Ball Packings
and
Fat Voronoi Diagrams

Don Sheehy

Ball Packings

Ball Packings

Packing unit balls into a square:

Ball Packings

Packing unit balls into a square:
Packing means disjoint interiors.

Ball Packings

Packing unit balls into a square:
Packing means disjoint interiors.
Maximal means no more will fit.

Ball Packings

Packing unit balls into a square:
Packing means disjoint interiors.
Maximal means no more will fit.

Goal: Find a small, maximal packing.

Ball Packings

Packing unit balls into a square:
Packing means disjoint interiors.
Maximal means no more will fit.

Goal: Find a small, maximal packing.

Greedy Algorithm:
- Find the **biggest** empty space.
- Add another ball there.

Problem:

Given n balls in a s x s box, how many total will there be after running the greedy algorithm?

Problem:

Given n balls in a s x s box, how many total will there be after running the greedy algorithm?

Upper Bound

Problem:

Given n balls in a s x s box, how many total will there be after running the greedy algorithm?

Upper Bound

Total volume is at most s^2.

Problem:

Given n balls in a s x s box, how many total will there be after running the greedy algorithm?

Upper Bound

Total volume is at most s^2.

$$\frac{s^2}{\pi}$$

Problem:

Given n balls in a s x s box, how many total will there be after running the greedy algorithm?

Upper Bound

Total volume is at most s^2.

$$\frac{s^2}{\pi}$$ (Packing Argument)

Problem:

Given n balls in a s x s box, how many total will there be after running the greedy algorithm?

Upper Bound

Total volume is at most s^2.

$$\frac{s^2}{\pi}$$ (Packing Argument)

Lower Bound

Problem:

Given n balls in a s x s box, how many total will there be after running the greedy algorithm?

Upper Bound

Total volume is at most s^2.

$$\frac{s^2}{\pi}$$ (Packing Argument)

Lower Bound

Each Voronoi cell is contained in a ball of radius 2.

Problem:

Given n balls in a s x s box, how many total will there be after running the greedy algorithm?

Upper Bound

Total volume is at most s^2.

$$\frac{s^2}{\pi}$$ (Packing Argument)

Lower Bound

Each Voronoi cell is contained in a ball of radius 2.

$$\frac{s^2}{4\pi}$$

Problem:

Given n balls in a s x s box, how many total will there be after running the greedy algorithm?

Upper Bound

Total volume is at most s^2.

$$\frac{s^2}{\pi}$$ (Packing Argument)

Lower Bound

Each Voronoi cell is contained in a ball of radius 2.

$$\frac{s^2}{4\pi}$$ (Covering Argument)

Problem:

Given n balls in a s x s box, how many total will there be after running the greedy algorithm?

Upper Bound

Total volume is at most s^2.

$$\frac{s^2}{\pi}$$ (Packing Argument)

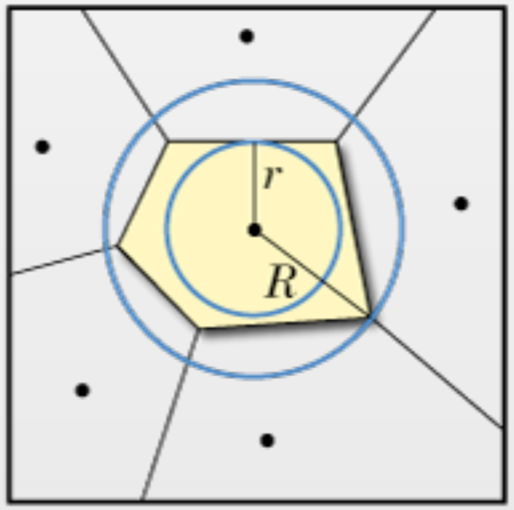
Lower Bound

Each Voronoi cell is contained in a ball of radius 2.

$$\frac{s^2}{4\pi}$$ (Covering Argument)

Why care about a factor of 4?
Because $4 = 2^d$.

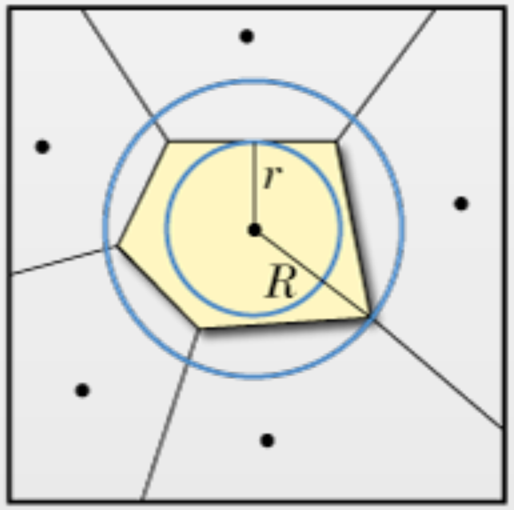
Voronoi Aspect Ratio



Voronoi Aspect Ratio

Ball Packing

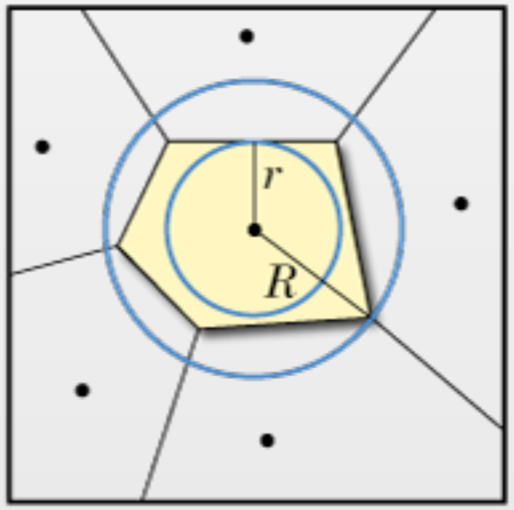
$$r \geq 1, R < 2$$



Voronoi Aspect Ratio

Ball Packing

$$r \geq 1, R < 2$$

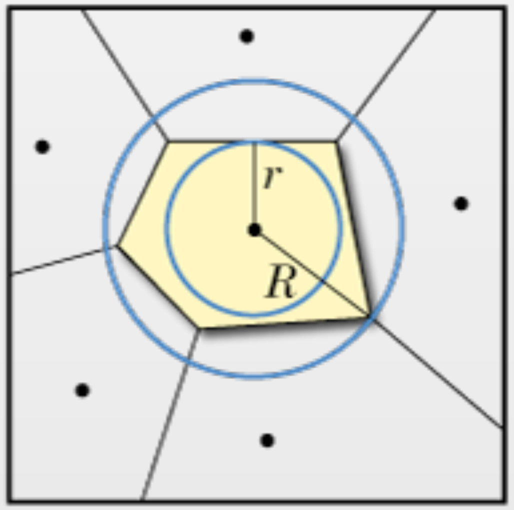


$$\frac{R}{r} < \text{constant}$$

Voronoi Aspect Ratio

Ball Packing

$$r \geq 1, R < 2$$



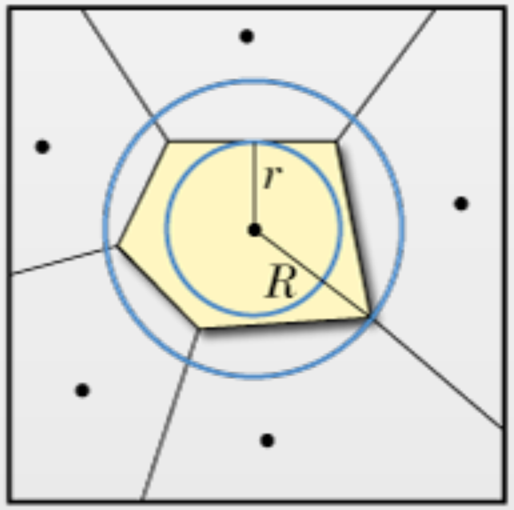
Meshing

$$\frac{R}{r} < \text{constant}$$

Voronoi Aspect Ratio

Ball Packing

$$r \geq 1, R < 2$$



Meshing

$$\frac{R}{r} < \text{constant}$$

Good Aspect Ratio Voronoi Diagram

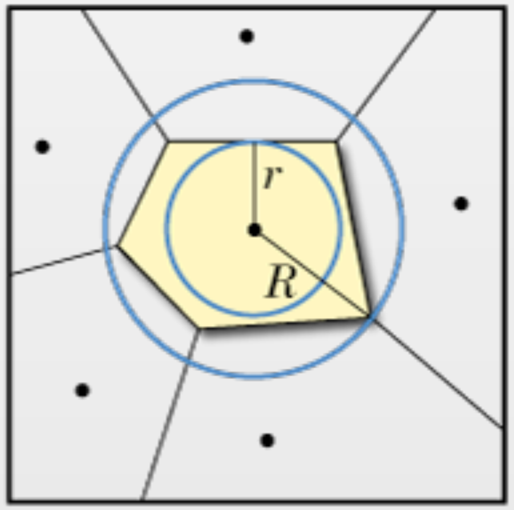
⇩

Good Quality Delaunay Triangulation

Voronoi Aspect Ratio

Ball Packing

$$r \geq 1, R < 2$$



Meshing

$$\frac{R}{r} < \text{constant}$$

Good Aspect Ratio Voronoi Diagram

⇩

Good Quality Delaunay Triangulation

Voronoi Refinement Meshing:

While any Voronoi cell has "bad" aspect ratio, add its farthest corner.

