

Beating the Spread: Time-Optimal Point Meshing

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Carnegie Mellon
(soon: INRIA)

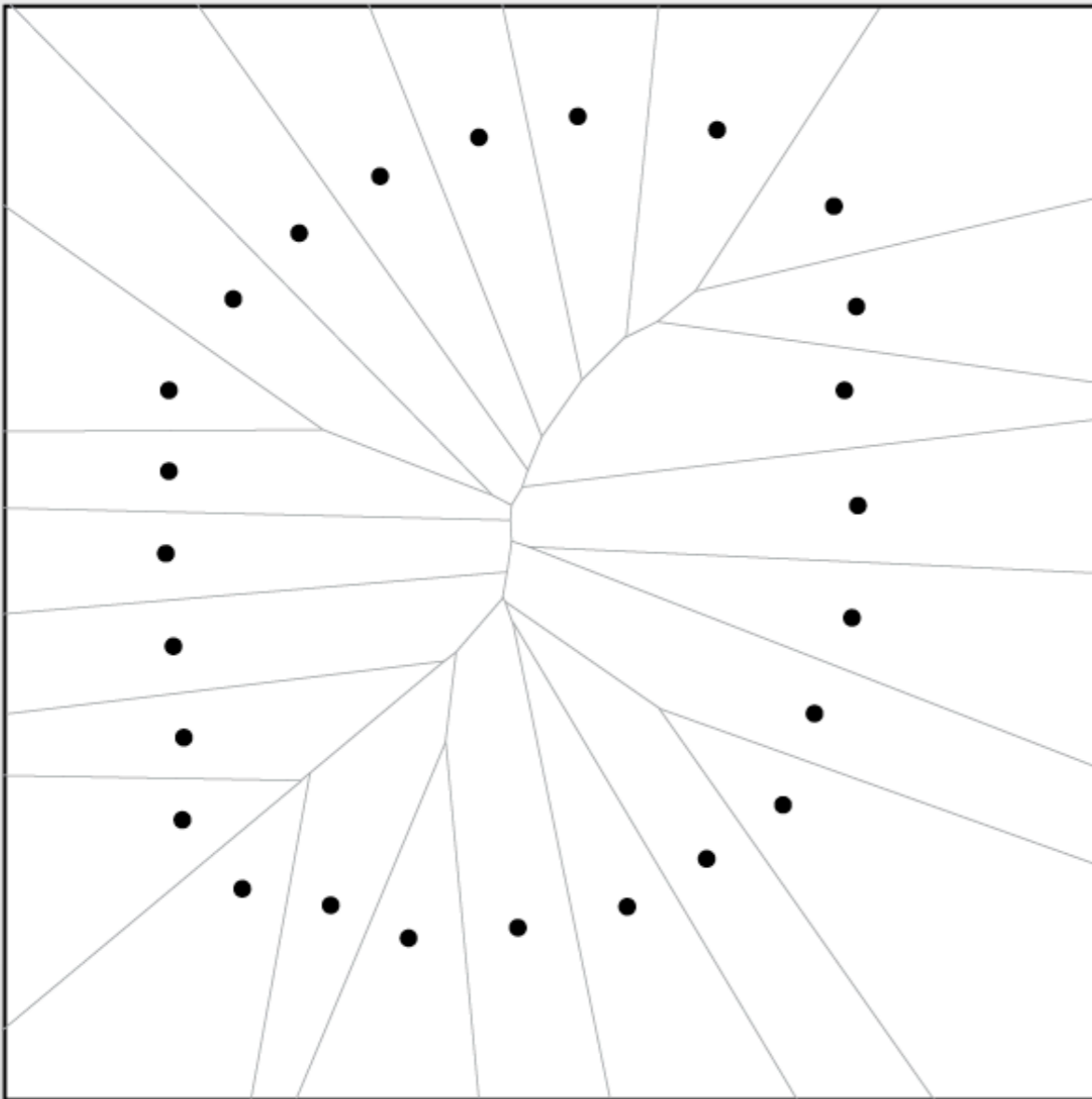
with Gary Miller and Todd Phillips at CMU

