edge from \( F \), incurring a congestion of \( k/|F| \) on at least one of these \( |F| \) edges, or (2) there is a path not containing any edge from \( F \), hence having a hop-length of at least \( T \). The lemma then follows by standard congestion + dilation lower bound arguments.

Perhaps surprisingly, the above bound does not apply to general (i.e., coding) protocols. Consider the instance in Figure 1. There, removing the single edge \( \{S, T\} \) increases the distance between any source-sink pair to 5, implying any routing protocol’s completion time is at least 5 on this instance. However, there exists a network coding protocol with completion time 3: Each source \( s_i \) sends its input to its neighbor \( S \) and all sinks \( t_j \) for \( i \neq j \) along the direct 3-hop path. Node \( S \) computes the XOR of all inputs, passes it to \( T \) who, in turn, passes it to all sinks \( t_j \), allowing each sink \( t_j \) to recover its source \( s_j \)'s packets by canceling all other factors in the XOR.

One can still recover a valid general (i.e., coding) lower bound by an appropriate strengthening of Lemma 1.4: one has to require that all sources be far from all sinks in the edge-deleted graph. This contrast serves as a good mental model for the differences between coding and routing protocols.
Figure 1: A family of instances with $k = 5$ pairs of terminals and completion-time coding gap of $5/3$. Each one of the $k$ sources $s_i$ has a path of 3 hops (in black and bold) connecting it to every sink $t_j$ for all $j \neq i$. Moreover, all sources $s_i$ neighbor some common node $S$, which also neighbors a node $T$, which in turn neighbors all sinks $t_j$.

**Lemma 1.5.** Let $\mathcal{M} = (G,S)$ be a simple $k$-unicast instance. Suppose that after deleting some edges $F \subseteq E$, any sink is at distance at least $T$ from any source; i.e., $\forall i,j \in [k]$ we have $d_{G \setminus F}(s_i,t_j) \geq T$. Then any (network coding) protocol for $\mathcal{M}$ has completion time at least $\min\{ T, k |F| / \ell \}$.

This lemma is also not hard to prove: we can assume all sources can share information among themselves for free (e.g., via a common controlling entity) since this makes the multiple-unicast instance strictly easier to solve; similarly, suppose that the sinks can also share information. Suppose that some coding protocol has completion time $T' < T$. Then all information shared between the sources and the sinks has to pass through some edge in $F$ at some point during the protocol. However, these edges can pass a total of $|F| \cdot T'$ packets of information, which has to be sufficient for the total of $k$ source packets. Therefore, $|F| \cdot T' \geq k$, which can be rewritten as $T' \geq k / |F|$.

Unfortunately, the bound of Lemma 1.5 is not always tight and it is instructive to understand when this happens. One key example known to us is the previously-mentioned instance studied in the influential distributed computing papers [9, 11, 33], where congestion and dilation seem to play key roles. We describe the instance in Figure 2. For these instance, the (coding) completion-time lower bound of Lemma 1.5 can be polynomially far from the optimal coding protocol’s completion time. Informally, this network was constructed precisely to give a $\tilde{\Omega}(\sqrt{n})$ completion time lower bound (which leads to the pervasive $\tilde{\Omega}(\sqrt{n} + D)$ lower bound which applies to many global problems in distributed computing [9]). The intuitive way to explain the $\tilde{\Omega}(\sqrt{n})$ lower bound is to say that one either has to communicate along a path of length $\sqrt{n}$ or all information needs to shortcut significant distance over the tree, which forces all information to pass through near the top of the tree, implying congestion of $\tilde{\Omega}(\sqrt{n})$. Lemma 1.5, however, can at best certify a lower bound of $\tilde{\Omega}(n^{1/4})$.

A more sophisticated argument is needed to certify the $\tilde{\Omega}(\sqrt{n})$ lower bound for this specific instance. Aforementioned papers [9, 11, 33] prove their results by implicitly using the technique we formalize as our moving cut in the following lemma (and proven in Section 2.2) on the instance of Figure 2.

**Lemma 1.6** (Moving cut). Let $\mathcal{M} = (G,S)$ be a $k$-unicast instance. Suppose that after increasing each edge $e$’s length from one to $\ell_e \in \mathbb{Z}_{\geq 1}$, we have

1. the total length increase, $\sum_{e \in E} c_e(\ell_e - 1)$ is less than $\sum_{i=1}^k q_i$, and
2. any sink is at distance at least $T$ from any source; i.e., $\forall i,j \in [k]$ we have $d_{\ell}(s_i,t_j) \geq T$.

Then any (coding) protocol for $\mathcal{M}$ has completion time at least $T$. 
It turns out that the statement of the certificate can be seen as a natural generalization of Lemma 1.5, which can be equivalently restated in the following way: “Suppose that after increasing each edge $e$’s length from one to $\ell_e \in \{1, T + 1\}$, we have that (1) $\sum_{e \in E} c_e (\ell_e - 1) < \sum_{i=1}^{k} q_i$, and (2) $d_{\ell}(s_i, t_j) \geq T$. Then any (coding) protocol $M$ has completion time at least $T$”. Dropping the restriction on $\ell_e$ recovers Lemma 1.6.

Strikingly, moving cut allows us not only to give tight certificates for the instance of Figure 2—it allows us to get almost-tight bounds for each multiple-unicast instance (up to polylogarithmic factors).

1.3.2 Upper Bounding the Coding Gap

Upper bounding the completion-time coding gap of multiple-unicast instance $M$ requires a routing protocol with completion time some $\tilde{O}(T)$ on $M$ and a certificate that all coding protocols have completion time at least $\tilde{\Omega}(T)$ on $M$.\footnote{We use $\tilde{O}$ and $\tilde{\Omega}$ to suppress $\log(k) \cdot \log(\sum_i q_i / \min_i q_i)$ terms.} In order to relate these quantities we start with a natural hop-bounded multicommodity flow LP relaxation, described in the introduction. Any routing protocol with completion time $T$ induces a “high-valued” solution for this LP. Moreover, when this LP has a “high” objective value of $\tilde{\Omega}(1)$, we can use standard LP rounding and network routing ideas to construct a routing protocol with completion time $\tilde{O}(T)$. The challenging part is proving that a low optimal LP value (implying that no routing protocol solves $M$ in $\tilde{O}(T)$) implies that no coding protocol can solve the instance in $\tilde{O}(T)$ time. To that end, we show that a low-valued solution to this LP’s dual implies a lower bound of $\tilde{\Omega}(T)$ for the best completion time of any protocol for $M$. In particular, we obtain this lower bound by converting this low-valued feasible dual solution into a moving cut.

1.3.3 Lower Bounding the Coding Gap

To complement our polylogarithmic upper bound on the completion time gap, we construct a multiple-unicast instance $M$ that exhibits a polylogarithmic completion-time coding gap. We achieve this by amplifying the gap via graph products, a powerful technique that was also used in prior work to construct extremal network coding examples [6, 7, 31].
We use a graph product that is essentially the same as the one from Braverman et al. [7] (with minor modifications). This product takes instances $I_1, I_2$ and intuitively replaces each edge of $I_1$ with a source-sink pair of a different copy of $I_2$. More precisely, multiple copies of $I_1$ and $I_2$ are created and interconnected. Edges of a copy of $I_1$ are replaced by the same session of different copies of $I_2$; similarly, sessions of a copy of $I_2$ replace the same edge in different copies of $I_1$. This regular structure allows for coding protocols in $I_1$ and $I_2$ to compose in a straightforward way to form a fast coding protocol in the product instance, giving us an easy way to upper bound coding completion times. To facilitate the routing lower bound, copies of instances are interconnected along a high-girth bipartite graph to prevent unexpectedly short paths from forming after the interconnection.

The power of our construction stems from the way we give a routing completion-time lower bound. However, before elaborating on this, it is instructive to review the approach of Braverman et al. [7]. They show that, given a multiple-unicast instance $I$ with a throughput coding gap of $1 + \varepsilon$, one can iteratively apply the aforementioned graph product with itself to boost the coding gap to $O(\log^c n)$ for some $c > 0$. While their coding protocols nicely compose, giving a coding possibility result, for the routing impossibility result they use a dual of the multicommodity flow LP (analogously to our CONCURRENTFLOW$_M(T)$, but without any hop restriction), since any feasible dual solution certifies a limit on the routing performance. Strikingly, in the throughput setting, a direct tensoring of dual LP solutions of $I_1$ and $I_2$ gives a satisfactory dual solution of the product instance. In more detail, a dual LP solution in $I$ assigns a positive length $\ell_I(e)$ to each edge in $I$; each edge of the product instance corresponds to two edges $e_1 \in I_1$ and $e_2 \in I_2$, and the direct tensoring $\ell_P((e_1, e_2)) = \ell_{I_1}(e_1) \cdot \ell_{I_2}(e_2)$ provides a feasible dual solution with an adequate objective value. To avoid creating edges in the product distance of zero $\ell_*$-length, they contract edges assigned length zero in the dual LP of either instance. Unfortunately for us, such contraction is out of the question when studying completion time gaps, as such contractions would shorten the hop length of paths, possibly creating short paths which do not correspond to short paths in the original instance.