The Local Feature Size

P: input
M: output

$f(x)$ = distance to second nearest point of P.

The Local Feature Size

P: input
M: output

$f(x)$ = distance to second nearest point of P.

Key fact about Greedy Voronoi refinement:

$$r_v \leq f(v) \leq k r_v$$

for all vertices v in M.

The Local Feature Size

P: input
M: output

f(x) = distance to second nearest point of *P*.

Key fact about Greedy Voronoi refinement:

$$r_v \leq f(v) \leq kr_v$$

for all vertices *v* in *M*.

The local feature size tells us how big the ball for *v* is.

The Upper Bound

The Upper Bound

Theorem: $|M| < \frac{(k+1)^d}{\Gamma_d} \int_B \frac{1}{f(x)^d} dx.$

The Upper Bound

Theorem: $|M| < \frac{(k+1)^d}{\Gamma_d} \int_B \frac{1}{f(x)^d} dx.$

proof.

The Upper Bound

Theorem: $|M| < \frac{(k+1)^d}{\Gamma_d} \int_B \frac{1}{f(x)^d} dx.$

proof.

$$\int_B \frac{1}{f(x)^d} dx = \sum_{v \in M} \int_{\mathrm{Vor}(v)} \frac{1}{f(x)^d} dx$$

The Upper Bound

Theorem: $|M| < \frac{(k+1)^d}{\Gamma_d} \int_B \frac{1}{f(x)^d} dx.$

proof.

$$\int_B \frac{1}{f(x)^d} dx = \sum_{v \in M} \int_{\mathrm{Vor}(v)} \frac{1}{f(x)^d} dx$$

$$> \sum_{v \in M} \int_{b_v} \frac{1}{f(x)^d} dx$$

The Upper Bound

Theorem: $|M| < \frac{(k+1)^d}{\Gamma_d} \int_B \frac{1}{f(x)^d} dx.$

proof.

$$\int_B \frac{1}{f(x)^d} dx = \sum_{v \in M} \int_{\mathrm{Vor}(v)} \frac{1}{f(x)^d} dx$$

$$> \sum_{v \in M} \int_{b_v} \frac{1}{f(x)^d} dx$$

Recall: $f(v) \leq k r_v$.

The Upper Bound

Theorem: $|M| < \frac{(k+1)^d}{\Gamma_d} \int_B \frac{1}{f(x)^d} dx$.

proof.

$$\int_B \frac{1}{f(x)^d} dx = \sum_{v \in M} \int_{\mathrm{Vor}(v)} \frac{1}{f(x)^d} dx$$

$$> \sum_{v \in M} \int_{b_v} \frac{1}{f(x)^d} dx$$

Recall: $f(v) \leq k r_v$.
Claim: If $x \in b_v$,
then $f(x) \leq (k+1) r_v$.

The Upper Bound

Theorem: $|M| < \frac{(k+1)^d}{\Gamma_d} \int_B \frac{1}{f(x)^d} dx$.

proof.

$$\int_B \frac{1}{f(x)^d} dx = \sum_{v \in M} \int_{\mathrm{Vor}(v)} \frac{1}{f(x)^d} dx$$

$$> \sum_{v \in M} \int_{b_v} \frac{1}{f(x)^d} dx$$

$$\geq \sum_{v \in M} \int_{b_v} \frac{1}{((k+1)r_v)^d} dx$$

Recall: $f(v) \leq k r_v$.
Claim: If $x \in b_v$,
then $f(x) \leq (k+1) r_v$.

The Upper Bound

Theorem: $|M| < \frac{(k+1)^d}{\Gamma_d} \int_B \frac{1}{f(x)^d} dx.$

proof.

$$
\begin{aligned}
\int_B \frac{1}{f(x)^d} dx &= \sum_{v \in M} \int_{\mathrm{Vor}(v)} \frac{1}{f(x)^d} dx \\
&> \sum_{v \in M} \int_{b_v} \frac{1}{f(x)^d} dx \\
&\geq \sum_{v \in M} \int_{b_v} \frac{1}{((k+1)r_v)^d} dx \\
&= \sum_{v \in M} \frac{r_v^d \Gamma_d}{((k+1)r_v)^d}
\end{aligned}
$$

Recall: $f(v) \leq k r_v$.
Claim: If $x \in b_v$,
then $f(x) \leq (k+1) r_v$.

The Upper Bound

Theorem: $|M| < \frac{(k+1)^d}{\Gamma_d} \int_B \frac{1}{f(x)^d} dx.$

proof.

$$\begin{aligned}
\int_B \frac{1}{f(x)^d} dx &= \sum_{v \in M} \int_{\mathrm{Vor}(v)} \frac{1}{f(x)^d} dx \\
&> \sum_{v \in M} \int_{b_v} \frac{1}{f(x)^d} dx \\
&\geq \sum_{v \in M} \int_{b_v} \frac{1}{((k+1)r_v)^d} dx \\
&= \sum_{v \in M} \frac{r_v^d \Gamma_d}{((k+1)r_v)^d} \\
&= |M| \frac{\Gamma_d}{(k+1)^d}.
\end{aligned}$$

Recall: $f(v) \leq k r_v$.
Claim: If $x \in b_v$,
then $f(x) \leq (k+1) r_v$.

The Lower Bound

The Lower Bound

Theorem: $|M| > \frac{1}{k^d \Gamma_d} \int_B \frac{1}{f(x)^d} dx.$

The Lower Bound

Theorem: $|M| > \frac{1}{k^d \Gamma_d} \int_B \frac{1}{f(x)^d} dx.$

proof.

The Lower Bound

Theorem: $|M| > \frac{1}{k^d \Gamma_d} \int_B \frac{1}{f(x)^d} dx.$

proof.

$$\int_B \frac{1}{f(x)^d} dx = \sum_{v \in M} \int_{\mathrm{Vor}(v)} \frac{1}{f(x)^d} dx$$

The Lower Bound

Theorem: $|M| > \frac{1}{k^d \Gamma_d} \int_B \frac{1}{f(x)^d} dx.$

proof.

$$\int_B \frac{1}{f(x)^d} dx = \sum_{v \in M} \int_{\mathrm{Vor}(v)} \frac{1}{f(x)^d} dx$$

Claim: If $x \in \mathrm{Vor}(v)$ then $f(x) \geq r_v$.

The Lower Bound

Theorem: $|M| > \frac{1}{k^d \Gamma_d} \int_B \frac{1}{f(x)^d} dx.$

proof.

$$\int_B \frac{1}{f(x)^d} dx = \sum_{v \in M} \int_{\mathrm{Vor}(v)} \frac{1}{f(x)^d} dx$$

$$\leq \sum_{v \in M} \int_{\mathrm{Vor}(v)} \frac{1}{r_v^d} dx$$

Claim: If $x \in \mathrm{Vor}(v)$ then $f(x) \geq r_v$.

The Lower Bound

Theorem: $|M| > \frac{1}{k^d \Gamma_d} \int_B \frac{1}{f(x)^d} dx.$

proof.

$$\int_B \frac{1}{f(x)^d} dx = \sum_{v \in M} \int_{\mathrm{Vor}(v)} \frac{1}{f(x)^d} dx$$

$$\leq \sum_{v \in M} \int_{\mathrm{Vor}(v)} \frac{1}{r_v^d} dx$$

Claim: If $x \in \mathrm{Vor}(v)$ then $f(x) \geq r_v$.

Claim: $\mathrm{Vor}(v) \subset k b_v$.

The Lower Bound

Theorem: $|M| > \frac{1}{k^d \Gamma_d} \int_B \frac{1}{f(x)^d} dx.$

proof.

$$\int_B \frac{1}{f(x)^d} dx = \sum_{v \in M} \int_{\mathrm{Vor}(v)} \frac{1}{f(x)^d} dx$$

$$\leq \sum_{v \in M} \int_{\mathrm{Vor}(v)} \frac{1}{r_v^d} dx$$

$$< \sum_{v \in M} \int_{kb_v} \frac{1}{r_v^d} dx$$

Claim: If $x \in \mathrm{Vor}(v)$ then $f(x) \geq r_v$.