Meshing Points

Input: $P \subset \mathbb{R}^d$

Output: $M \supset P$ with a “nice” Voronoi diagram

$$n = |P|, m = |M|$$

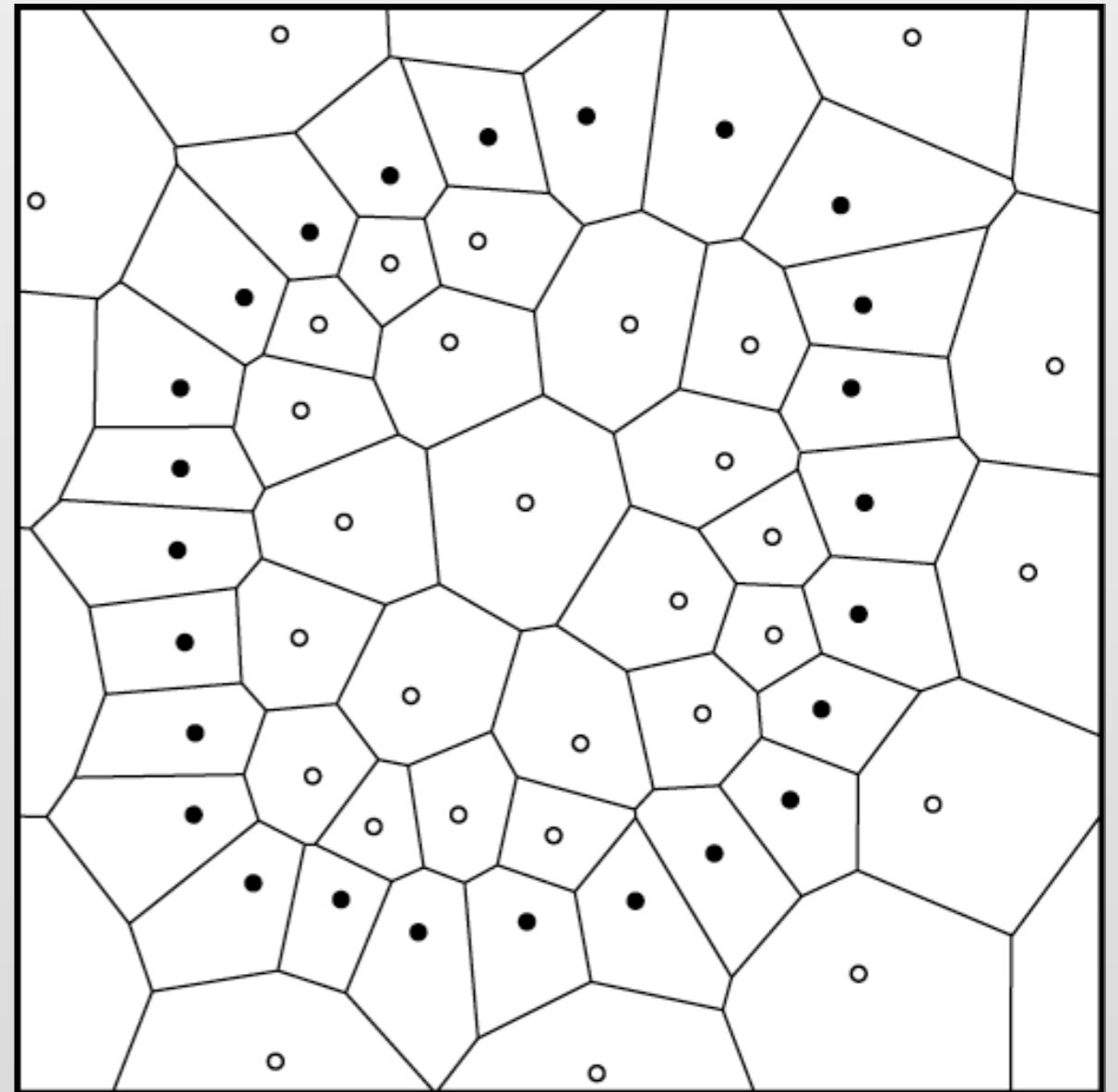
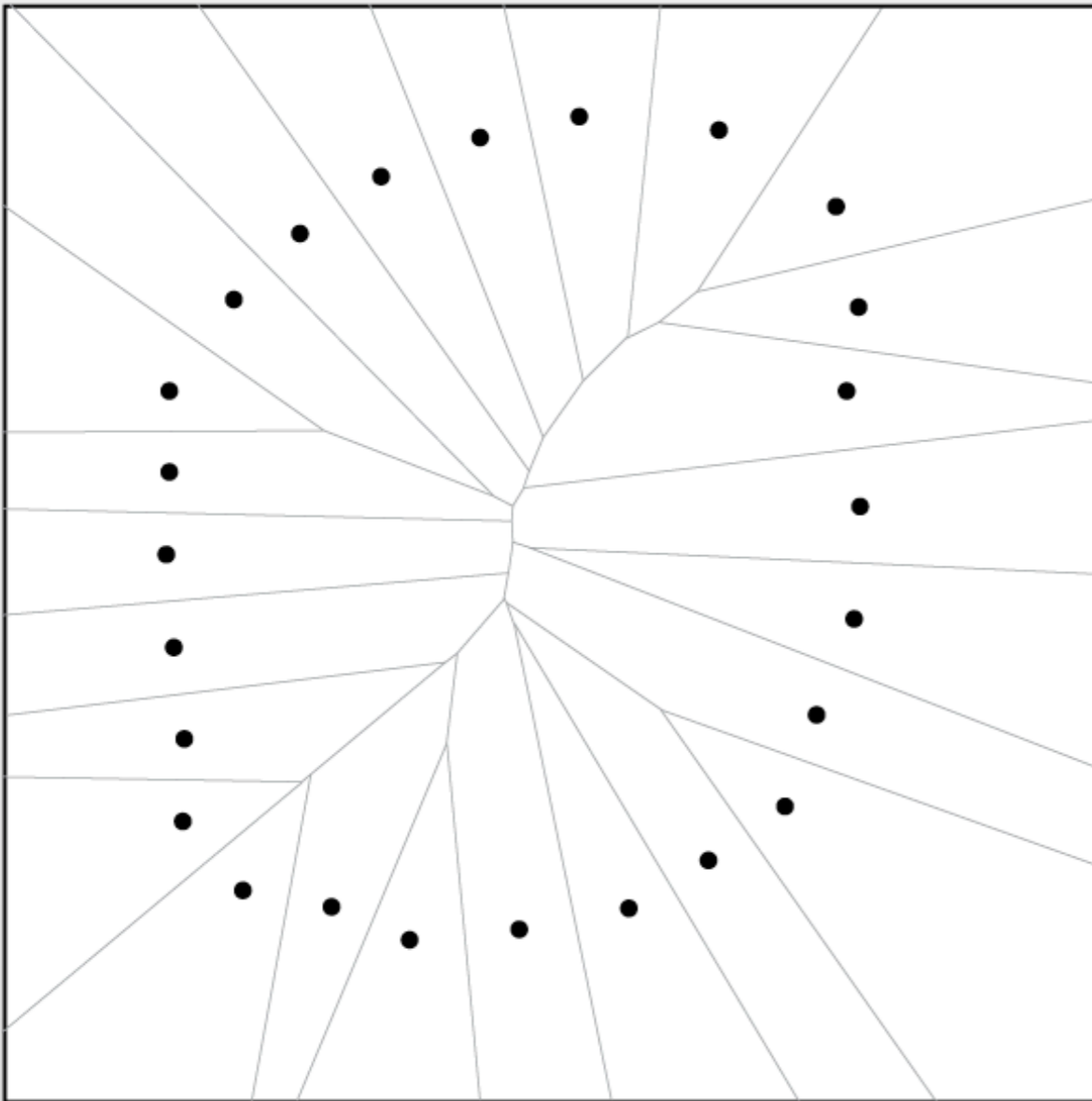


