Solution Set 2

Due 11:59pm, Monday, September 27th

Collaboration is allowed on this homework. You may discuss the problems with your colleagues, but each student must prepare and submit a separate assignment. Please list the names of the people you worked with:

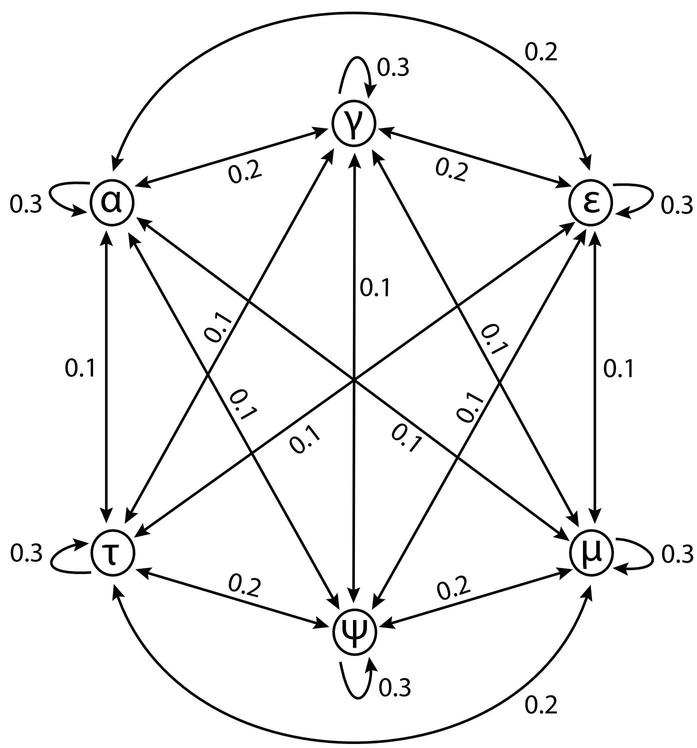
Please provide answers to all questions. In order to obtain full credit, explain your reasoning on each question and show all intermediate steps leading to your solution.

You may submit your assignments in any of the following ways:

- Download the assignment in pdf format and print it out. Write your solutions in the space provided. Scan your handwritten solution and upload it to Canvas. Attach additional pages, if needed.
- Download the assignment in pdf format. Use the commenting features in adobe or a similar tool to enter your solution directly on the pdf. Upload the completed assignment, annotated with your solutions.
- Download the assignment in latex format. Enter your solutions in latex format, compile the assignment, and turn in the resulting pdf.

1. Agents Mulder and Scully recently recovered a set of samples of alien blood. During analysis of the blood, Scully discovered a molecule that is similar to DNA, but is composed of 6 nucleotides unknown to modern science. She determined that 3 bases, which she calls the jupitomines, α, γ, and ϵ, pair with the other three bases, which she calls the plutomines, τ, ψ and μ, respectively. You have been hired by Area 51 to model the evolution of the alien species on a genetic level. Your first task is to build a Markov model of nucleotide substitution in this alien DNA-like molecule.

   (a) Based on her lab observations, Scully reports that each base is more likely to be replaced by a different base from the same class, than by a base from the other class. For a given pair of bases, x and y, base x is replaced by base y with probability 0.2, if x and y are members of the same class, and with probability 0.1, if x and y are members of different classes. Draw the topology of the Markov chain and label the edges with transition probabilities. Do not forget to add self loops, if needed.



(b) Give the transition matrix.

	α	γ	ϵ	τ	ψ	μ
α	0.3	0.2	0.2	0.1	0.1	0.1
γ	0.2	0.3	0.2	0.1	0.1	0.1
ϵ	0.2	0.2	0.3	0.1	0.1	0.1
τ	0.1	0.1	0.1	0.3	0.2	0.2
ψ	0.1	0.1	0.1	0.2	0.3	0.2
μ	0.1	0.1	0.1	0.2	0.2	0.3

(c) Mulder and Scully call in from the field to report that the aliens seems to mutate right in front of their eyes over a period of minutes. Assuming that your time step is 1 minute, determine the transition matrix for your Markov chain after 2 minutes.

This should be the 2-step transition matrix, $P \times P$.