Claim: $\mathrm{Vor}(v) \subset kb_v$.

The Lower Bound

Theorem: $|M| > \frac{1}{k^d \Gamma_d} \int_B \frac{1}{f(x)^d} dx.$

proof.

$$\int_B \frac{1}{f(x)^d} dx = \sum_{v \in M} \int_{\mathrm{Vor}(v)} \frac{1}{f(x)^d} dx$$

$$\leq \sum_{v \in M} \int_{\mathrm{Vor}(v)} \frac{1}{r_v^d} dx$$

$$< \sum_{v \in M} \int_{kb_v} \frac{1}{r_v^d} dx$$

$$= \sum_{v \in M} \frac{k^d r_v^d \Gamma_d}{r_v^d}$$

Claim: If $x \in \mathrm{Vor}(v)$ then $f(x) \geq r_v$.

Claim: $\mathrm{Vor}(v) \subset kb_v$.

The Lower Bound

Theorem: $|M| > \frac{1}{k^d \Gamma_d} \int_B \frac{1}{f(x)^d} dx.$

proof.

$$\begin{aligned}
\int_B \frac{1}{f(x)^d} dx &= \sum_{v \in M} \int_{\mathrm{Vor}(v)} \frac{1}{f(x)^d} dx \\
&\leq \sum_{v \in M} \int_{\mathrm{Vor}(v)} \frac{1}{r_v^d} dx \\
&< \sum_{v \in M} \int_{k b_v} \frac{1}{r_v^d} dx \\
&= \sum_{v \in M} \frac{k^d r_v^d \Gamma_d}{r_v^d} \\
&= |M| k^d \Gamma_d.
\end{aligned}$$

Claim: If $x \in \mathrm{Vor}(v)$ then $f(x) \geq r_v$.

Claim: $\mathrm{Vor}(v) \subset k b_v$.

Degree Bounds

How many neighbors can any point have?

Degree Bounds

How many neighbors can any point have?

$$\sum_{q \in Q} \mathrm{vol}(b_q) > \sum_{q \in Q} \mathrm{vol}\left(\frac{b_p}{k}\right) = |Q| \left(\frac{r_p}{k}\right)^d \Gamma_d.$$

Degree Bounds

How many neighbors can any point have?

$$\sum_{q \in Q} \mathrm{vol}(b_q) > \sum_{q \in Q} \mathrm{vol}\left(\frac{b_p}{k}\right) = |Q| \left(\frac{r_p}{k}\right)^d \Gamma_d.$$

$$\sum_{q \in Q} \mathrm{vol}(b_q) < \mathrm{vol}(3kb_p) = (3kr_p)^d \Gamma_d.$$

Degree Bounds

How many neighbors can any point have?

$$\sum_{q\in Q} \mathrm{vol}(b_q) > \sum_{q\in Q} \mathrm{vol}\left(\frac{b_p}{k}\right) = |Q|\left(\frac{r_p}{k}\right)^d \Gamma_d.$$

$$\sum_{q\in Q} \mathrm{vol}(b_q) < \mathrm{vol}(3kb_p) = (3kr_p)^d \Gamma_d.$$

$$\Rightarrow |Q| < 3^d$$

Degree Bounds

How many neighbors can any point have?

$$\sum_{q\in Q} \mathrm{vol}(b_q) > \sum_{q\in Q} \mathrm{vol}\left(\frac{b_p}{k}\right) = |Q| \left(\frac{r_p}{k}\right)^d \Gamma_d.$$

$$\sum_{q\in Q} \mathrm{vol}(b_q) < \mathrm{vol}(3kb_p) = (3kr_p)^d \Gamma_d.$$

$$\Rightarrow |Q| < 3^d$$

Total Voronoi Edges < $3^d |M|$

Degree Bounds

How many neighbors can any point have?

$$\sum_{q \in Q} \mathrm{vol}(b_q) > \sum_{q \in Q} \mathrm{vol}\left(\frac{b_p}{k}\right) = |Q| \left(\frac{r_p}{k}\right)^d \Gamma_d.$$

$$\sum_{q \in Q} \mathrm{vol}(b_q) < \mathrm{vol}(3kb_p) = (3kr_p)^d \Gamma_d.$$

$$\Rightarrow |Q| < 3^d$$

Same tricks as before!
(Packing Argument)

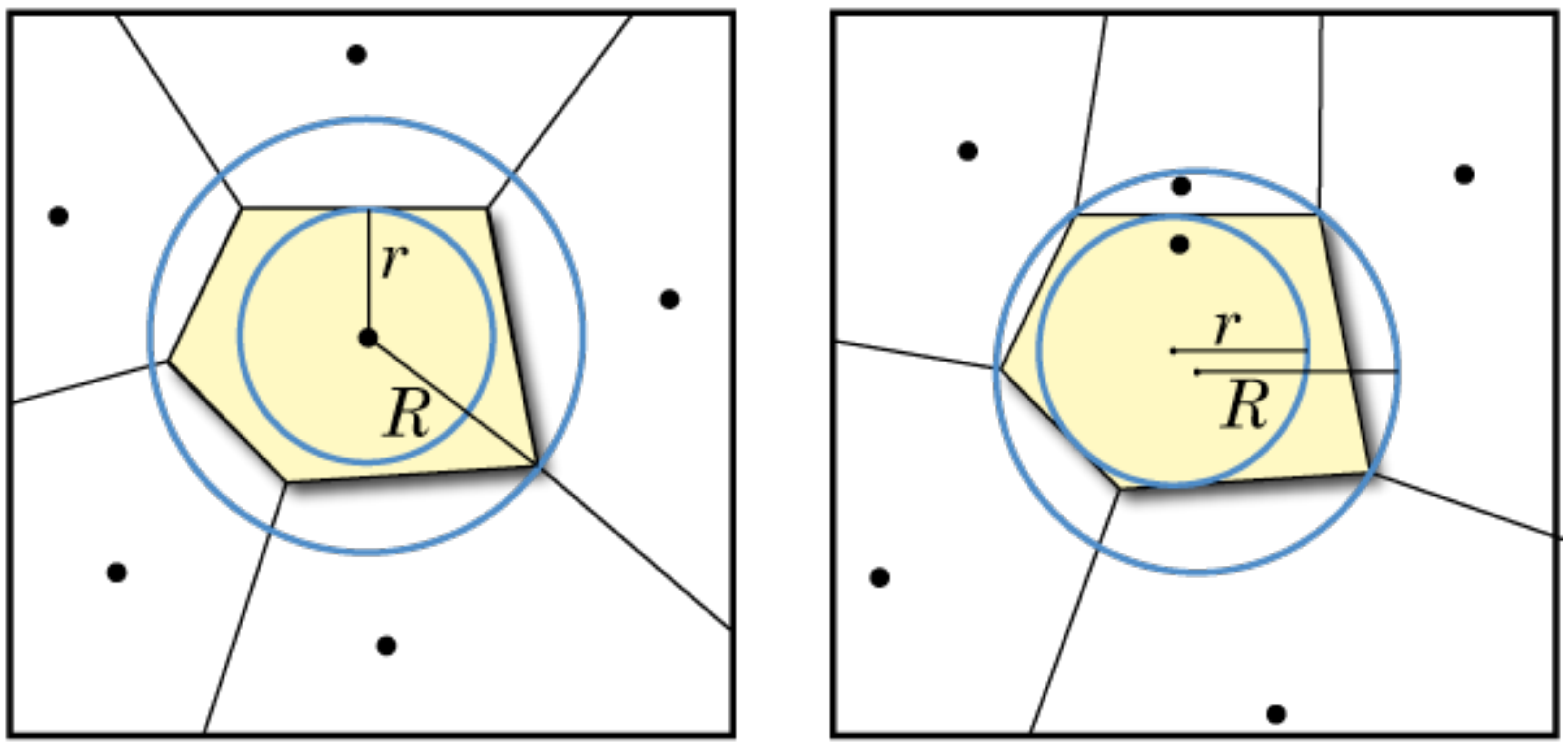
Total Voronoi Edges $< 3^d |M|$

Fat Voronoi Diagrams

Idea: Let the center shift.