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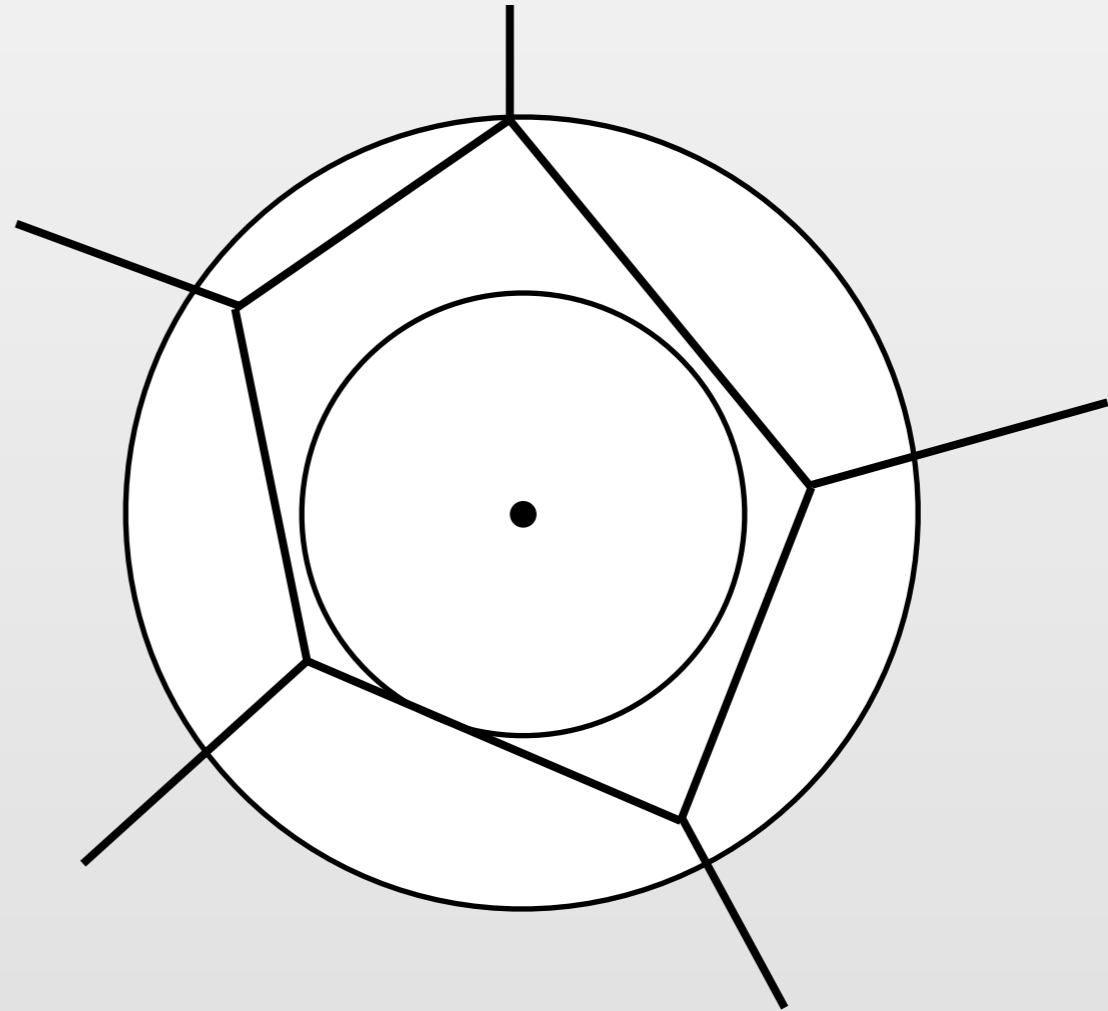
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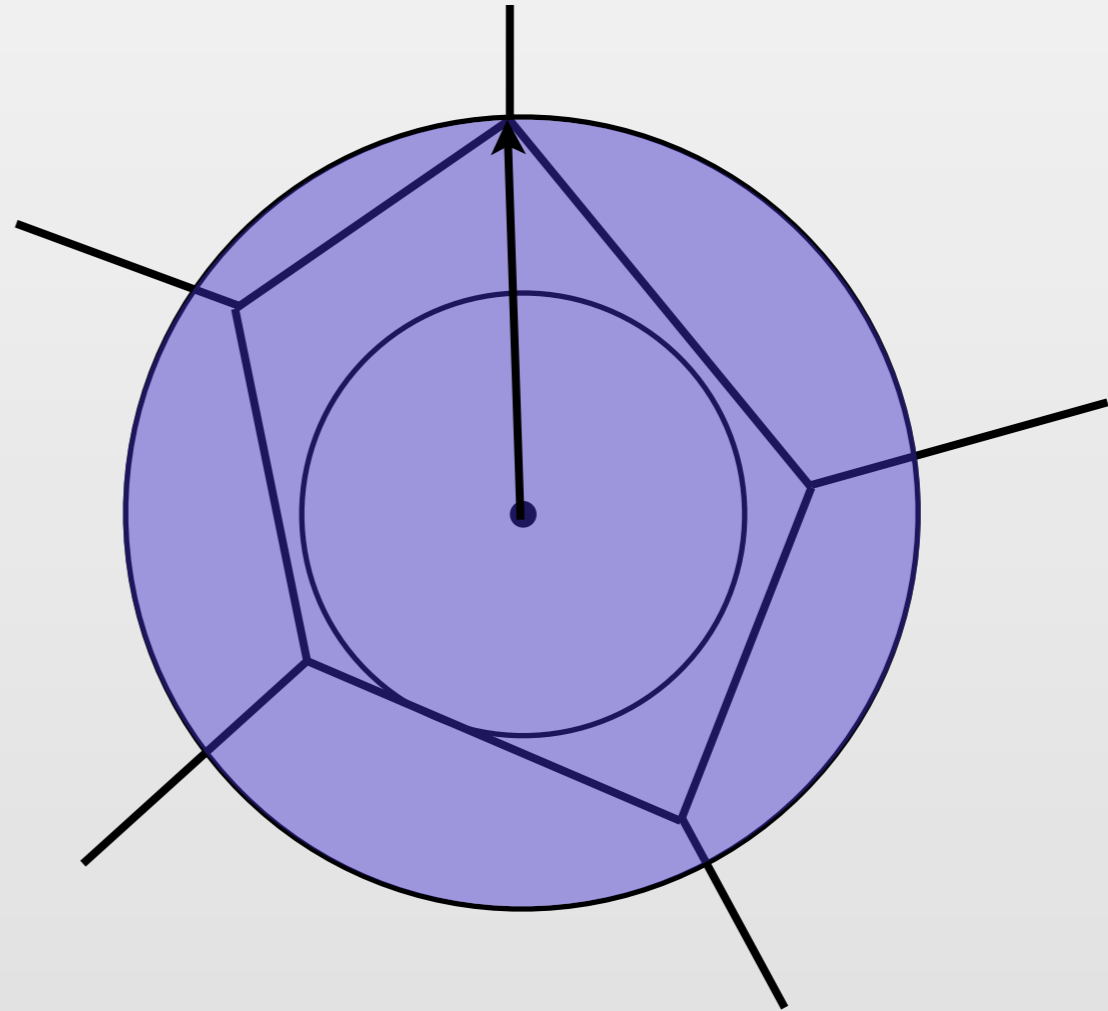
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$$\text{aspect ratio} = \frac{R}{r} \leq \tau$$