Worse yet, any approach that uses the dual of our $T$-hop-bounded LP CONCURRENTFLOW$_M(T)$ is bound to fail in the completion-time setting. Say we are given two instances $I_1, I_2$, both of which have routing completion time at least $T$ and expect that the product instance $I_+$ to have routing completion time at least $T^2$ by some construction of a feasible dual LP solution. Such a claim cannot be directly argued since a source-sink path in the product instance that traverses, say, $T - 1$ different copies of $I_2$ along a path of hop-length $T + 1$ could carry an arbitrary large capacity unrelated to any routing completion-time bound from the preconditions, since the hop-bounded LP solution on $I_2$ only takes short paths into account. In particular, paths of hop-length at most $T$. Since there is no direct way to compose the dual LP solutions, we are forced to use a different style of analysis from the one of [7], which in turn forces our construction to become considerably more complicated.

Our approach to bounding the routing performance in the product instance is via Lemma 1.4: We keep a list of edges $F$ along with each instance and ensure (i) that all source-sink distances in the $F$-deleted instance are large and (ii) that the ratio of the number of sessions $k$ to $|F|$ is large. We achieve property (i) by interconnecting along a high-girth graph and treating the replacement of edges in $F$ in a special way. Property (ii) is ensured by making the inner instance $I_2$ significantly larger than the outer instance $I_1$, which forces the product instance to create extra copies of $I_1$ and increases the final number of sessions.

The main challenge in our approach becomes controlling the size of the product instance. To achieve this, we affix to each instance a relatively complicated set of parameters (e.g., coding time, number of edges, number of sessions, etc.) and study how these parameters change upon applying the graph product. Choosing the right set of parameters is key – they allow us to properly quantify the size escalation. The instances are then asymmetrically combined (in the sense that $I_1 \neq I_2$) in a way that the coding gap grows doubly-exponentially and the size of the graph grows triply-exponentially, yielding the desired polylogarithmic coding gap.
2 Upper Bounding the Coding Gap

In this section we bound the coding gap by \( O\left(\log(k) \cdot \log\left(\frac{\sum_{i=1}^{k} q_i}{\min_{i=1}^{k} q_i}\right)\right) \). Given a multiple-unicast instance \( \mathcal{M} \) we thus want to upper bound its routing completion time and lower bound its coding completion time. To characterize these quantities we start with a natural hop-bounded multicommodity flow LP relaxation \textsc{ConcurrentFlow}_{\mathcal{M}}(T)\) as a proxy to test whether a network has a routing protocol of completion time at most \( T \). The LP, introduced in Section 2.1, requires sending a flow of magnitude \( q_i \) between each source-sink pair \( (s_i, t_i) \), with the additional constraints that (1) the combined congestion of any edge \( e \) is at most \( T \cdot c_e \) where \( c_e \) is the capacity of the edge (as only \( c_e \) packets can use this edge during any of the \( T \) time steps of a routing protocol), and (2) the flow is composed of only short paths, of at most \( T \) hops. When the \textsc{ConcurrentFlow}_{\mathcal{M}}(T)\) has a “high” objective value of \( \Omega(1) \), we can use standard LP rounding and network routing ideas to construct a routing protocol with completion time \( \tilde{O}(T) \).

The challenging part is proving that a low optimal LP value implies that no (coding) protocol can solve the instance in \( \tilde{O}(T) \) time. More thoroughly, we take the dual LP \textsc{Cut}_{\mathcal{M}}(T)\) and convert a feasible solution with a low objective value into a information-theoretic certificate of impossibility. Section 2.2 introduces the framework to prove such an impossibility result and finally Section 2.3 explains the transformation of a low-value dual LP solution to a moving cut.

2.1 Hop-Bounded Concurrent Flow LP

In this section we introduce the hop-bounded concurrent flow LP, \textsc{ConcurrentFlow}_{\mathcal{M}}(T), which is an LP relaxation of the problem of routing a multiple-unicast instance \( \mathcal{M} = (G, \mathcal{S}) \) in \( T \) rounds. We discuss the dual of this LP, \textsc{Cut}_{\mathcal{M}}(T), in Section 2. For each \( i \in [k] \), we denote by \( \mathcal{P}_i(T) \triangleq \{ p : s_i \xrightarrow{} t_i \mid |p| \leq T, p \text{ is simple}\} \) the set of simple paths of hop-length at most \( T \) connecting \( s_i \) and \( t_i \) in \( G \).

![Figure 3: The concurrent flow LP relaxation and its dual.](image)

A routing protocol solving \( \mathcal{M} \) in \( T \) rounds almost by definition yields a solution to \textsc{ConcurrentFlow}_{\mathcal{M}}(T)\) of value \( z \geq 1 \). One can formally see this by tracing each sub-packet from the input as it gets forwarded throughout the network. Note that being able to perform such a sub-packet trace is essentially equivalent to the routing restriction. Such a sub-packet originating from \( s_i \) furnishes a \( s_i \xrightarrow{} t_i \) path \( p \) of at most \( T \) hops and of flow value \( f_i(p) = 1 \). Combining all \( q_i \) such sub-packets yields \( \sum_{p \in \mathcal{P}_i(T)} f_i(p) \geq 1 \cdot q_i \). The congestion bound \( \sum_{i \in [k]} \sum_{p \in \mathcal{P}_i(T)} f_i(p) \leq T \cdot c_e \) also follows naturally from the per-round edge capacity.

In this section we observe that the “converse” is also true; i.e., given a feasible solution to \textsc{ConcurrentFlow}_{\mathcal{M}}(T)\) of value at least \( \Omega(1) \) implies a routing protocol for \( \mathcal{M} \) in time \( O(T) \). The issue is that the feasible LP solution provides fractional paths, hence requiring us to round (integralize) the LP solution. To this end, we use the following theorem of Srinivasan and Teo [35].

Lemma 2.1 ([35], Theorem 2.4, paraphrased). Let \( \mathcal{M} \) be a multiple-unicast instance and for each \( i \in [k] \) let \( \mathcal{D}_i \) be a distribution over \( s_i \xrightarrow{} t_i \) paths of hop-length at most \( L \). Suppose that the product
distribution $\prod D_i$ has expected congestion for each edge at most $L$. Then there exists a sample $\omega \in \prod D_i$ (i.e., a choice of $a$ from $D_i$ between each $s_i \rightarrow t_i$) with (maximum) congestion $O(L)$.

Using the above lemma to round the LP and appealing to the celebrated $O(\text{congestion} + \text{dilation})$ routing theorem of Leighton et al. [27], we directly get the following result.

**Lemma 2.2.** Let $z \leq 1$, $\{f_i(p) \mid i \in [k], p \in P_i(T)\}$ be a feasible solution to the $\text{CONCURRENTFLOW}_M(T)$ LP. Then there exists an integral routing protocol with completion time $O(T/z)$.

**Proof.** Consider an optimal solution to this $\text{CONCURRENTFLOW}_M(T)$. Clearly, picking for each pair $(s_i, t_i)$ some $q_i$ paths in $P_i(T)$ with each $p \in P_i(T)$ picked with probability $f_i(p)/(q_i + \sum_{p \in P_i(T)} f_i(p)) \leq f_i(p)/z$ yields an expected congestion at most $T \cdot c_e/z$ for each edge $e$. That is, thinking of $G$ as a multigraph with $c_e$ copies per edge, each such parallel edge has congestion $T/z$. On the other hand, each such path has length at most $T \leq T/z$ (since $z \leq 1$). Therefore, by Lemma 2.1, there exist choices of paths for each pair of (maximum) congestion and hop-bound (i.e., dilation) at most $O(T/z)$.

But then, using $O(\text{congestion} + \text{dilation})$ routing [27] this implies an integral routing protocol with completion time $O(T/z)$, as claimed.$\square$

### 2.2 Moving Cut

In this section we present our completion-time lower bound certificates, which we term the **moving cut**. Informally, given an underlying multiple-unicast instance, a moving cut is a sequence of nested cuts (represented as vertex sets) each containing all of the sources, i.e., $\{s_i\}_{i \in [k]} \subseteq B_0 \subseteq B_1 \subseteq B_2 \subseteq \ldots$, and where none of the first $T$ sets contain any of the sinks, i.e., $B_t \cap \{t_i\}_{i = 1} = \emptyset$. A good mental model for the moving cut is to consider a “wave” propagating in time from the sinks to the sources at a pace of one cut per time step (i.e., the wave is at $B_t$ at time $t$).

In order to prove that all protocols must use more than $T$ rounds to complete the unicast, we measure the amount of input information (i.e., originating from $\{s_i\}_{i}$) that can be “ahead” of $B_t$ at time $t$; i.e., the amount of information available to $V \setminus B_t$. At $t = 0$ this amount is clearly 0, and we are clearly done if we can prove that at $t = T$ that amount does not contain all input information. Formalizing this idea allows us to prove the following lemma.