	α	γ	ϵ	τ	ψ	μ
α	0.2	0.19	0.19	0.14	0.14	0.14
γ	0.19	0.2	0.19	0.14	0.14	0.14
ϵ	0.19	0.19	0.2	0.14	0.14	0.14
τ	0.14	0.14	0.14	0.2	0.19	0.19
ψ	0.14	0.14	0.14	0.19	0.2	0.19
μ	0.14	0.14	0.14	0.19	0.19	0.2

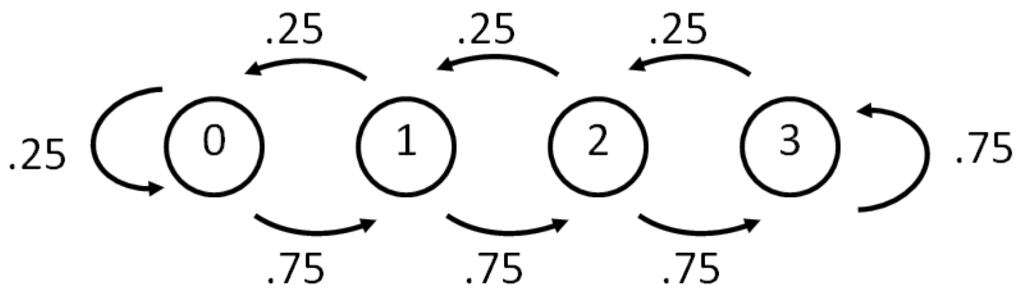
(d) What is the probability that a given base is the same as it was two minutes ago?

0.2, the diagonal values in the $P^{(2)}$ *matrix.*

(e) What is the probability that a specific plutomine will be replaced by a jupitomine (*any* jupitomine) after two minutes have elapsed?

The probability of making a transition from a specific plutomine to each of the jupitomines is 0.14. *Therefore, the probability of making a transition from a specific plutomine to any of the jupitomines is* $3 \cdot 0.14 = 0.42$.

2. Suppose our drunk on the railroad track has a particular yen to move to the right. Perhaps the bar at the right end of the track has better beer. We could model his progress toward the bar with a Markov chain like this:



(a) Write down the transition probability matrix for the Markov chain depicted graphically above.

	0	1	2	3
0	0.25	0.75	0.00	0.00
1	0.25	0.00	0.75	0.00
2	0.00	0.25	0.00	0.75
3	0.00	0.00	0.25	0.75

(b) Does this Markov model have a stationary state probability distribution?
If so,

i. Calculate the stationary probability for each state. Show your work.

ii. Is this stationary distribution unique? How do you know?

If not, explain why not.

$\varphi \cdot P$ yields the following equations:

$$\begin{aligned}
\varphi_0^* &= \frac{1}{4}\varphi_0^* + \frac{1}{4}\varphi_1^* \\
\varphi_1^* &= \frac{3}{4}\varphi_0^* + \frac{1}{4}\varphi_2^* \\
\varphi_2^* &= \frac{3}{4}\varphi_1^* + \frac{1}{4}\varphi_3^* \\
\varphi_3^* &= \frac{3}{4}\varphi_2^* + \frac{3}{4}\varphi_3^*
\end{aligned}$$

These reduce to:

$$\begin{aligned}
3\varphi_0^* &= \varphi_1^* \\
3\varphi_1^* &= \varphi_2^* \\
3\varphi_2^* &= \varphi_3^*
\end{aligned}$$

$$\begin{aligned}
\varphi_1^* &= 3\varphi_0^* \\
\varphi_2^* &= 3\varphi_1^* = 9\varphi_0^* \\
\varphi_3^* &= 3\varphi_2^* = 27\varphi_0^*
\end{aligned}$$

Since the probability state distribution must sum to one, we obtain

$$\varphi_0^* + 3\varphi_0^* + 9\varphi_0^* + 27\varphi_0^* = 1$$

or $\varphi_0^* = 1/40$. *Thus, the steady state distribution must be*

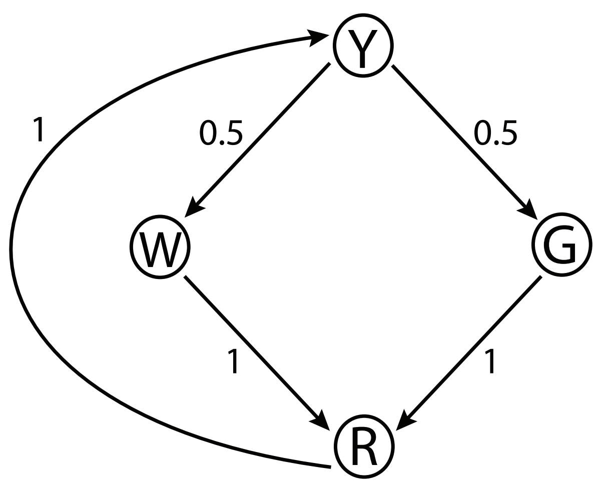
$$\varphi^* = (0.025, 0.075, 0.225, 0.675).$$

This is a finite, aperiodic, irreducible ("connected") Markov chain, so it has a unique stationary distribution.