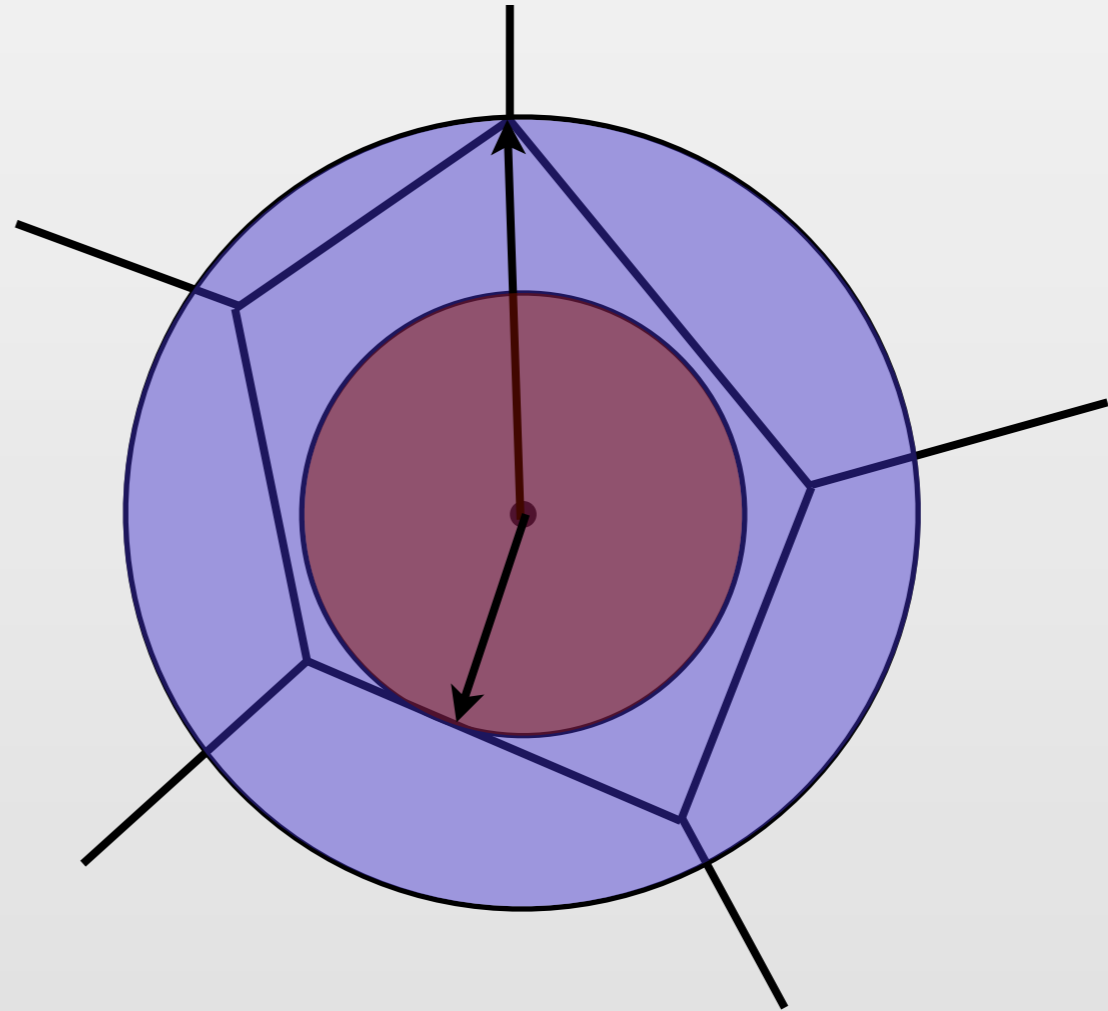
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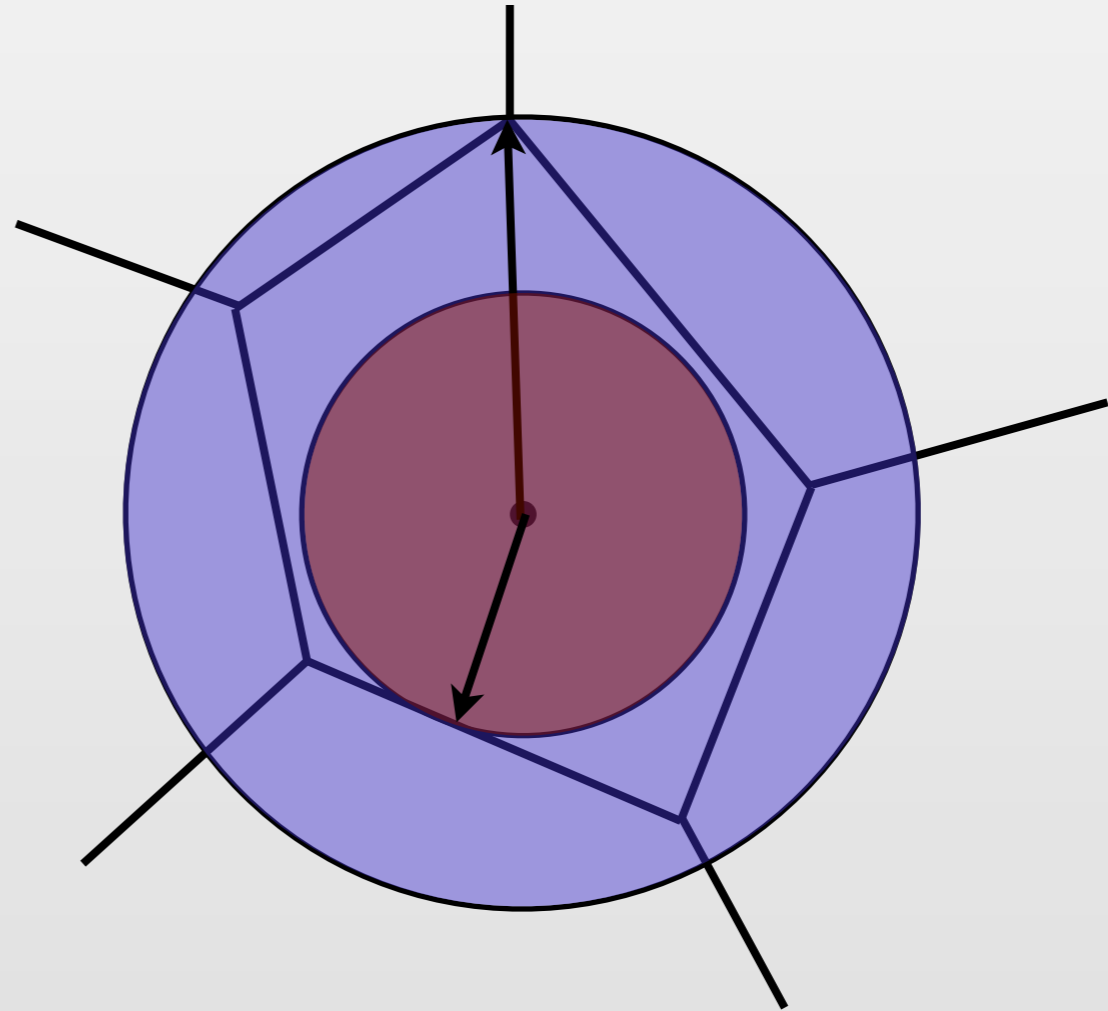
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$$\tau \geq 2 + \varepsilon$$