**Lemma 1.6** (**Moving cut**). Let $M = (G, S)$ be a $k$-unicast instance. Suppose that after increasing each edge $e$’s length from one to $\ell_e \in \mathbb{Z}_{\geq 1}$, we have

1. the total length increase, $\sum_{e \in E} c_e (\ell_e - 1)$ is less than $\sum_{i=1}^k q_i$, and
2. any sink is at distance at least $T$ from any source; i.e., $\forall i, j \in [k]$ we have $d_e(s_i, t_j) \geq T$.

Then any (coding) protocol for $M$ has completion time at least $T$.

**Proof.** We will show via simulation that a protocol solving $M$ in at most $T - 1$ rounds would be able to compress $\sum_{i=1}^k q_i$ random bits to a strictly smaller number of bits, thereby leading to a contradiction. Our simulation proceeds as follows. We have two players, Alice and Bob, who control different subsets of nodes. In particular, if we denote by $A_r \triangleq \{v \in V \mid \min_i d_e(s_i, v) \leq r\}$ the set of nodes at distance at most $r$ from any source, then during any round $r \in \{0, 1, \ldots, T - 1\}$ all nodes in $A_r$ are “spectated” by Alice. By spectated we mean that Alice gets to see all of these nodes’ private inputs and received transmissions during the first $r$ rounds. Similarly, Bob, at time $r$, spectates $B_r \triangleq V \setminus A_r$. Consequently, if at round $r$ a node $u \in A_r$ spectated by Alice sends a packet to a node $v \in V$, then Bob will see the contents of that packet if and only if Bob spectates the node $v$ at round $r + 1$. That is, this happens only if $u \in A_r$ and $v \in B_{r+1} = V \setminus A_{r+1}$. Put otherwise, Bob can receive a packet from Alice along edge $e$ during times $r \in [\min_i d_e(s_i, u), \min_i d_e(s_i, v) - 1]$. Therefore, the number of rounds transfer can happen along edge $e$ is at most $\min_i d_e(s_i, v) - \min_i d_e(s_i, u) - 1 \leq \ell_e - 1$. Hence, the
maximum number of bits transferred from Alice to Bob via $e$ is $c_e(\ell_e - 1)$. Summing up over all edges, we see that the maximum number of bits Bob can ever receive during the simulation is at most $\sum_{e \in E} c_e(\ell_e - 1) < \sum_{i=1}^k q_i$. Now, suppose Alice has some $\sum_{i=1}^k q_i$ random bits. By simulating this protocol with each source $s_i$ having (a different) $q_i$ of these bits, we find that if all sinks receive their packet in $T$ rounds, then Bob (who spectates all $t_j$ at time $T - 1$, as $min, d_\ell(s_i, t_j) \geq T$ for all $j$) learns all $\sum_{i=1}^k q_i$ random bits while receiving less than $\sum_{i=1}^k q_i$ bits from Alice – a contradiction.

2.3 From Dual Solution to Moving Cut

In the previous section we showed that high objective value for the primal LP implies an upper bound on the routing time for the given instance. In this section we show that low objective value of the primal LP — implying a feasible dual LP solution of low value — yields a moving cut for some sub-instance, allowing us to relate the fastest routing protocol to the fastest coding protocol.

By definition, a low-value feasible solution to the dual LP $\text{CUT}_M(T)$ assigns non-negative lengths $\ell : E \rightarrow \mathbb{R}_{\geq 0}$ such that (1) the $c$-weighted sum of $\ell$-lengths is small, i.e., $\sum_{e \in E} c_e \ell_e = \tilde{O}(1/T)$, as well as (2) if $\ell_i$ is the $\ell$-length of the $T$-hop-bounded $\ell$-shortest path between $s_i$ and $t_i$, then $\sum_{i \in [k]} q_i \cdot h_i \geq 1$.

Our coding lower bound of Lemma 1.6 needs a common lower bound on all $d(s_i, t_j)$ and not a lower bound on $\sum_{i} q_i \cdot d(s_i, t_i)$. In particular, to present a moving cut for an instance induced by some subset of sessions with indices $I \subseteq [k]$ we need to lower bound $d(s_i, t_i)$ for all $i \in I$. Lemma 2.3, proven in Appendix A using a “continuing” bucketing argument, does just this.

**Lemma 2.3.** Given sequences $h_1, \ldots, h_k, q_1, \ldots, q_k \in \mathbb{R}_{\geq 0}$ with $\sum_{i=1}^k q_i \cdot h_i \geq 1$ there exists a non-empty subset $I \subseteq [k]$ with $\min_{i \in I} h_i \geq \frac{1}{\alpha_{\text{gap}}} \frac{1}{\sum_{i=1}^k q_i}$ for $\alpha_{\text{gap}} = O\left(\log \frac{\sum_{i=1}^k q_i}{\min_{i \in I} q_i}\right)$.

Next, to get even closer to a moving cut as defined in Lemma 2.4, we appropriately scale $\ell$ by defining $\tilde{\ell} = \ell + [\ell \cdot T \cdot \sum_i q_i] \in \mathbb{Z}_{\geq 1}$. The condition $h_i \geq \tilde{\Omega}(1/k)$ now converts to $d_\tilde{\ell}(s_i, t_i) \geq \tilde{\Omega}(T)$ with the sum-bound $\sum_{e \in E} c_e(\tilde{\ell}_e - 1) < \tilde{O}(\sum_i q_i)$. Note that $h_i$ refers to $T$-hop-bounded distances w.r.t. $\ell$ and that $d_\tilde{\ell}$ refers to unbounded-hop distances w.r.t. $\ell$.

**Lemma 2.4.** Let $M = (G, S)$ be a $k$-unicast instance with feasible solution $\{h_i, \ell_e \mid i \in [k], e \in E\}$ to $\text{CUT}_M(T)$ with objective value $T \sum_{e \in E} c_e \ell_e = z$. Then there exist indices $I \subseteq [k]$ and integral edge lengths $\ell : E \rightarrow \mathbb{Z}_{\geq 1}$ that satisfy both $\sum_{e \in E} c_e (\tilde{\ell}_e - 1) \leq z \cdot \alpha_{\text{gap}} \cdot \sum_{i \in I} q_i$ with $\alpha_{\text{gap}} = O\left(\log \frac{\sum_{i=1}^k q_i}{\min_{i \in I} q_i}\right)$ as in Lemma 2.3 and $d_\tilde{\ell}(s_i, t_i) \geq T$ for all $i \in I$.

**Proof.** Let $I \subseteq [k]$ be a subset of indices as guaranteed by Lemma 2.3. Define $\tilde{\ell}_e = \ell_e + [\ell_e \cdot \alpha_{\text{gap}} \cdot T \cdot \sum_{i \in I} q_i]$ for all $e \in E$ and note that $\tilde{\ell}_e \in \mathbb{Z}_{\geq 1}$. By definition of $\tilde{\ell}$ and $T \sum_{e} c_e \ell_e = z$, we get the first condition,

$$\sum_{e \in E} c_e (\tilde{\ell}_e - 1) \leq \sum_{e} c_e \ell_e \cdot \alpha_{\text{gap}} \cdot T \cdot \sum_{i \in I} q_i = z \cdot \alpha_{\text{gap}} \cdot \sum_{i \in I} q_i.$$

We now show that $d_\tilde{\ell}(s_i, t_i) > T$ for $i \in I$. Consider any simple path $p$ between $s_i \rightarrow t_i$. Denote by $\tilde{\ell}(p)$ and $\ell(p)$ the length with respect to $\tilde{\ell}$ and $\ell$, respectively. It is sufficient to show that $\tilde{\ell}(p) > T$. If $p \notin P_i(T)$, i.e., the hop-length of $p$ (denoted by $|p|$) is more than $T$. Then $\tilde{\ell}(p) \geq |p| > T$ since $\tilde{\ell}_e \geq 1 \forall e \in E$. On the other hand, if $p \in P_i(T)$, then by our choice of $I$ as in Lemma 2.3 and the definition of $h_i$, we have that $\ell(p) \geq h_i \geq \frac{1}{\alpha_{\text{gap}}} \frac{1}{\sum_{i \in I} q_i}$, hence

$$\tilde{\ell}(p) \geq \ell(p) \cdot \alpha_{\text{gap}} \cdot T \cdot \sum_{i \in I} q_i = \frac{1}{\alpha_{\text{gap}}} \frac{1}{\sum_{i \in I} q_i} \cdot \alpha_{\text{gap}} \cdot T \cdot \sum_{i \in I} q_i = T.$$

\qed
Finally, to obtain a moving cut we need to move from a bound on \( d_t(s_i, t_i) \) for all \( i \in I \) to a bound on \( d_t(s_i, t_j) \) for all \( i, j \in I' \subseteq I \). To obtain lower bounds of the latter kind, we use the following metric decomposition lemma (whose proof is deferred to Section 2.4).