Fat Voronoi Diagrams

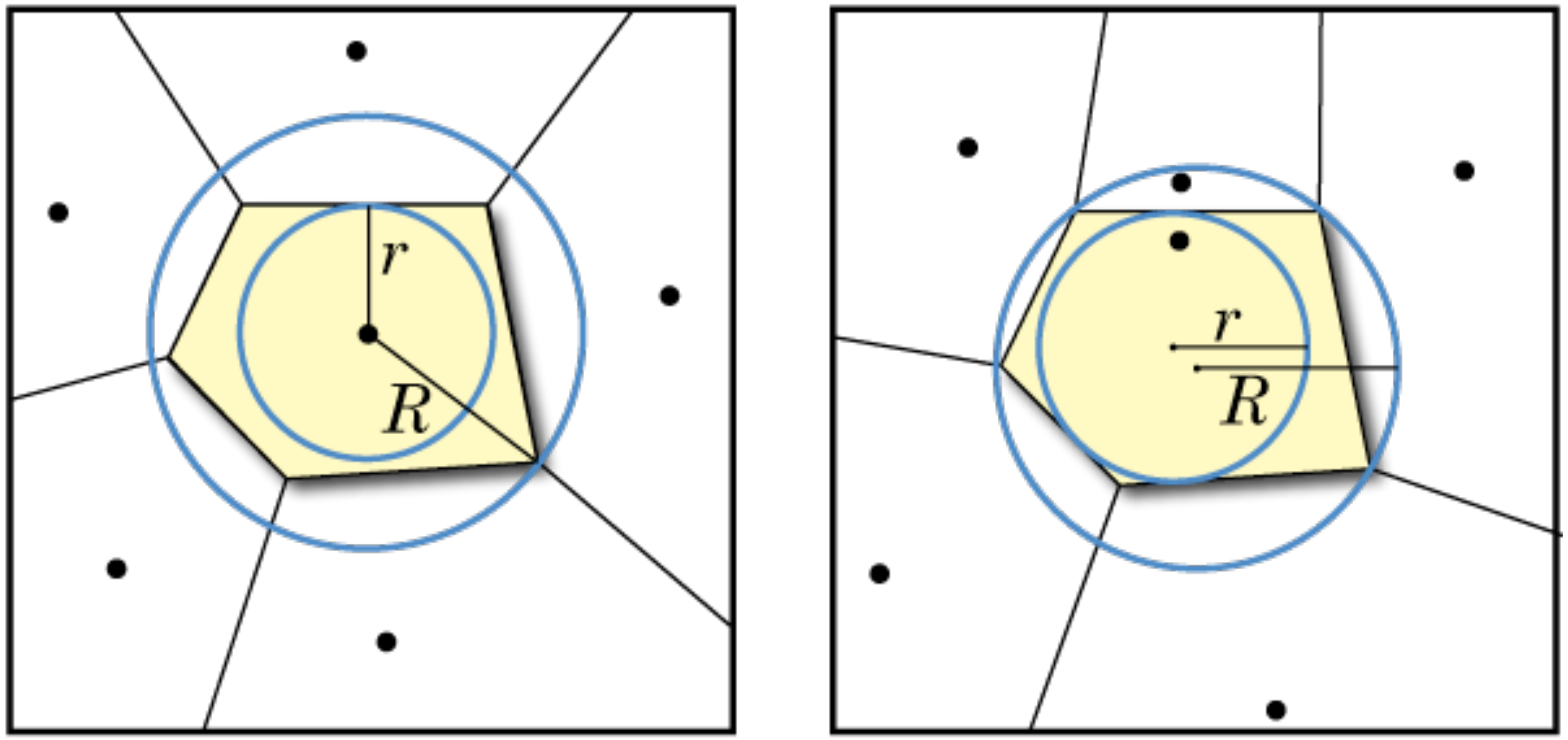
Idea: Let the center shift.



Good Aspect Ratio   Fat

Fat Voronoi Diagrams

Idea: Let the center shift.



Good Aspect Ratio   Fat

Fat Voronoi Conjecture:

The number of neighbors of any cell in a fat Voronoi diagram is $2^{O(d)}$.

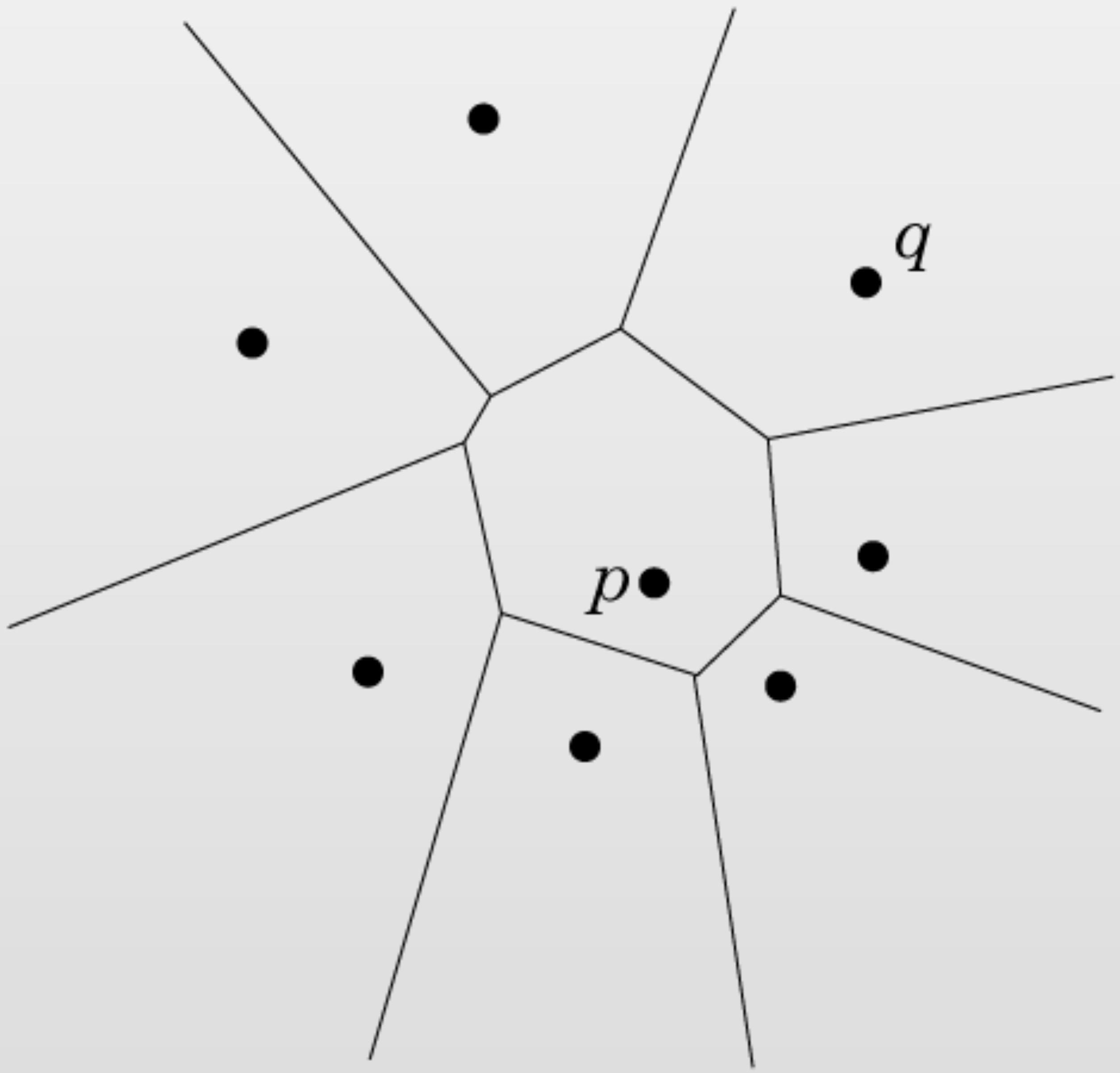
Fat Voronoi diagrams in the plane.

Fat Voronoi diagrams in the plane.

Here’s the trick.