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Hides dimension terms

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For some point sets, $m = \Omega(n \log \Delta)$

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$$m = \int_{\Omega} \frac{1}{\text{fs}_P(x)^d}$$

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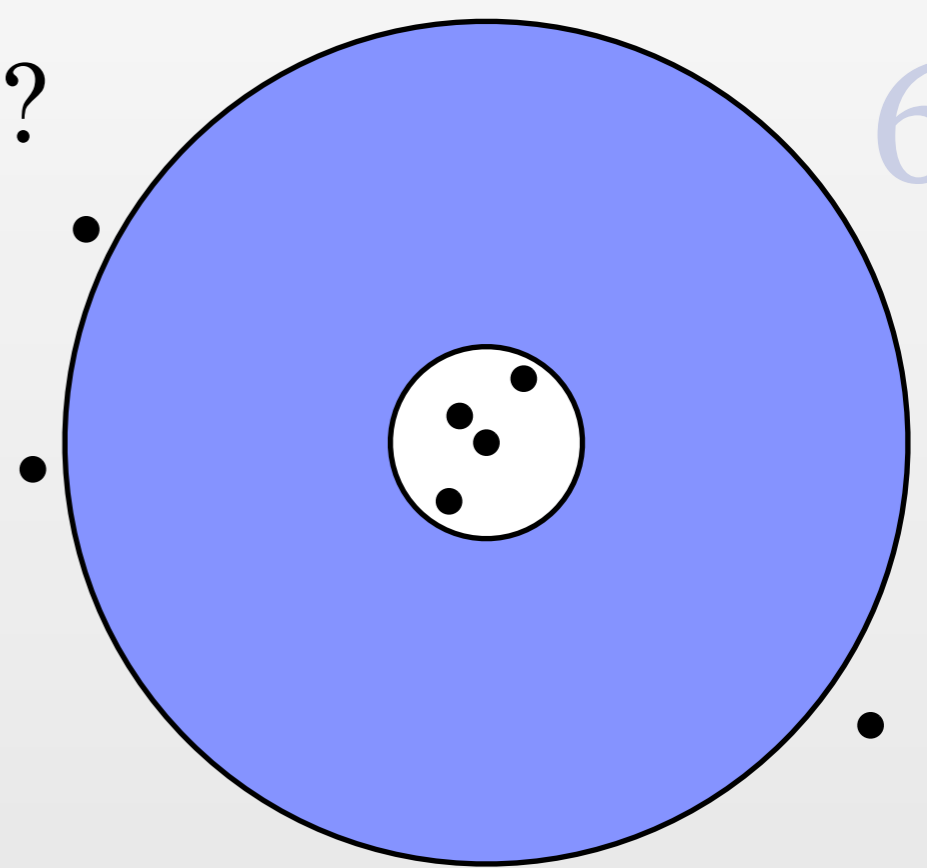
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$m = O(n)$ as long as there are no big empty annuli with 2 or more points inside [MPS08].