Lemma 2.5. Let \((X, d)\) be a metric space and let \( T > 0 \). Given \( k \) pairs \( \{(s_i, t_i)\}_{i \in [k]} \) of points in \( X \) with at most \( n \) distinct points in \( \bigcup_{i \in [k]} \{s_i, t_i\} \) and pairwise distances at least \( d(s_i, t_i) \geq T \), there exists a subset of indices \( I \subseteq [k] \) of size \( |I| \geq \frac{k}{8} \) such that \( d(s_i, t_j) \geq \frac{T}{O(\log k)} \) for all \( i, j \in I \).

Lemmas 2.3, 2.4 and 2.5 now allow us to construct, given feasible dual solution of low value, a moving cut for an instance induced by a subset of the sessions. This implies a lower bound on the completion time of the induced instance – and therefore of the entire instance. Combined with the upper bound on routing time if the primal (and therefore the dual) LP has high value, we obtain our positive result — a polylogarithmic upper bound on the network coding gap for completion time.

Theorem 1.2. The network coding gap for completion-time of any \( k \)-unicast instance is at most

\[
O \left( \log(k) \cdot \log \left( \sum_{i} q_i / \min_i q_i \right) \right).
\]

Proof. For notational simplicity, let \( \alpha_{\text{gap}} = O \left( \log \left( \sum_{i} \frac{q_i}{\min_i q_i} \right) \right) \) be as in Lemma 2.4. Let \( T \) be an integer.

We will show that if \( \text{CONCURRENTFLOW}_M(T) \) has optimal value less than \( z \leq 1/8\alpha_{\text{gap}} \), then the optimal completion time of \( M \) is at least \( T^* \geq T/O(\log k) \). This implies that for \( T = O(T^* \cdot \log k) \), the LP \( \text{CONCURRENTFLOW}_M(T) \) has objective value at least \( 1/8\alpha_{\text{gap}} \), which by Lemma 2.2 implies that the fastest routing protocol has completion time

\[
T_R \leq O(T \cdot \alpha_{\text{gap}}) \leq O(T^* \cdot \log k \cdot \alpha_{\text{gap}}) = O \left( T^* \cdot \log k \cdot \log \frac{\sum_{i=1}^{k} q_i}{\min_{i=1}^{k} q_i} \right),
\]

which implies our desired bound. It remains to show that \( \text{CONCURRENTFLOW}_M(T) \) having value less than \( 1/10\alpha_{\text{gap}} \) implies \( T^* \geq T/O(\log k) \).

By Lemma 2.4, if \( \text{CONCURRENTFLOW}_M(T) \) has value less than \( z = 1/9\alpha_{\text{gap}} < 1/8\alpha_{\text{gap}} \), then there exist some integral edge lengths \( \ell : E \rightarrow \mathbb{Z}_{\geq 1} \) and a subset of indices \( I \subseteq [k] \) satisfying \( \sum_{e \in E} \cdot (\ell_e - 1) < \sum_{i \in I} q_i / 8 \) and \( d_t(s_i, t_i) \geq T \) for all \( i \in I \). But then, appealing to Lemma 2.5 with the graphic metric defined by \( \ell \) and each pair \( (s_i, t_i) \) repeated \( q_i \) times, this implies that there exists a multiset of indices \( I' \) in \([k]\) such that \( d_t(s_i, t_j) \geq T/O(\log k) \) for all \( i, j \in I' \) and such that \( |I'| \geq \sum_i q_i / 8 \). Therefore, taking each pair \( (s_i, t_i) \) indexed by \( I' \) at least once, we find a subset of sessions \( I'' \subseteq [k] \) such that \( \sum_{i \in I''} q_i \geq \sum_{i \in [k]} q_i / 8 \) and \( d_t(s_i, t_j) \geq T/O(\log k) \) for all \( i, j \in I'' \). But then, by Lemma 1.6, we find that the sub-instance induced by these sessions has completion time at least \( T/O(\log k) \), and therefore so does \( M \). The theorem follows.

\( \square \)

2.4 From Pairwise to All-Pairs Distances

It remains to prove Lemma 2.5. Its proof can be made fairly simple if one relies on padded decompositions [15], which partition the metric space into parts of diameter at most \( \Delta \) while keeping each ball of radius \( \Delta/O(\log k) \) in the same part with constant probability. Our proof partitions the metric space into parts of diameter at most \( \Delta \Delta T \), separating all \( (s_i, t_i) \) pairs, and then randomly colors the parts into 2 colors. We only keep \( s_i \)'s with the first color and \( t_j \)'s with the second color. This in expectation keeps a constant number of terminals, while keeping them at least \( \Delta/O(\log k) \) apart.

We start by introducing some section-specific notation. Let \((X, d)\) be a metric space. A partition of \( X \) is a collection \( \{X_1, \ldots, X_t\} \) where \( X_i \subseteq X \), \( \bigcup_{i=1}^{t} X_i = X \), and \( X_i \cap X_j = \emptyset, i \neq j \). We refer to
where each edge is labeled by $I$ of the coding gaps of $P$. Let $B(x, \rho) = \{y \in X \mid d(x, y) \leq \rho\}$ denote the ball of radius $\rho \geq 0$ around $x \in X$ and let the diameter of a set of points $U \subseteq X$ be denoted by $\text{diam}(U) \triangleq \max_{x,y \in U} d(x, y)$.

**Definition 2.6.** Let $(X, d)$ be a metric space on $k$ points. We say that a distribution $\mathcal{P}$ over partitions $P = (X_1, \ldots, X_t)$ of $X$ (it can differ between different partitions) is a $\Delta$-padded decomposition if

1. $\text{diam}(X_i) \leq \Delta$ for all $i \in [t]$ with probability 1, and
2. for every $x \in X$ and $\rho \geq 0$ we have that $\Pr_{\mathcal{P}}[B(x, \rho) \subset P] \leq \frac{\rho}{\Delta} C_{\text{pad}} \ln k$, for constant $C_{\text{pad}}$.

In words, each part of the partition has diameter at most $\Delta$ and the probability of a any point $x$ in the metric being at distance less than $\rho$ from a different part than its own part is at most $\frac{\rho}{\Delta} C_{\text{pad}} \ln k$.

Such decompositions were presented, for example, by Gupta et al. [15].

**Lemma 2.7 ([15]).** Any metric $(X, d)$ admits a $\Delta$-padded decomposition, for any $\Delta > 0$.

We are now ready to prove Lemma 2.5.

**Proof of Lemma 2.5.** First note that we can focus on the metric space induced by the $k$ distinct points. Let $\mathcal{P}$ be a distribution over $\Delta$-padded decompositions $P = \{X_1, X_2, \ldots, X_t\}$ with $\Delta \leq T - 1$. We first note that for all $i \in [k]$, $s_i$ and $t_i$ are contained in different parts since the diameter of each part $X_i$ is at most $\Delta = T$ and $d(s_i, t_i) \geq T$. Furthermore, letting $\beta_{\text{gap}} \triangleq 2C_{\text{pad}} \ln 2k$ and $\rho \triangleq \frac{1}{\beta_{\text{gap}}}$ we have that $\Pr[B(s_i, \rho) \subset P] \geq \frac{1}{2}$. Let $I' \subseteq [k]$ be the subset of indices $i$ with $B(s_i, \rho) \subseteq P$. Then we have $\Pr[i \in I'] \geq \frac{1}{2}$ for all $i \in [k]$.

Flip a fair and independent head/tails coin for each part in $P$. Let $U \subseteq X$ be the set of points in parts for which the associated coin came out heads, and $V \subseteq X$ be the analogous set for tails. Then for each $i \in I'$ we have that $\Pr[s_i \in U \text{ and } t_i \in V] = \frac{1}{4}$. Let $I \subseteq I'$ be the subset of indices $i$ with $s_i \in U$ and $t_i \in V$, giving $\Pr[i \in I] \geq \Pr[i \in I'] \Pr[i \in I \mid i \in I'] = \frac{1}{2} \cdot \frac{1}{4} = \frac{1}{8}$ for all $i \in I$. We also have that $d(s_i, t_j) > \rho = \frac{T-1}{\beta_{\text{gap}}}$ for all $i, j \in I$, since $B(s_i, \rho) \subseteq U$ for all $i \in I \subseteq I'$ and $\{t_j\}_{j \in I \cap U} = \emptyset$. Therefore, this random process yields a subset of indices $I \subset [k]$ such that $d(s_i, t_j) > \frac{T-1}{\beta_{\text{gap}}}$, of expected size at least $\mathbb{E}[|I|] = \mathbb{E}[\sum_{i \in k} \Pr[i \in I]] \geq \frac{k}{8}$. As some realization of the randomness yields a set $I$ of size at least $\mathbb{E}[|I|]$, the lemma follows.