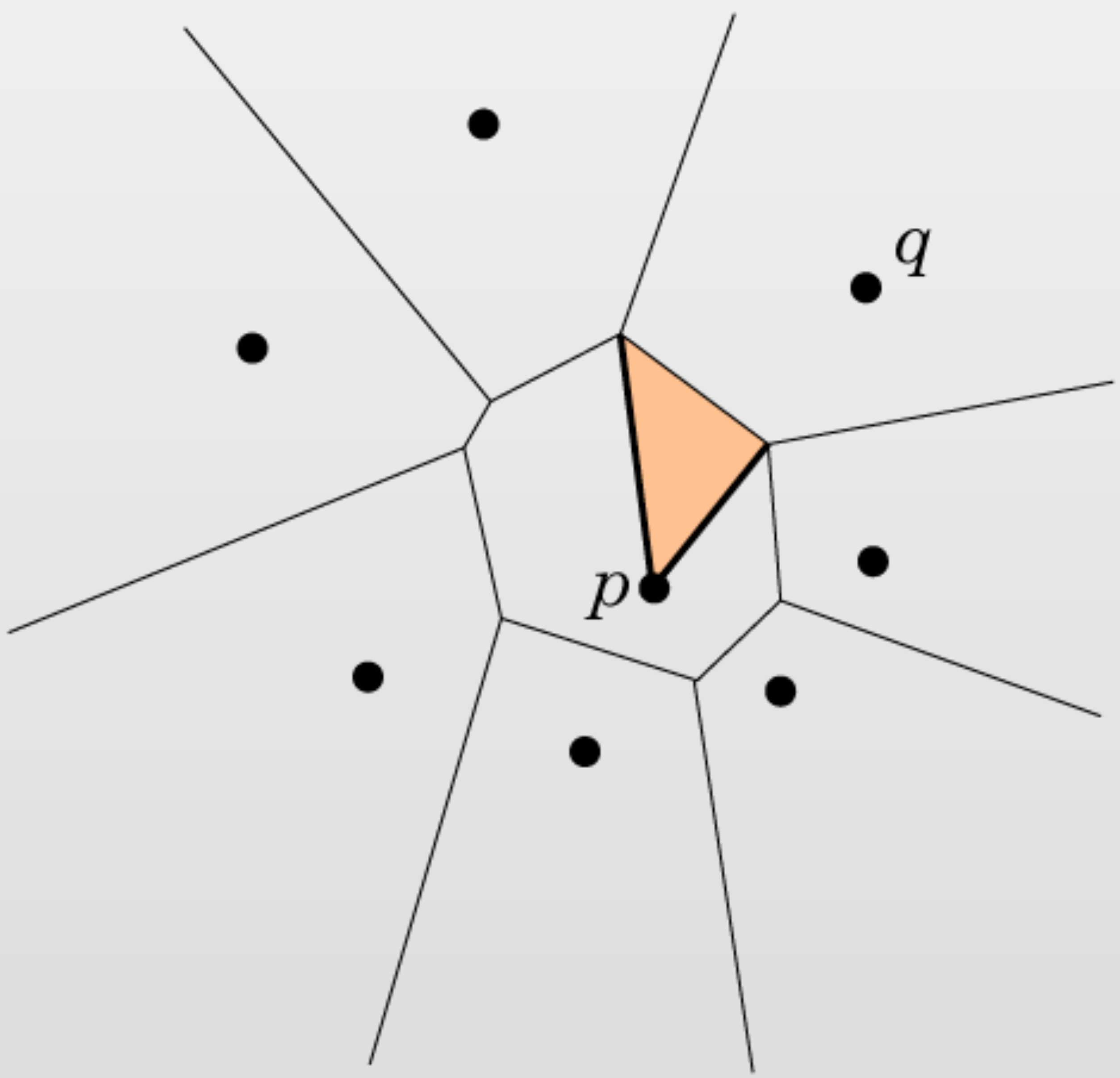
Fat Voronoi diagrams in the plane.

Here's the trick.



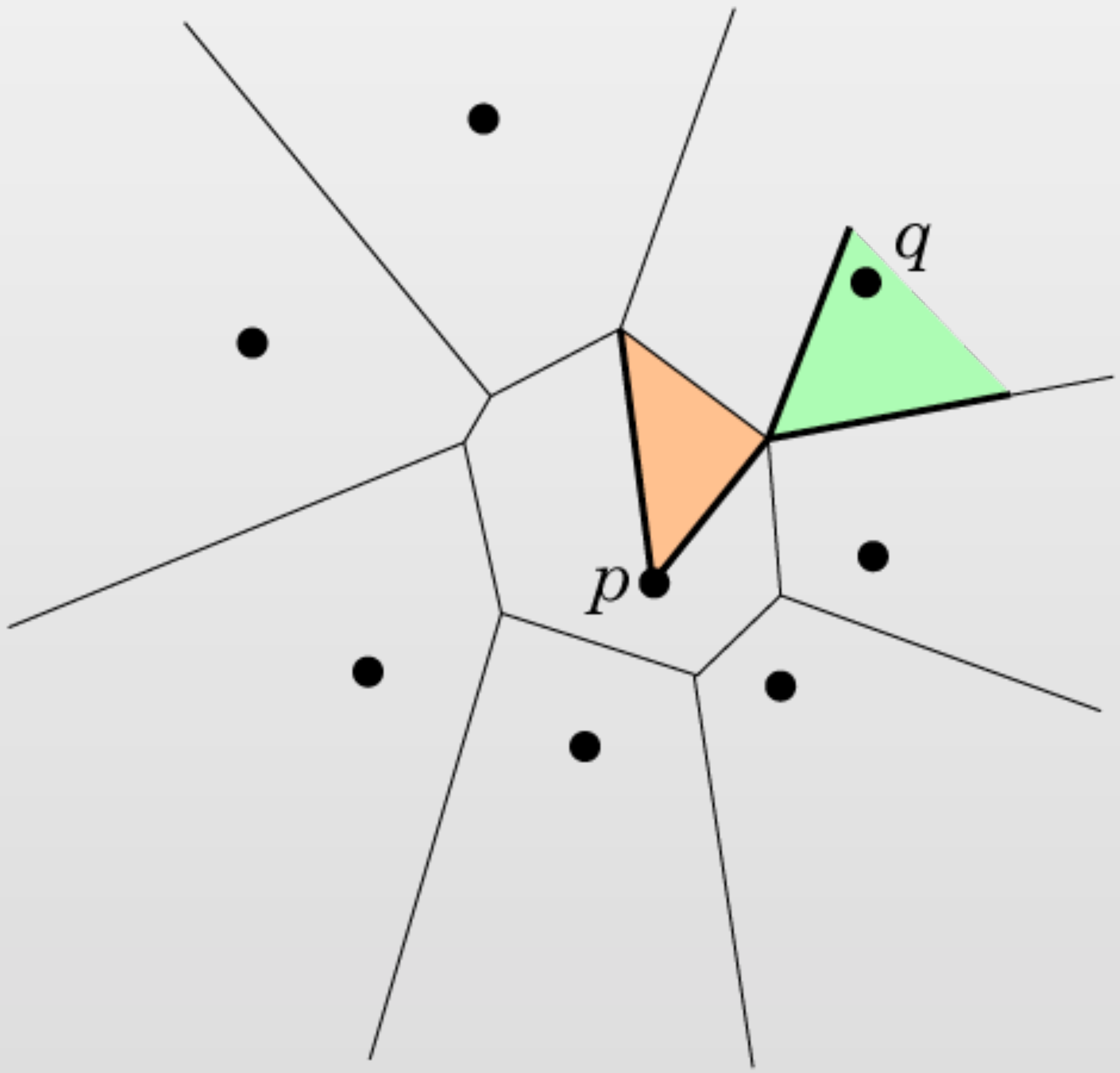
Fat Voronoi diagrams in the plane.

Here's the trick.



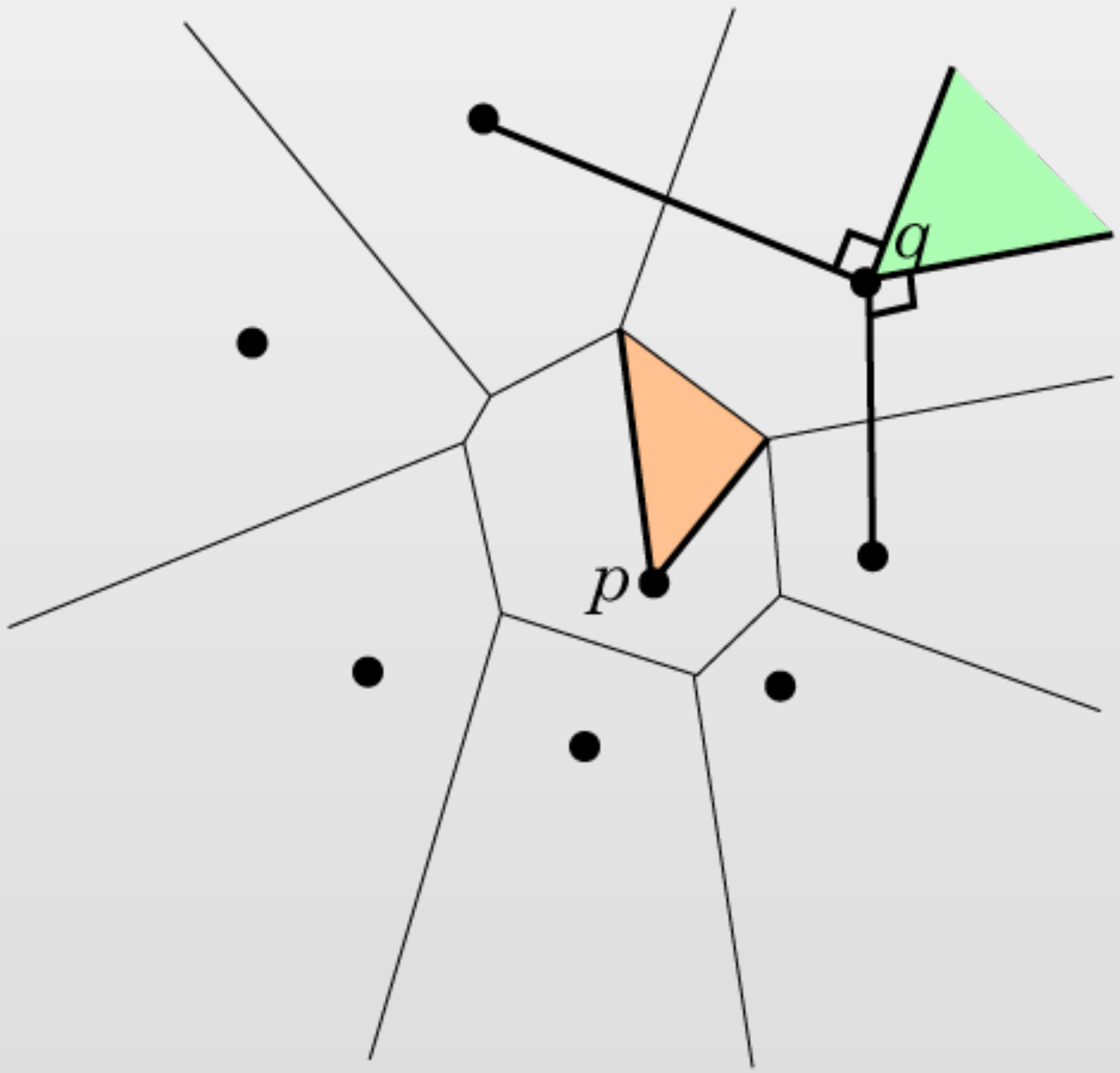
Fat Voronoi diagrams in the plane.