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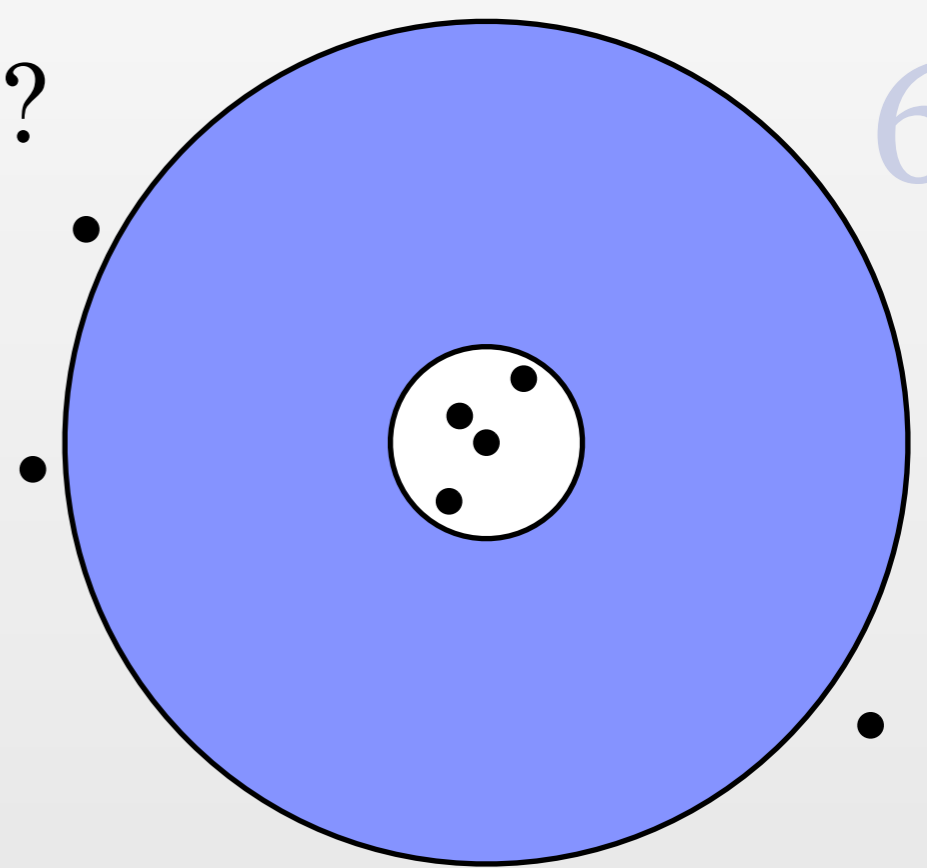
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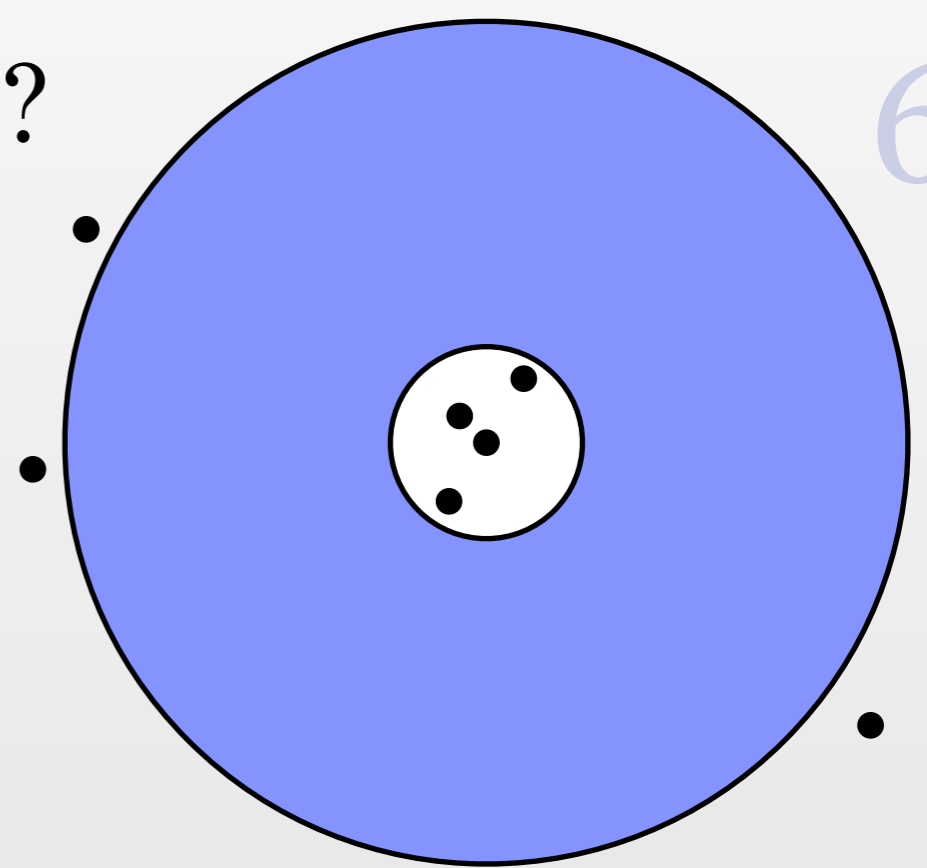
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How many simplices?