## 3 Polylogarithmic Coding Gaps

In this section we construct a family of multiple-unicast instances with polylogarithmic completion-time coding gap. More precisely, we construct instances where the coding gap is at least $(5/3)^2$ and the size (both the number of edges and sessions) is bounded by $2^{O(t^2)}$. Here we give a bird’s eye view of the construction and leave the details to subsequent subsections. We clarify that all big-O bounds like $f = O(g)$ mean there exists a universal constant $c > 0$ s.t. $f \leq c \cdot g$ for all admissible values (in particular, there is no assumption on $f$ or $g$ being large enough).

We use the graph product of [7] as our main tool. Given two multiple-unicast instances $I_1, I_2$ (called the outer and inner instance, resp.) we create a new instance $I_+$ where the coding gap is the product of the coding gaps of $I_1$ and $I_2$. The product is guided by a colored bipartite graph $B = (V_1, V_2, E)$ where each edge is labeled by $(\chi_1, \chi_2) = (\text{edge in } I_1, \text{session in } I_2)$. Precisely, we create $|V_1|$ copies of $I_1$, $|V_2|$ copies of $I_2$ and for each edge $(a, b) \in E(B)$ with label $(\chi_1, \chi_2)$ we replace the edge $\chi_1$ in the $a^{th}$ copy of $I_1$ with session $\chi_2$ in the $b^{th}$ copy of $I_2$.

To prove a lower bound on the coding gap, one needs to upper bound the coding completion time and lower bound the coding completion time. The former is easy: the coding protocols nicely compose. The latter, however, is more involved. Our main tool is Lemma 1.4, which necessitates (i) keeping
track of cut edges $F$ along each instance $I$ such that all source-sink pairs of $I$ are well-separated after edges in $F$ are deleted, and (ii) keeping the ratio $r \triangleq \frac{k}{|F|}$, number of sessions to cut edges, high. We must ensure that the properties are conserved in the product instance $I_\star$. For (i), i.e., to disallow any short paths from forming as an unexpected consequence of the graph product, we choose $B$ to have high girth. Also, we replace edges $F$ in the outer instances with paths rather than connecting them to a session in the inner instance. Issue (ii) is somewhat more algebraically involved but boils down to ensuring that the ratio of sessions to cut edges (i.e., $r$) in the inner instance is comparable to the size (i.e., number of edges) of the outer instance itself. Note that makes the size of the outer instance $I_1$ insignificant when compared to the size of the inner instance $I_2$.

We recursively define a family of instances by parametrizing them with a “level” $i \geq 0$ and a lower bound on the aforementioned ratio $r$, denoting them by $I(i, r)$. We start for $i = 0$ with the $5/3$ instance of Figure 1 where we can control the ratio the aforementioned ratio $r$ by changing the number of sessions $k$ (at the expense of increasing the size). Subsequently, an instance on level $i$ is defined as a product two of level $i - 1$ instances with appropriately chosen parameters $r$. One can show that the coding completion time for a level $i$ instance is at most $2^i$ and routing completion time is at least $3^i$, hence giving a coding gap of $(5/3)^i$. Furthermore, we show that the size of $I(i, r)$ is upper bounded by $r^{2^{O(i^2)}}$, giving us the full result.

Finally, we note an important optimization to our construction and specify in more detail how $I(i, r)$ is defined. Specifically, it is defined as the product of $I_1 \triangleq I(i - 1, 3r)$ being the outer instance and $I_2 \triangleq I(i - 1, m_1/f_1)$ being the inner instance, where $m_1$ and $f_1$ are the number of total and cut edges of $I_1$. This necessitates the introduction and tracking of another parameter $u \triangleq m/f$ to guide the construction. We remark that this might be necessary since if one uses a looser construction of $I_2 \triangleq I(i - 1, m_1)$ the end result $I(i, r)$ would be of size $r^{2^{O(i^2)}}$ and give a coding gap of $\exp \left( \frac{\log \log k}{\log \log \log k} \right)$, just shy of a polylogarithm.

### 3.1 Gap Instances and Their Parameters

In this section we formally define the set of instance parameters we will track when combining the instances.

A gap instance $I = (G, S, F)$ is a multiple-unicast instance $M = (G, S)$ over a connected graph $G$, along with an associated set of cut edges $F \subseteq E(G)$. We only consider gap instances where the set of terminals is disjoint, i.e., $s_i \neq s_j, s_i \neq t_j, t_i \neq t_j$ for all $i \neq j$. Furthermore, edge capacities and demands are one; i.e., $c_e = 1 \forall e \in E(G)$, and $q_i = 1 \forall (s_i, t_i, q_i) \in S$. A gap instance $I = (G, S, F)$ has parameters $(a, b, f, k, m, r, u)$ when:

- $M$ admits a network coding protocol with completion time at most $a$.
- Let $d_{G \setminus F}(\cdot, \cdot)$ be the hop-distance in $G$ after removing all the cut edges $F$. Then for all terminals $i \in [k]$ we have that $d_{G \setminus F}(s_i, t_i) \geq b$.
- The number of cut edges is $f = |F|$.
- The number of sessions is $k = |S|$.
- The graph $G$ has at most $m$ edges; i.e., $|E(G)| \leq m$.
- $r$ is a lower bound on the ratio between number of sessions and cut edges; i.e., $k/f \geq r$.
- $u$ is an upper bound on the ratio between number of total edges and cut edges; i.e., $m/f \leq u$.

We note that the parameters of a gap instance immediately imply a lower bound on the optimal routing completion time. The following observation follows from Lemma 1.4.
Observation 3.1. Let \( I \) be a gap instance with parameters \( (a,b,f,k,m,r,u) \) and \( b \leq r \). Then the routing completion time for \( (G,S) \) is at least \( b \).

As an application of the above observation, we obtain another proof of the lower bound of the routing completion time for the family of instances of Figure 1. More generally, letting the cut edges be the singleton \( F = \{ (S,T) \} \), we obtain a family of gap instances with the following parameters.

Fact 3.2. The family of gap instances of Figure 1 have parameters \((3,5,1,k,\theta(k^2),k,\theta(k^2))\) for \( k \geq 5 \).

The above family of gap instances will serve as our base gap instances in a recursive construction which we describe in the following section.

### 3.2 Graph Product of Two Gap Instances

In this section we present the graph product that combines two instances to obtain one a with higher coding gap.

**Definition 3.3.** Colored bipartite graphs are families of bipartite graphs \( \mathcal{B}(n_1,n_2,m,k,g) \). Graphs \( B = (V_1,V_2,E) \in \mathcal{B}(n_1,n_2,m,k,g) \) are bipartite graphs with \( |V_1| = n_1 \) (resp. \( |V_2| = n_2 \)) nodes on the left (resp., right), each of degree \( m \) (resp., \( k \)), and these graphs have girth at least \( g \). In addition, edges of \( B \) are colored using two color functions, edge color \( \chi_1 : E(B) \to [m] \) and session color \( \chi_2 : E(B) \to [k] \), which satisfy the following.

- \( \forall v \in V_1 \), the edge colors of incident edges form a complete set \( \{ \chi_1(e) \mid e \ni v \} = [m] \).
- \( \forall v \in V_2 \), the session colors of incident edges form a complete set \( \{ \chi_2(e) \mid e \ni v \} = [k] \).
- \( \forall v \in V_1 \), the session colors of incident edges are unique \( |\{ \chi_2(e) \mid e \ni v \}| = 1 \).
- \( \forall v \in V_2 \), the edge colors of incident edges are unique, i.e, \( |\{ \chi_1(e) \mid e \ni v \}| = 1 \).

The size of the colored bipartite graphs will determine the size of the derived gap instance obtained by performing the product along a colored bipartite graph. The following gives a concrete bound on the size and, in turn, allows us to control the growth of the gap instances obtained this way.

**Lemma 3.4** ([7]). \( \forall r,m,g \geq 3 \), there exists a colored bipartite graph \( B \in \mathcal{B}(n_1,n_2,m,k,2g) \) with \( n_1, n_2 \leq (9mk)^{g+3} \).

**Performing the product along a colored bipartite graph.** Having defined colored bipartite graphs, we are now ready to define the graph product of \( I_1 \) and \( I_2 \) along \( B \).

For \( i \in \{1,2\} \) let \( I_i = (G_i,S_i,F_i) \) be a gap instance with parameters \((a_i,b_i,f_i,k_i,m_i,r_i,u_i)\) and let \( B = (n_1,n_2,2(m_1-f_1),k_2,g) \) be a colored bipartite graph with girth \( g \leq 2b_1b_2 \). We call \( I_1 \) the outer instance and \( I_2 \) the inner instance. Denote the product gap instance \( I_+ \hat{=} T(I_1,I_2,B) \) by the following procedure:

- Replace each non-cut edge \( e = \{u,v\} \in E(G_1) \backslash F_1 \) with two anti-parallel arcs \( \bar{e} = (u,v), \bar{e} = (v,u) \) and let \( \bar{E}(G_1) = \{ \bar{e}_1, \bar{e}_2, \ldots, \bar{e}_{2(m_1-f_1)} \} \) be the set of all such arcs.
- Construct \( n_1 \) copies of \((V(G_1),\bar{E}(G_1))\) and \( n_2 \) copies of \( G_2 \). Label the \( i^{th} \) copy as \( G_1^{(i)} \) and \( G_2^{(i)} \).
- Every cut edge \( e \in F_1 \) and every \( i \in [n_1] \) replace edge \( e \) in \( G_1^{(i)} \) by a path of length \( a_2 \) with the same endpoints. Let \( f^{(i)}_e \) be an arbitrary edge on this replacement path.
For every \((i,j) \in E(B)\) where \(i \in [n_1], j \in [n_2]\) with edge color \(\chi_1\) and session color \(\chi_2\) do the following. Let \(e^{(i)}_x = (x,y)\) be the \(\chi_1\)th arc in \(G^{(i)}_1\) and let \((s,t)\) be the \(\chi_2\)th terminal pair in \(G^{(j)}_2\). Merge \(x\) with \(s\) and \(y\) with \(t\); delete \(e^{(i)}_x = (x,y)\) from \(G^{(i)}_1\).