Here's the trick.



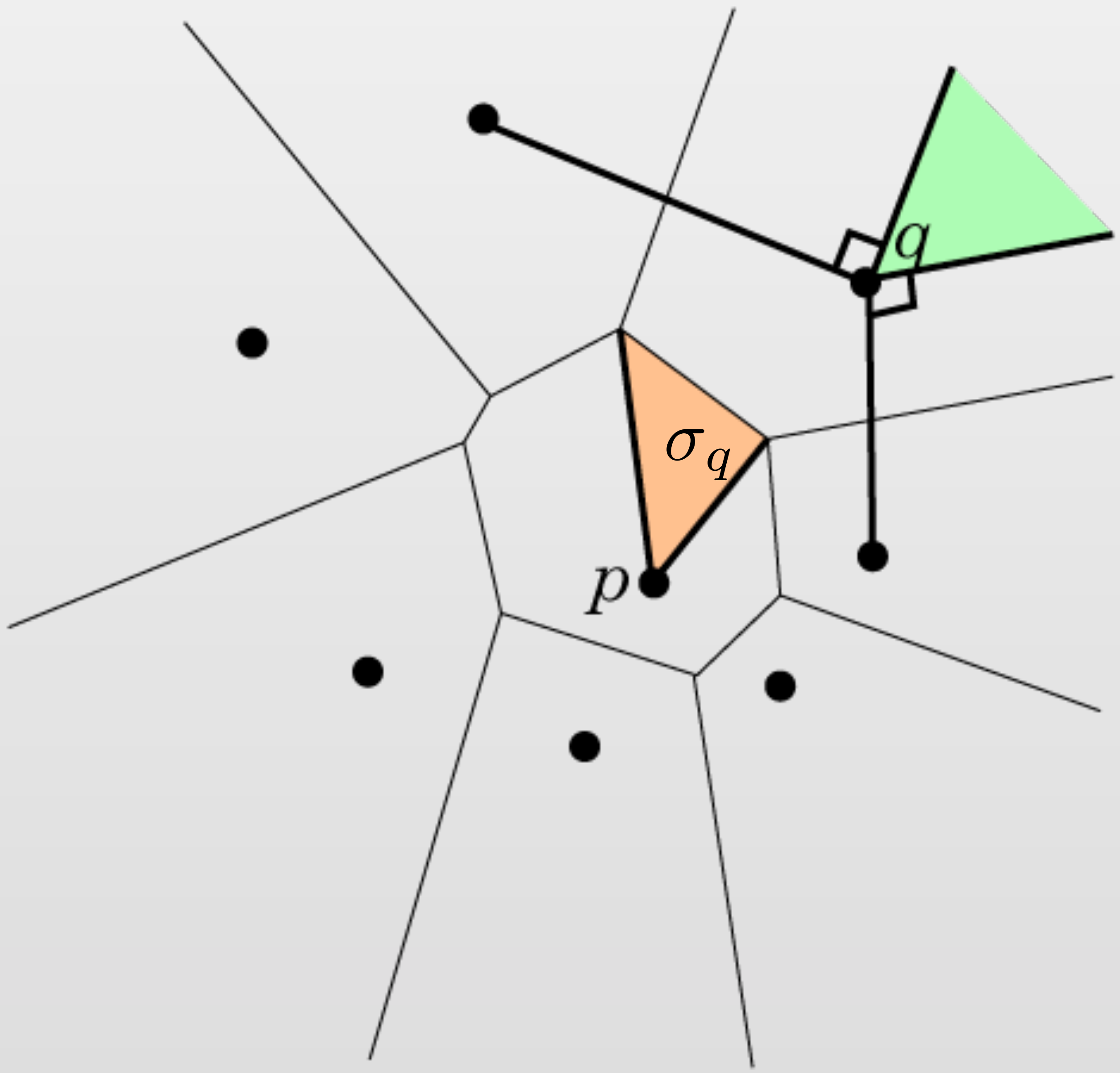
Fat Voronoi diagrams in the plane.

Here's the trick.



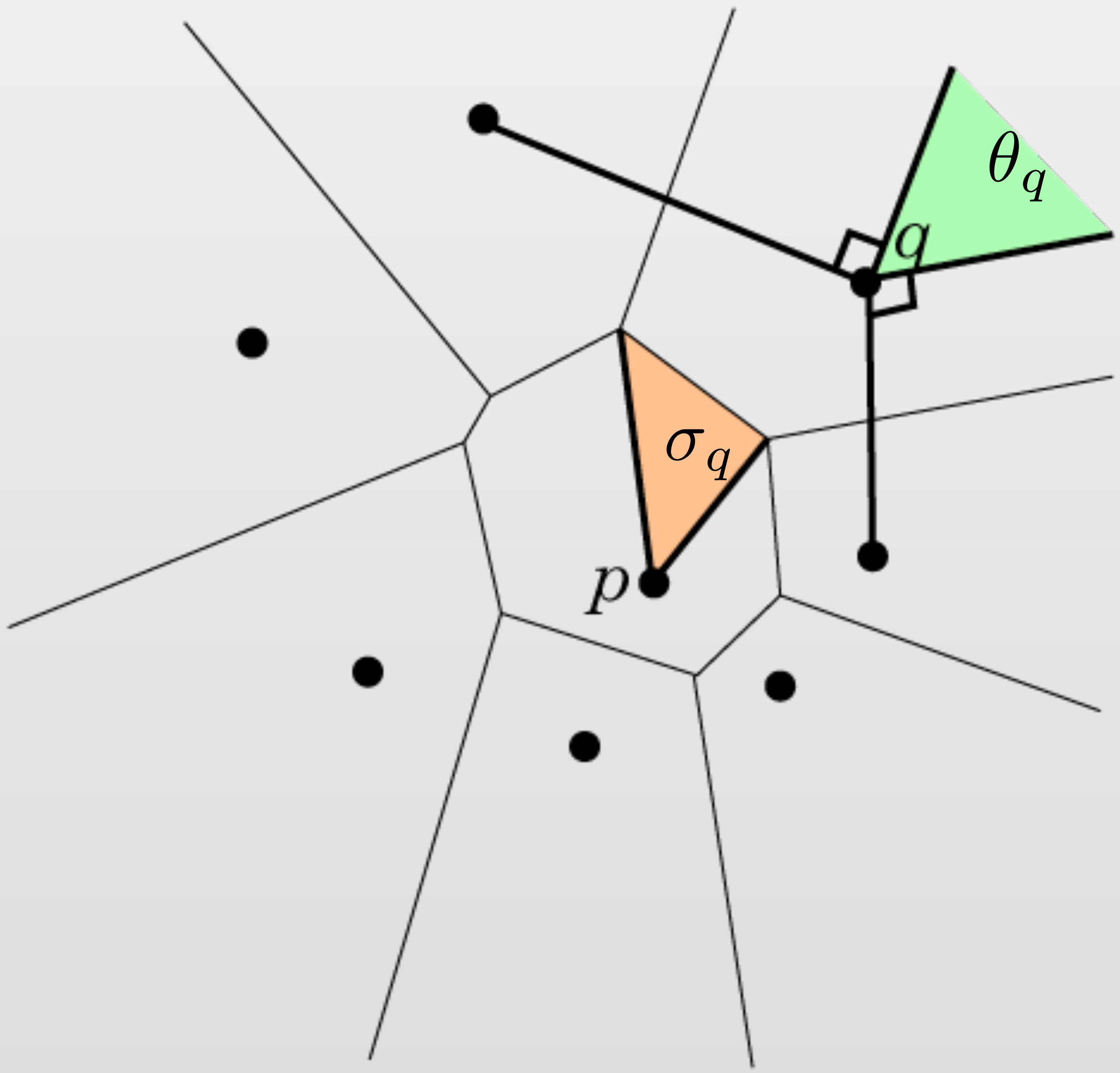
Fat Voronoi diagrams in the plane.