Only $O(m)$ simplices.
Compare to $m^{\lceil d/2 \rceil}$ for general Delaunay triangulations.
Constants depend on aspect ratio.



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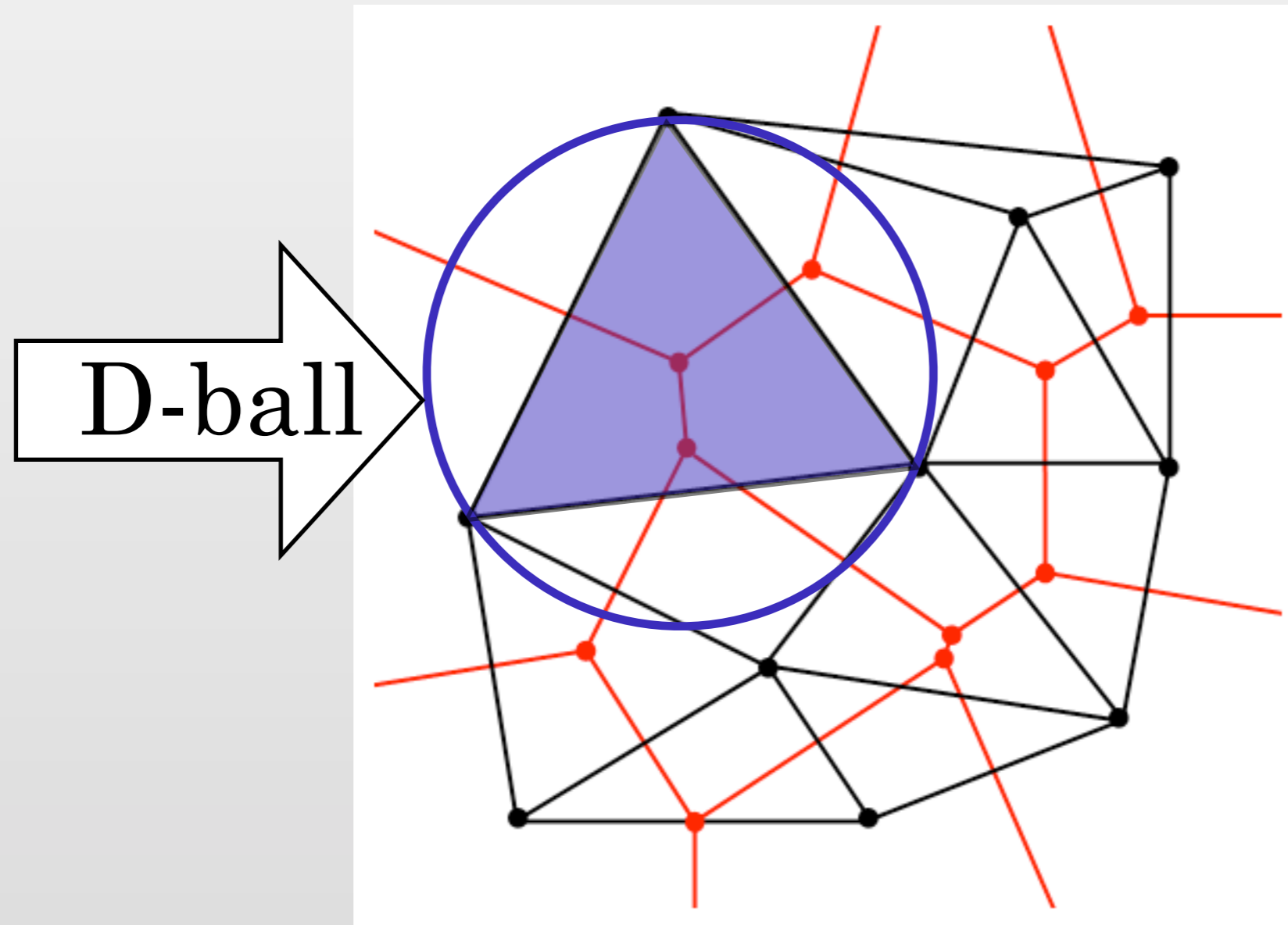
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Finishing post-process.