- For each session in the outer instance \((s_i,t_i,q_i) = 1) \in S_1\) add a new session \((s^{(j)}_i,t^{(j)}_i,1)\) in \(G^{(j)}_1\) for \(j \in [n_1]\) to the product instance.

- The cut edges \(F_+\) in the product instance \(I_+\) consist of the the union of the following: (i) one arbitrary (for concreteness, first one) edge from all of the \(a_2\)-length paths that replaced cut edges in \(G^{(i)}_1\), i.e., \(\{e^{(i)}_e | e \in F_1, i \in [n_1]\}\), and (ii) all cut edges in copies of \(G_2\), i.e., \(\{e^{(i)}_e | e \in F_2, i \in [n_2]\}\).

We now give bounds on how the parameters change after combining two instances. First, we note that by composing network coding protocols for \(I_1\) and \(I_2\) in the natural way yields a network coding protocol whose completion time is at most the product of these protocols’ respective completion times.

**Lemma 3.5.** (Coding completion time) The product instance \(I_+\) admits a network coding protocol with completion time at most \(a_1a_2\).

Less obviously, we show that if we choose a large enough girth \(g\) for the colored bipartite graph, we have that the \(b\) parameter of the obtained product graph is at least the product of the corresponding \(b\) parameters of the inner and outer instances.

**Lemma 3.6** (Routing completion time). Let \(I_+ = (G_+, S_+, F_+)\) be the product instance using a colored bipartite graph \(B\) of girth \(\hat{g} = 2b_1b_2\) and let \(d_{G_+}\) be the hop-distance in \(G_+\) with all the edges of \(F_+\) deleted. We have that \(d_{G_+} = \min(b_1b_2, \frac{g}{2}) = b_1b_2\) for all \((s_i,t_i,q_i) \in S_+\).

**Proof.** Let \(p\) be a path in \(G_+ \setminus F_+\) between some terminals \(s_i \sim t_i\) that has the smallest hop-length among all \((s_i,t_i,q_i) \in S_+\). We want to show that \(|p| \geq \min(b_1b_2, \frac{g}{2})\).

First, let \(q\) be the path in the colored bipartite graph \(B\) that corresponds to \(p\). There are some technical issues with defining \(q\) since merging vertices in the graph product has the consequence that some \(v \in V(G_+)\) belong to multiple nodes \(V(B)\). To formally specify \(q\), we use the following equivalent rephrasing of the graph product that will generate an “expanded instance” \(G'_+\). Instead of “merging” two vertices \(u, v\) as in \(G_+\), connect then with an edge \(e\) of hop-length \(h(e) = 0\) and add \(e\) to \(G'_+\). Edges from \(G_+\) have hop-length \(h(e) = 1\) and are analogously added to \(G'_+\). The path \(p\) can be equivalently specified as the path between \(s_i \sim t_i\) in \(G_+ \setminus F_+\) that minimizes the distance \(d_q(s_i, t_i)\). Now, each vertex \(V(G'_+)\) belongs to exactly one vertex \(V(B)\), hence the path \(q\) in \(B\) corresponding to \(p\) is well-defined. Note that \(p\) is a closed path in \(G_+\) and \(q\) is a closed path in \(B\).

Suppose that \(q\) spans a non-degenerate cycle in \(B\). Then \(|p| \geq \frac{|q|}{2} \geq \frac{g}{2}\), where the last inequality \(|q| \geq g\) is due to the girth of \(G\). The first inequality \(|p| \geq \frac{|q|}{2}\) is due to the fact that when \(q\) enters a node \(v \in V_2(B)\), a node representing an inner instance, the corresponding path \(p\) had to traverse at least one inner instance edge before its exit since the set of terminals is disjoint and a path can enter/exit inner instances only in terminals.

Suppose now that \(q\) does not span a cycle in \(B\), therefore the set of edges in \(q\) span a tree \(T\) in \(B\) and \(q\) is simply the (rotation of the unique) Eulerian cycle of that tree. Notation-wise, let \(v \in V_1(B)\) be the node in the colored bipartite graph \(B\) that contains the critical terminals \(s_i\) and \(t_i\) and suppose that \(T\) is rooted in \(v\). If the depth of \(T\) is 1 (i.e., consists only of \(v \in V_1(B)\) and direct children \(w_1, \ldots, w_j \in V_2(B)\)), then \(p\) must correspond to a \(s_i \sim t_i\) walk in \(v\), where each (non-cut) edge traversal is achieved by a non-cut walk in the inner instance \(w_j\) between a set of inner terminals. Note that every \(s_i \sim t_i\) non-cut walk has hop-length at least \(b_1\) and each non-cut walk in the inner instance has hop-length at least \(b_2\), for a cumulative \(b_1 \cdot b_2\).
Finally, suppose that $T$ has depth more than 1, therefore there exists two $v, w \in V(T)$ and $v, w \in V_1(B)$. Since $T$ is traversed via an Eulerian cycle, the path $p$ passes through two terminals of $(s_j, t_j, \cdot) \in S_i$. Let $p'$ be the natural part of $p$ going from $s_j \leadsto t_j$, e.g., obtained by clipping the path corresponding to the subtree of $q$ in $S$. Furthermore, let $p''$ be the part of $p$ connecting $v$ and $w$ and is disjoint from $p'$. From the last paragraph we know that $|p''| \geq 1$ since it passes through at least one $u \in V_2(B)$. Also, by minimality of $s_j \leadsto t_j$ we have that $d_h(s_j, t_j) \geq d_h(s_i, t_i)$. Now we have a contradiction since $d_h(s_i, t_i) = |p| \geq |p'| + |p''| \geq 1 + d_h(s_j, t_j)$.

Combining Lemmas 3.5 and 3.6 together with some simple calculations (deferred to Appendix B), we find that the product instance is a gap instance with the following parameters.

**Lemma 3.7.** For $i \in \{1, 2\}$ let $I_i = (G_i, S_i, F_i)$ be a gap instance with parameters $(a_i, b_i, f_i, k_i, m_i, r_i, u_i)$ with $\frac{m_i}{a_i} \geq 2$ and $a_i \geq 2$; let $B \in B(n_1, n_2, 2(m_1 - f_1), k_2, 2b_1b_2)$ be a colored bipartite graph. Then $I_1 \triangleq T(G_1, G_2, B)$ is a gap instance with parameters $a_+ \triangleq a_1a_2, b_+ \triangleq b_1b_2, f_+ \triangleq n_1f_1 + n_2f_2, k_+ \triangleq n_1k_1, m_+ \triangleq a_2n_1f_1 + n_2m_2, r_+ \triangleq r_1 \frac{1}{1+2a_2/m_2}, u_+ \triangleq u_2 \frac{1+a_2/2}{1+r_2/(2m_2)}$. Moreover, $\frac{m_+}{a_+} \geq 2$ and $a_+ \geq 2$.

### 3.3 Iterating the Graph Product

Having bounded the parameters obtained by combining two gap instances, we are now ready to define a recursive family of gap instances from which we obtain our polylogarithmic completion-time network coding gap.

**Definition 3.8.** We recursively define a collection of gap instances $I(i, r))_{i \geq 0, r \geq 5}$, and denote its parameters by $(a_{i,r}, b_{i,r}, f_{i,r}, k_{i,r}, m_{i,r}, r_{i,r}, u_{i,r})$. For the base case, we let $I(0, r)$ be the gap instance of Fact 3.2 with parameters $(3, 5, 1, r, r, r, r)$. For $i + 1 > 0$ we define $I(i + 1, r) \triangleq T(I_i, I_2)$. Here, $I_1 \triangleq I(i, 3r)$ and $I_2 \triangleq I(i, u_{i,3r})$, with parameters $(a_1, \ldots, u_1)$ and $(a_2, \ldots, u_2)$, respectively.

In other words, $I_1$ is defined such that $r_1 = 3r_+ + I_2$ such that $r_2 = u_1$. In Appendix B we study the growth of the parameters of our gap instance families. Two parameters that are easy to bound for this construction are the following.