Here's the trick.



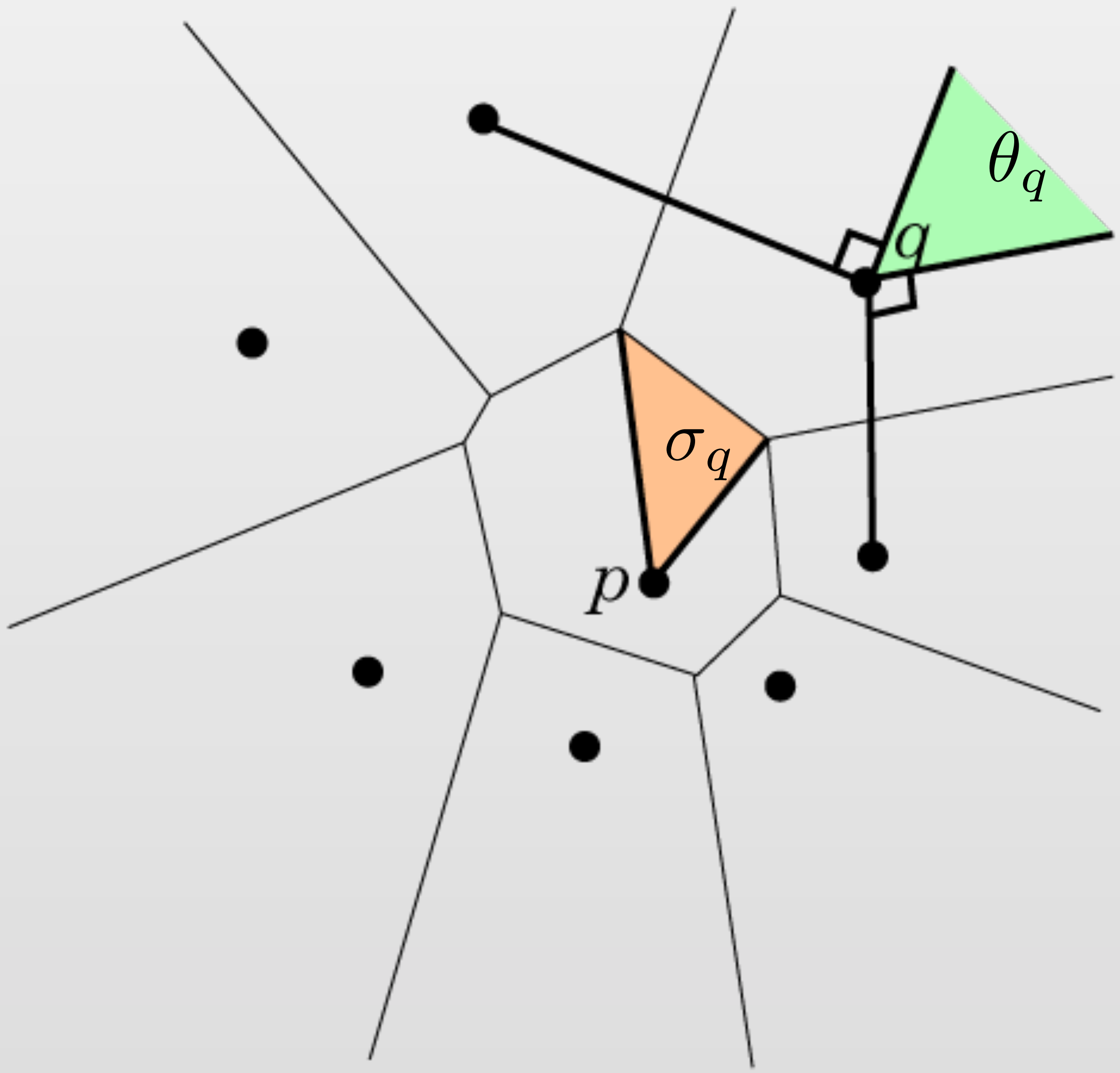
Fat Voronoi diagrams in the plane.

Here's the trick.



Fat Voronoi diagrams in the plane.

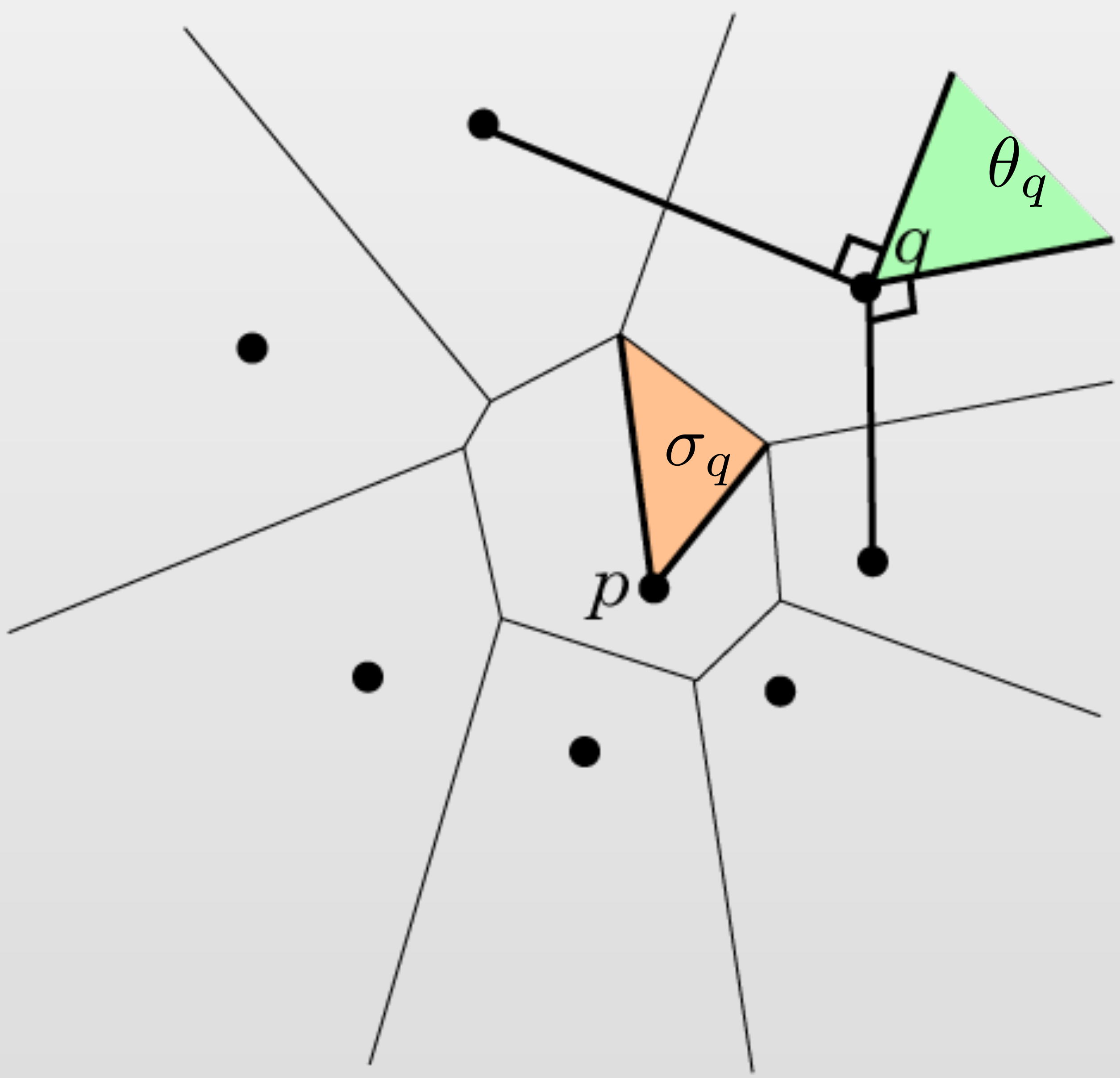
Here's the trick.



$$2\pi = \sum_{q \in Q} \sigma_q = \sum_{q \in Q} \theta_q$$

Fat Voronoi diagrams in the plane.