3. Esmeralda practices magic on Monday, Wednesday and Friday mornings. She has 4 wands: a Red wand that turns professors into lobsters, a Green wand that turns professors into parrots, a Yellow wand that turns professors into lions, and a White wand that turns professors into polar bears. No matter which wand Esmeralda uses, the professor always reverts to human form at midnight on the following day.

   Esmeralda has a procedure for deciding which wand to use. If the wand she used last was white or green, then she chooses the Red wand. If the last wand she used was red, then she chooses the Yellow wand. If she used the Yellow wand on the previous occasion, then she flips a fair coin. If it comes up heads, she chooses the White wand; otherwise, she chooses the Green wand.

   (a) Construct a Markov chain that models Esmeralda's wand usage. Each state corresponds to a wand color and transitions represent the probability of Esmeralda's next selection, given the wand she used last time. Draw the topology of the Markov chain and label the edges with transition probabilities. Do not forget to add self loops, if needed.



   (b) Give the transition matrix for this Markov chain.

	W	G	Y	R
W	0.0	0.0	0.0	1.0
G	0.0	0.0	0.0	1.0
Y	0.5	0.5	0.0	0.0
R	0.0	0.0	1.0	0.0

(c) Esmeralda likes bright colors, so she uses the Red wand ($p = 0.75$) or the Yellow wand ($p = 0.25$) on the first Monday of the semester.

Calculate the probabilities of seeing the various animals (a lobster, a parrot, a lion, or a polar bear) if you look into the professor's office on the first Monday, Wednesday, and Friday of the semester. What are the probabilities on the second Monday, Wednesday, and Friday of the semester? (Calculate the probabilities of these events for each of the six days separately. Remember, Esmerelda only practices magic on Mondays, Wednesdays, and Fridays.)

	Bear	Parrot	Lion	Lobster
Monday 1	0.000	0.000	0.25	0.75
Wednesday 1	0.125	0.125	0.75	0.00
Friday 1	0.375	0.375	0.00	0.25
Monday 2	0.000	0.000	0.25	0.75
Wednesday 2	0.125	0.125	0.75	0.00
Friday 2	0.375	0.375	0.00	0.25

(d) Polar bears and lions are carnivores; parrots and lobsters are not. Assuming that carnivores always eat their prey (i.e., you), what is the safest day to visit the professor, Monday, Wednesday, or Friday? Why?

The safest day is the day that has the highest probability of finding an animal that is not a carnivore, i.e., a lobster or a parrot. This is the probability of being in state R or state G.

This probability is:

Monday	0.750	$= \varphi_G(\text{Mon}) + \varphi_R(\text{Mon})$
Wednesday	0.125	$= \varphi_G(\text{Wed}) + \varphi_R(\text{Wed})$
Friday	0.625	$= \varphi_G(\text{Fri}) + \varphi_R(\text{Fri})$

So Monday is the safest day.

(e) Does this Markov model have a stationary state probability distribution?
If so,

i. Calculate the stationary probability for each state. Show your work.
ii. Is this stationary distribution unique? How do you know?

If not, explain why not.

This Markov chain does not have a unique, stationary distribution, because it is periodic. It is possible to solve this linear system of equations:

$$\begin{aligned} \varphi_W^* &= 0.5\varphi_Y^* \\ \varphi_G^* &= 0.5\varphi_Y^* \\ \varphi_Y^* &= \varphi_R^* \\ \varphi_R^* &= \varphi_W^* + \varphi_G^* \\ 1 &= \varphi_R^* + \varphi_Y^* + \varphi_G^* + \varphi_W^* \end{aligned}$$

but the resulting distribution, $(\varphi_W^*, \varphi_G^*, \varphi_Y^*, \varphi_R^*) = (\frac{1}{6}, \frac{1}{6}, \frac{1}{3}, \frac{1}{3})$, *depends on the initial state distribution and is not unique.*

(f) Derive the 3-step transition matrix for the Markov chain that models Esmeralda's wand use on the same day every week. As before, each state corresponds to a wand color. Transitions represent the probability of using a particular wand a week in the future, given the wand she used today. That is, your model should give the probability that Esmeralda will use a particular wand next Monday, given the wand she used this Monday, and so on.