**Observation 3.9.** For any $i \geq 0$ and $r \geq 5$, we have $a_{i,r} = 3^{2^i}$ and $b_{i,r} = 5^{2^i}$.

A less immediate bound, whose proof is also deferred to Appendix B, is the following bound on the number of edges of the gap instances.

**Lemma 3.10.** We have that $\log m_{i,r} \leq 2^{O(2^i)} \log r$ for all $i \geq 0, r \geq 5$.

### 3.4 LowerBounding the Coding Gap

We are now ready to prove this section’s main result – a polylog($k$) completion-time coding gap.

**Theorem 1.3.** There exists an absolute constant $c > 0$ and an infinite family of $k$-unicast instances for $k$ sufficiently large whose completion-time coding gap is at least $\Omega(\log^c k)$.

**Proof.** For each $i \geq 0$ and $r \geq 5^2$, consider $I_{i,r}$ as defined above. By Observation 3.9 this gap instance has coding completion time at most $a_{i,r} = 3^{2^i}$. Moreover, also by Observation 3.9, this instance has $b_{i,r} = 5^{2^i}$, and so by Observation 3.1 its routing completion time is at least $5^{2^i}$. Hence the completion-time coding gap of $I_{i,r}$ is at least $(5/3)^{2^i}$. It remains to bound this gap in terms of $k$.

As the terminals of $I_{i,r}$ are disjoint, we have that $k$ is upper bounded by the number of nodes of $I_{i,r}$, which is in turn upper bounded by $m_{i,r}$, as $I_{i,r}$ is connected and not acyclic. That is, $k \leq m_{i,r}$.
But by Lemma 3.10, we have that \( \log m_{i,r} \leq 2O^{(2')} \cdot \log r = 2O^{(2')} \cdot O(2^i) \leq 2O^{(2')} \leq 2c'2^i \), for some universal constant \( c' > 0 \). Therefore, stated in terms of \( k \), the completion-time coding gap is at least
\[
(5/3)^{2^i} = 2^{2^i} \log 5/3 = (2^{c'}2^i)^{\log 5/3} = (\log m_{i,r})^c \geq \log c^c k,
\]
where \( c = \frac{\log 5/3}{c'} > 0 \) is a universal constant, as claimed.

\[ \square \]

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Appendix

A Deferred Proofs of Section 2

Here we prove Lemma 2.3, restated here for ease of reference.

**Lemma 2.3.** Given sequences \( h_1, \ldots, h_k, q_1, \ldots, q_k \in \mathbb{R}_{\geq 0} \) with \( \sum_{i=1}^k q_i \cdot h_i \geq 1 \) there exists a non-empty subset \( I \subseteq [k] \) with \( \min_{i \in I} h_i \geq \frac{1}{\sum_{i=1}^k q_i} \) for \( \alpha \) gap = \( O \left( \log \frac{\sum_{i=1}^k q_i}{\min_{i=1}^k q_i} \right) \).

**Proof.** Suppose (without loss of generality) that \( h_1 \geq h_2 \geq \ldots h_k \) and assume for the sake of contradiction that none of the sets [1], [2], ..., [k] satisfy the condition. In other words, if we let \( q([j]) \triangleq \sum_{i=1}^j q_j \), then \( h_i < \frac{1}{\alpha \cdot \sum_{i=1}^k q_i} \) for all \( i \in [k] \). Multiplying both sides by \( q_i \) and summing them up, we get that
\[
1 \leq \sum_{i=1}^k q_i h_i < \frac{1}{\alpha} \sum_{i=1}^k q_i \sum_{i=1}^k q_i = \frac{1}{\alpha} \sum_{i=1}^k q_i [\frac{q_i}{q([i])}].
\]
Reordering terms, this implies \( \sum_{i=1}^k q_i [\frac{q_i}{q([i])}] > \alpha \).

Define \( f(x) \) as \( 1/q_1 \) on \([0, q_1)\); \( 1/(q_1 + q_2) \) on \([q_1, q_1 + q_2)\); ...; \( 1/q([i]) \) on \([q([i-1]), q([i]))\) for \( i \in [k] \). Now we have
\[
\int_0^{q([k])} f(x) = \frac{q_1}{q_1} + \frac{q_2}{q_1 + q_2} + \frac{q_3}{q_1 + q_2 + q_3} + \ldots + \frac{q_k}{q([k])}.
\]
However, since \( f(x) \leq 1/x \)
\[
\int_0^{q([k])} f(x) = \int_0^{q_1} f(x) \, dx + \int_{q_1}^{q([k])} f(x) \, dx 
\leq 1 + \int_{q_1}^{q([k])} \frac{1}{x} \, dx = 1 + \ln \frac{q([k])}{q_1}.
\]
Hence by setting \( \alpha \triangleq 1 + \ln \frac{q([k])}{q_1} \) we reach a contradiction and finish the proof. \[ \square \]

B Deferred Proofs of Section 3

Here we provide the deferred proofs of lemmas of Section 3, restated below for ease of reference.

**Lemma 3.5.** (Coding completion time) The product instance \( I_+ \) admits a network coding protocol with completion time at most \( a_1 a_2 \).
Proof. Suppose that there exists a network coding protocol with completion time \( t_i \leq a_i \) that solves \((G_i,S_i)\) for \( i \in \{1,2\} \). Functionally, each round in the outer instance \((G_1,S_1)\) consists of transmitting \( c_e \) bits of data from \( u \) to \( v \) for all arcs \((u,v)\) where \( \{u,v\} \in E(G_1) \). This is achieved by running the full \( t_2 \) rounds of the inner instance protocol over all copies of the instances which pushes \( q_i \) bits from \( s_i \) to \( t_i \) for all \((s_i,t_i,q_i)\) and all copies of the inner instance. The reason why such inner protocol pushes the information across each arc \((u,v)\) is because \( u \) is merged with some \( s_i \), \( v \) is merged with \( t_i \), and with \( q_i = c_e \) for some \((s_i,t_i,q_i) \in S_2 \) and some copy of the inner instance. In conclusion, by running \( t_1 \) outer rounds, each consisting of \( t_2 \) inner rounds, we get a \( t_1 t_2 \leq a_1 a_2 \) round protocol for the product instance. \( \square \)

**Lemma 3.7.** For \( i \in \{1,2\} \) let \( I_i = (G_i,S_i,F_i) \) be a gap instance with parameters \((a_i,b_i,f_i,k_i,m_i,r_i,u_i)\) with \( \frac{m_i}{f_i} \geq 2 \) and \( a_i \geq 2 \); let \( B \in \mathcal{B}(n_1,n_2,2(m_1-f_1),k_2,2b_2b_2) \) be a colored bipartite graph. Then \( I_+ \defeq T(G_1,G_2,B) \) is a gap instance with parameters \( a_+ \defeq a_1 a_2, b_+ \defeq b_1 b_2, f_+ \defeq n_1 f_1 + n_2 f_2, k_+ \defeq n_1 k_1, m_+ \defeq a_2 n_1 f_1 + n_2 m_2, r_+ \defeq r_1 \frac{1}{1+2n_1/r_2}, u_+ \defeq u_2 \frac{1+a_2/2}{1+2n_1/2} \). Moreover, \( \frac{m_+}{f_+} \geq 2 \) and \( a_+ \geq 2 \).

**Proof of Lemma 3.7.** First, the set of terminals in the product instance \( I_+ \) is disjoint, as distinct terminals of copies of the outer instance \( I_1 \) have their edges associated with distinct terminals source-sink pairs of the inner instance \( I_2 \). Consequently, no two terminals of the outer instance are associated with the same node of the same copy of an inner instance. The capacities and demands of \( I_+ \) are one by definition. Now we turn to bounding the gap instance’s parameters.

Parameters \( a_+ \) and \( b_+ \) are directly argued by Lemma 3.5 and Lemma 3.6. Furthermore, \( f_+, k_+, m_+ \) are obtained by direct counting, as follows.

Recall that the cut edges of the outer instance get replaced with a path of length \( a_2 \). Since there are \( n_1 \) copies of outer instances, each having \( f_1 \) cut edges, this contributes \( a_2 n_1 f_1 \) edges to \( m_+ \). The non-cut edges of the outer instance get deleted and serve as a merging directive, hence they do not contribute to \( m_+ \). Finally, each edge of the inner instance gets copied into \( I_+ \), contributing \( n_2 m_2 \) as there are \( n_2 \) copies of the inner instance.