Here's the trick.

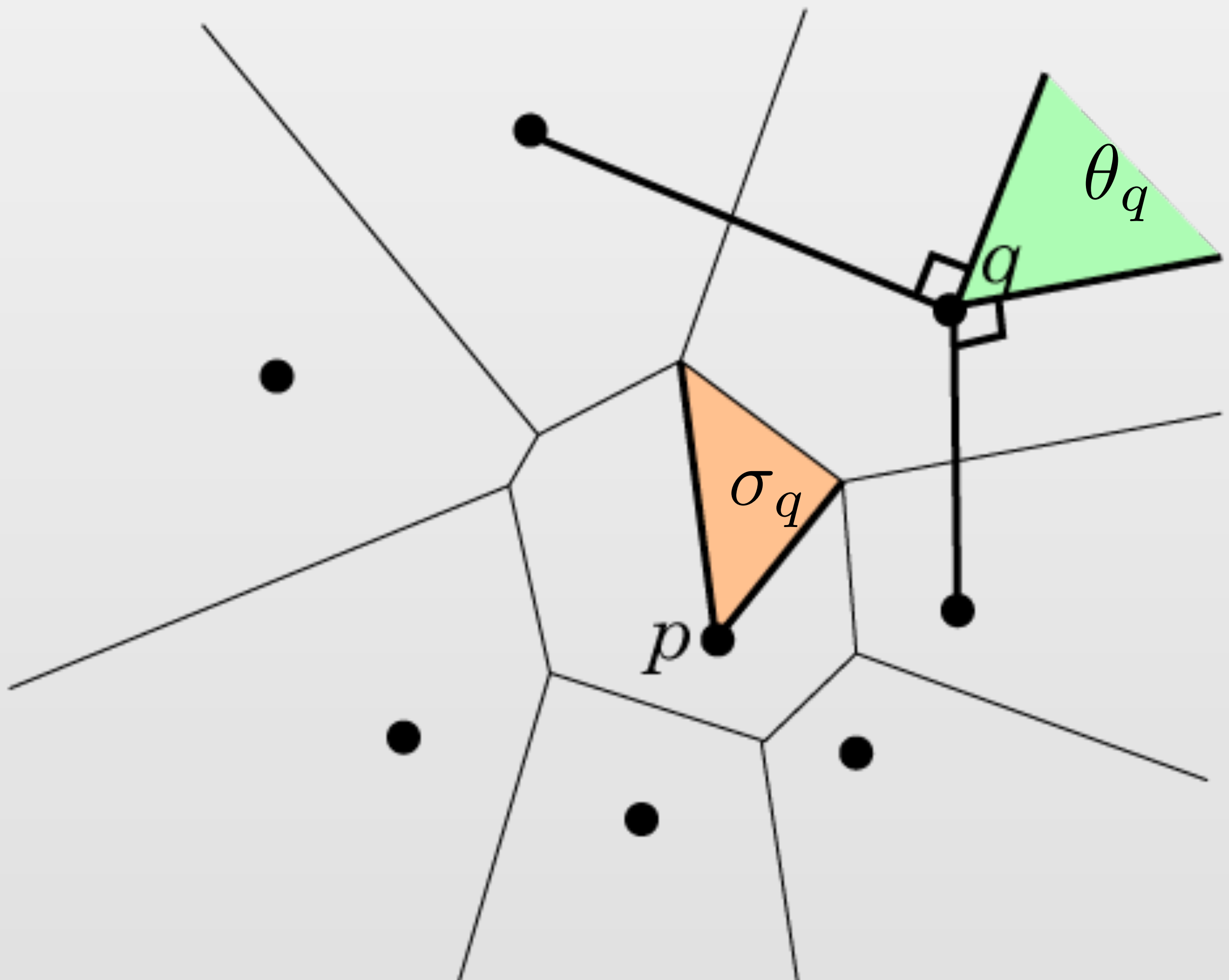


$$2\pi = \sum_{q \in Q} \sigma_q = \sum_{q \in Q} \theta_q$$

Is this new?

Fat Voronoi diagrams in the plane.

Here's the trick.



$$2\pi = \sum_{q \in Q} \sigma_q = \sum_{q \in Q} \theta_q$$

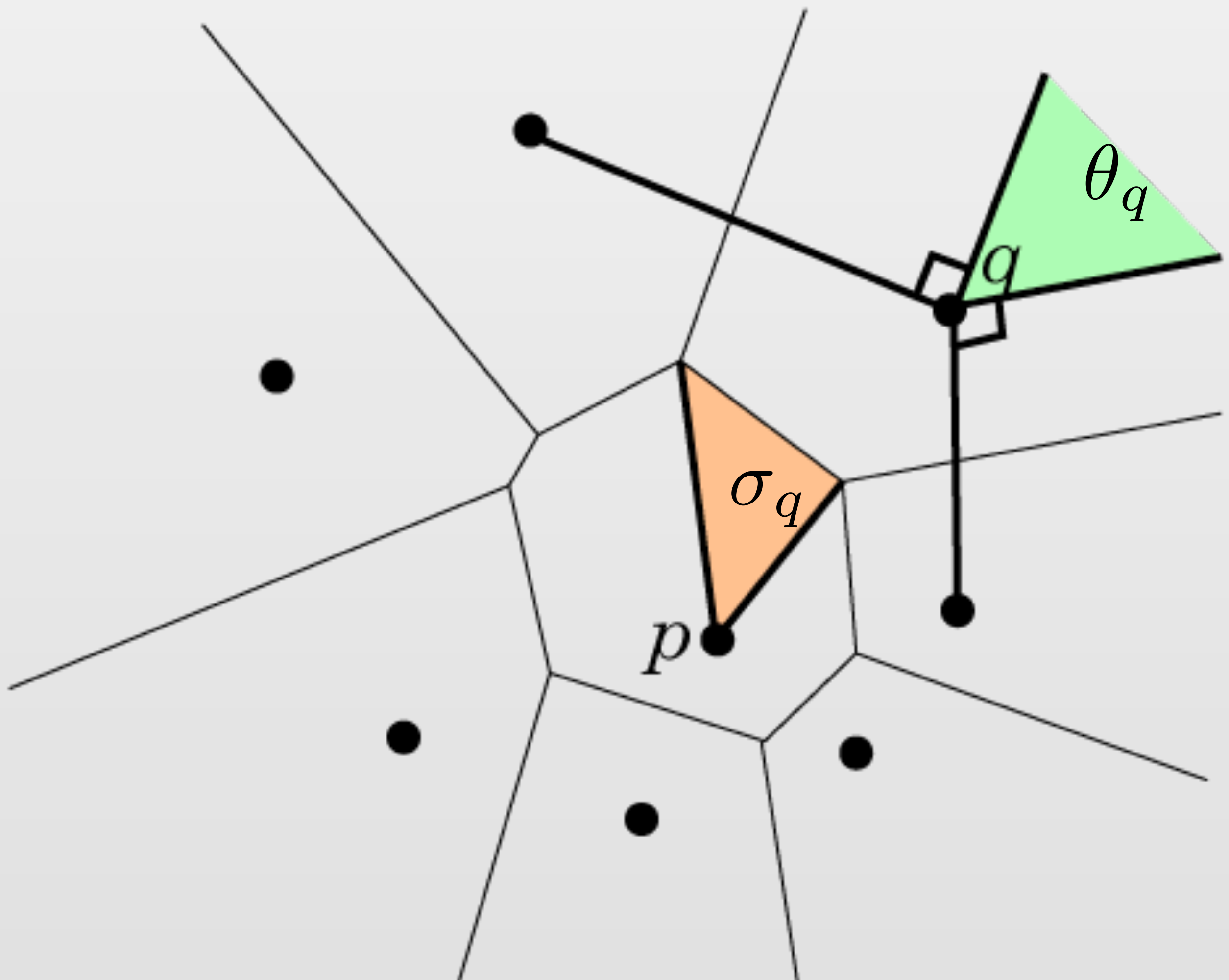
Is this new?

Punchline:

Fat Voronoi Conjecture Holds in the plane.

Fat Voronoi diagrams in the plane.

Here's the trick.



3D?

$$2\pi = \sum_{q \in Q} \sigma_q = \sum_{q \in Q} \theta_q$$

Is this new?

Punchline:

Fat Voronoi Conjecture Holds in the plane.

Thank you.