	W	G	Y	R
W	0.5	0.5	0.0	0.0
G	0.5	0.5	0.0	0.0
Y	0.0	0.0	1.0	0.0
R	0.0	0.0	0.0	1.0

(g) Use your transition matrix and the first week probabilities from 3(c) to calculate the probability of seeing the various animals (a lobster, a parrot, a lion, or a polar bear) if you look into the professor's office on the Monday, Wednesday, and Friday of the second week of the semester. Did your results agree with the results of Question 3c?

Day in 2nd week: $\varphi(\text{Day 1st week}) \cdot P^3$

Monday: $(0, 0, 0.25, 0.75) \cdot P^3 = (0.0, 0.0, 0.25, 0.75)$

Wednesday: $(0.125, 0.125, 0.750, 0.000) \cdot P^3 = (0.125, 0.125, 0.75, 0)$

Friday: $(0.375, 0.375, 0.000, 0.250) \cdot P^3 = (0.375, 0.375, 0.000, 0.250)$

Yes, applying the 3-step transition matrix, P^3, once has the same effect as applying the first Markov chain three times, as we did in the second part of question 3(c).

(h) Suppose that you look into the professors office on the 1st and 2nd Wednesdays of the semester (after carefully making sure that the door is closed). What is the probability that you see a carnivore on both occasions?

Seeing a carnivore in the professor's office is represented by states Y (i.e., Esmerelda used the Yellow wand to turn the professor into a lion) and W (i.e., Esmerelda used the White wand to turn the professor into a polar bear) in the Markov chain model. Therefore, the probability of seeing a carnivore on the first and on the second Wednesday is the probability that the model visits states Y or W at the second time point AND also visits states Y or W three time points later (or at the next time point in the 3-step model).

So, for the first Wednesday, visiting state Y is $\varphi_Y(Wed_1)$ and visiting state W is $\varphi_W(Wed_1)$. The probability of visiting either state Y or W on the second Wednesday, given state x the previous Wednesday is $\varphi_x(Wed_1) \cdot (P^3_{xY} + P^3_{xW})$. However, to see a carnivore on both occassions, we only consider the Y and W states on the first Wednesday, yielding:

$$\begin{aligned} & \varphi_Y(\text{Wed}_1) \cdot (P^3_{YY} + P^3_{YW}) + \varphi_W(\text{Wed}_1) \cdot (P^3_{WW} + P^3_{WY}) \\ = \; & 0.75 \cdot (1.0 + 0.0) + 0.125 \cdot (0.5 + 0.0) \\ = \; & 0.8125. \end{aligned}$$

(i) If you visit the professor's office on the 1st and 2nd Wednesdays of the semester, what is the probability that you will have been eaten by a carnivore before the end of the second week of the semester? (At first, questions (h) and (i) may seem very similar. It may helpful to consider how the events in the two questions differ.)

There are two ways to get eaten in the first two weeks of classes. The first is to encounter a carnivore in the first week (states Y or W). The probability of one of these events occuring is

$$\varphi_Y(\mathrm{Wed}_1) + \varphi_W(\mathrm{Wed}_1).$$

The second possibility is to not encounter a carnivore in the first week (states R or G), allowing you to survive long enough to meet a carnivore and be eaten in the second week. The probability of not meeting a carnivore in the first week and *meeting a carnivore in the second week is*

$$\varphi_R(\mathrm{Wed}_1) \cdot (P^3_{RY} + P^3_{RW}) + \varphi_G(\mathrm{Wed}_1) \cdot (P^3_{GW} + P^3_{GY}).$$

The probability of observing a lobster (red wand) in the professor's office on Wednesday is zero, so the first term can be ignored.
Combining the probabilities of being eaten in the first and second weeks, we obtain

$$\begin{aligned} & \varphi_Y(\mathrm{Wed}_1) + \varphi_W(\mathrm{Wed}_1) + \varphi_G(\mathrm{Wed}_1) \cdot (P^3_{GW} + P^3_{GY}) \\ = \; & (0.125 + 0.75) + (0.125 \cdot (0.5 + 0.0)) \\ = \; & 0.9375. \end{aligned}$$