For \( r_+ \) we need to show it is a lower bound on \( k_+/f_+ \). We note that \( |E(B)| = n_1 \cdot 2(m_1-f_1) = n_2 k_2 \) and proceed by direct calculation:

\[
\frac{k_+}{f_+} = \frac{n_1 k_1}{n_1 f_1 + n_2 f_2} = \frac{k_1}{f_1} \cdot \frac{1}{1 + \frac{n_2 f_2}{n_1 f_1}} \geq \frac{k_1}{f_1} \cdot \frac{1}{1 + \frac{2(m_1-f_1) f_2}{f_1}} = \frac{k_1}{f_1} \cdot \frac{1}{1 + \frac{2u_1}{r_2}} = r_+.
\]

For \( u_+ \) we need to show it is an upper bound on \( m_+/f_+ \). Note that \( k_2 \leq m_2 \) since the set of terminals is disjoint and the graph is connected.

\[
\frac{m_+}{f_+} = \frac{n_2 m_2 + a_2 n_1 f_1}{n_2 f_2 + n_1 f_1} = \frac{m_2}{f_2} \cdot \frac{1 + a_2 \frac{n_1 f_1}{n_2 m_2}}{1 + \frac{n_1 f_1}{n_2 f_2}} \leq \frac{m_2}{f_2} \cdot \frac{1 + a_2 \frac{k_2}{2(m_1-f_1) f_2}}{1 + \frac{k_2}{2(m_1-f_1) f_2}} \leq u_2 \cdot \frac{1 + a_2 \cdot \frac{1}{2}}{1 + \frac{1}{2 u_1}} = u_+.
\]

Here the last inequality relies on \( m_1/f_1 \geq 2 \) and on \( k_2 \leq m_2 \), which follows from the set of terminals being disjoint and the graph \( G_2 \) being connected.

For the final technical conditions, note that \( a_+ \geq 2 \) is clear from \( a_+ = a_1 a_2 \geq 4 \geq 2 \). Finally, \( \frac{m_+}{f_+} \geq 2 \) follows from the following:

\[
\frac{m_+}{f_+} = \frac{a_2 n_1 f_1 + n_2 m_2}{n_1 f_1 + n_2 f_2} = a_2 \frac{n_1 f_1}{n_1 f_1 + n_2 f_2} + \frac{m_2}{n_2 f_2} \frac{n_2 f_2}{n_1 f_1 + n_2 f_2} \geq 2 \left( \frac{n_1 f_1}{n_1 f_1 + n_2 f_2} + \frac{n_2 f_2}{n_1 f_1 + n_2 f_2} \right) = 2. \]
B.1 Upper Bounding $m_{i,r}$

For readability, we sometimes write $u(i, r)$ instead of $u_{i,r}$ and similarly for $m(i, r)$. Also, we note that the technical conditions $a_{i,r} \geq 2$ and $\frac{m_{i,r}}{f_{i,r}} \geq 2$, which clearly hold for $i = 0$, hold for all $i > 0$, due to Lemma 3.7. Finally, we note that by Lemma 3.7, since $r_2 = u_1$ and $a_2 \geq 1$, we have that $u_+ \geq u_2$ and so for all gap instances in the family we have $u_{i,r} \geq u_{i-1,u(i-1,3r)} \geq 5$.

**Lemma B.1.** The parameter of $I(i, r)$ for any $i \geq 0, r \geq 5$ satisfy the following.

(i) $\frac{k_{i+1,r}}{f_{i+1,r}} \geq r$,

(ii) $u_{i+1,r} \leq 3^{2i} \cdot u_{i,u(i,3r)}$ and

(iii) $\log m(i+1, r) \leq O(5^{2i+1}) \cdot \log(m_{i,3r} \cdot m_{i,u(i,3r)})$.

**Proof.** Claim (i) follows from Lemma 3.7, as follows.

$$
\frac{k_{i+1,r}}{f_{i+1,r}} \geq r_{i+1,r} = r_{i,3r} \cdot \frac{1}{1 + 2u_{i,3r}/r_{i,u(i,3r)}} \geq 3r \cdot \frac{1}{1 + 2u_{i,3r}/u_{i,3r}} = 3r \cdot \frac{1}{3} = r.
$$

We now prove claims (ii) and (iii). Fix $i, r$ and define $I_1 \triangleq I(i, 3r)$ (with parameters $(a_1, \ldots, u_1)$) and $I_2 \triangleq I(i, u_i, r)$ (with parameters $(a_2, \ldots, u_2)$). We have $u(i + 1, r) = u_2 \frac{1 + a_2/2}{1 + r/2} \leq u_2 \frac{1 + a_2/2}{1 + 1/2} \leq u_2 \cdot a_2$ (Lemma 3.7), with $a_2 = a_{i,u(i,3r)} = 3^{2i}$ and $u_2 = u(i, u(i, 3r))$ from the iterated tensoring process. Therefore, we conclude that $u(i + 1, r) \leq 3^{2i} \cdot u_{i,u(i,3r)}$, as claimed.

We now prove Claim (iii). The corresponding colored bipartite graph $B \in B(n_1, n_2, 2(m_1 - f_1), k_2, 2b_1 b_2)$ used to produce the product $I_{i,r}$ has max$(n_1, n_2) \leq (2(m_1 - f_1)k_2)^{O(b_1 b_2)}$, by Lemma 3.4. Therefore, as $k_2 \leq m_2$, we have that max$(n_1, n_2) \leq (m_1 \cdot m_2)^{O(b_1 b_2)}$. This implies the following recurrence for $m_{i,r}$.

$$
m_{i+1,r} = a_2 n_1 f_1 + n_2 m_2 \leq a_2 \max(n_1, n_2) m_1 m_2.
$$

Taking out logs, we obtain the desired bound.

$$
\log m_{i+1,r} \leq \log a_2 + \log \max(n_1, n_2) + \log m_1 m_2
$$

$$
= O(2^i) + O(b_1 b_2) \log(m_1 m_2) + \log(m_1 m_2)
$$

$$
= O(2^i) + O(5^{2i}) \log(m_1 m_2)
$$

$$
= O(5^{2i}) \cdot \log(m_{i,3r} \cdot m_{i,u(i,3r)}).
$$

Given Lemma B.1 we obtain the bound on $u_{i,r}$ in terms of $i$ and $r$.

**Lemma B.2.** We have that $\log u_{i, r} \leq 2^{O(2^i)} \log r$ for all $i \geq 0, r \geq 5$.

**Proof.** By Lemma B.1, we have the recursion $u(i + 1, r) \leq 3^{2^i} \cdot u(i, u(i, 3r))$ with initial condition $u(0, r) = O(r^2)$, by Fact 3.2. Taking out logs, we obtain $\log u(i + 1, r) \leq O(2^i) + \log u(i, u(i, 3r))$ and $\log u(0, r) = O(\log r)$. We prove via induction that $\log u(i, r) \leq \frac{1}{c}(c^2)^{2^i} \cdot \log r$ for some sufficiently large $c > 0$. In the base case $\log u(0, r) = O(\log r) \leq \frac{1}{c}(c^2)^{2^0} \cdot \log r = c \log r$. For the inductive step we have:

$$
\log u(i + 1, r) \leq O(2^i) + \log u(i, u(i, 3r))
$$

$$
\leq O(2^i) + \frac{1}{c}(c^2)^{2^i} \log u(i, 3r)
$$

$$
\leq O(2^i) + \frac{1}{c}(c^2)^{2^i} \cdot \frac{1}{c} c^{2^i} \log 3r
$$

$$
\leq O(2^i) + \frac{1}{c^2}(c^2)^{2^{i+1}} \log 3r
$$

$$
\leq \frac{1}{c}(c^2)^{2^{i+1}} \log r.
$$
where the last inequality holds for \( i \geq 1 \) and \( r \geq 5 \) and a sufficiently large \( c > 0 \).

Plugging in the bound of Lemma B.2 and Lemma B.1 we can prove inductively the upper bound on the number of edges of \( I_{i,r} \) in terms of \( i \) and \( r \) given by Lemma 3.10, restated here.

**Lemma 3.10.** We have that \( \log m_{i,r} \leq 2^{O(2^i)} \log r \) for all \( i \geq 0, r \geq 5 \).

**Proof.** By Lemma B.1, we have the recursion \( \log m(i+1,r) \leq O(5^{2^{2^i}}) \cdot \log(m_{i,3r} \cdot m_{i,u(i,3r)}) \) with initial condition \( \log m(0,r) = O(\log r) \), by Fact 3.2. We prove via induction that \( \log m_{i,r} \leq c^{2^i} \log r \) for a sufficiently large universal constant \( c > 0 \). In the base case \( \log m_{0,r} \leq O(\log r) \leq c \log r = c^{2^0} \log r \).

For the inductive step, using Lemma B.2 to bound \( \log u(i,3r) \), we have:

\[
\log m_{i+1,r} \leq O(5^{2^i}) \cdot (\log m_{i,3r} + \log m_{i,u(i,3r)}) \\
\leq O(5^{2^i}) \cdot \left( c^{2^i} \log 3r + c^{2^i} \log u(i, 3r) \right) \\
\leq O(5^{2^i}) \cdot \left( c^{2^i} \log 3r + c^{2^i} 2^{O(2^i)} \log 3r \right) \\
= c^{2^i} \cdot O(5^{2^i}) \cdot 2^{O(2^i)} \log 3r \\
\leq c^{2^i+1} \log r,
\]

where the last inequality holds for \( i \geq 1 \) and \( r \geq 5 \) and a sufficiently large \( c > 0 \).

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**References**