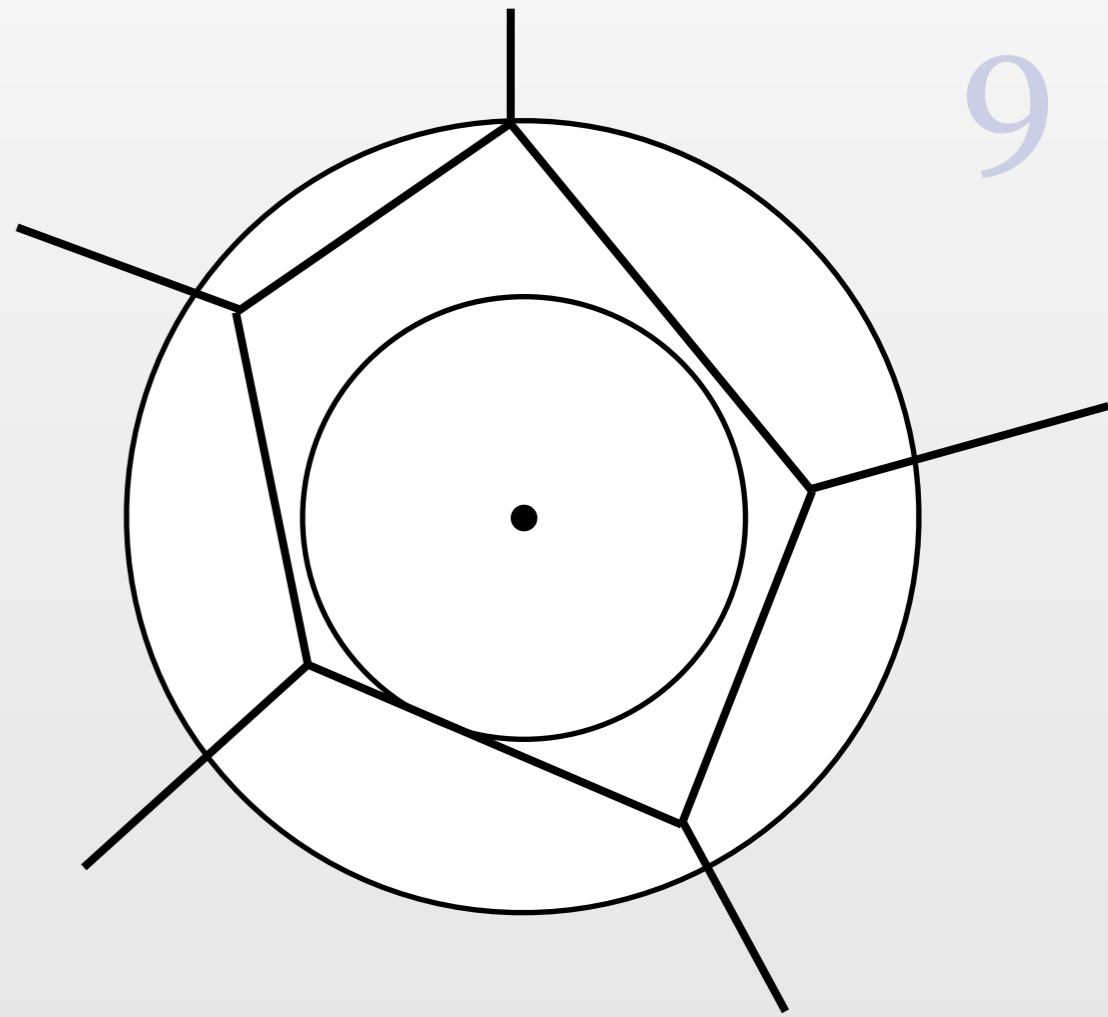
(easy)

Key fact: Size is $O(n)$

The Delaunay Triangulation is the dual of the Voronoi Diagram.

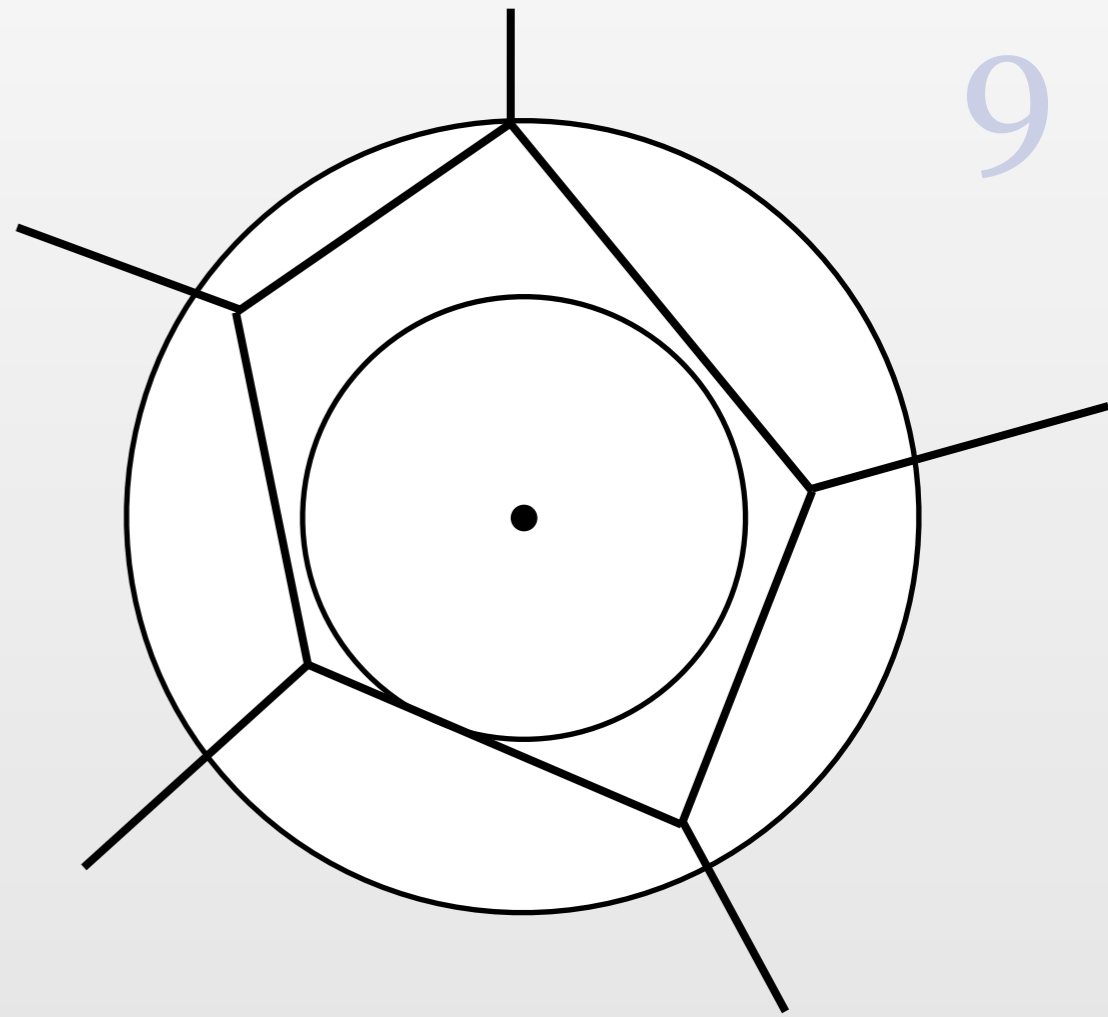


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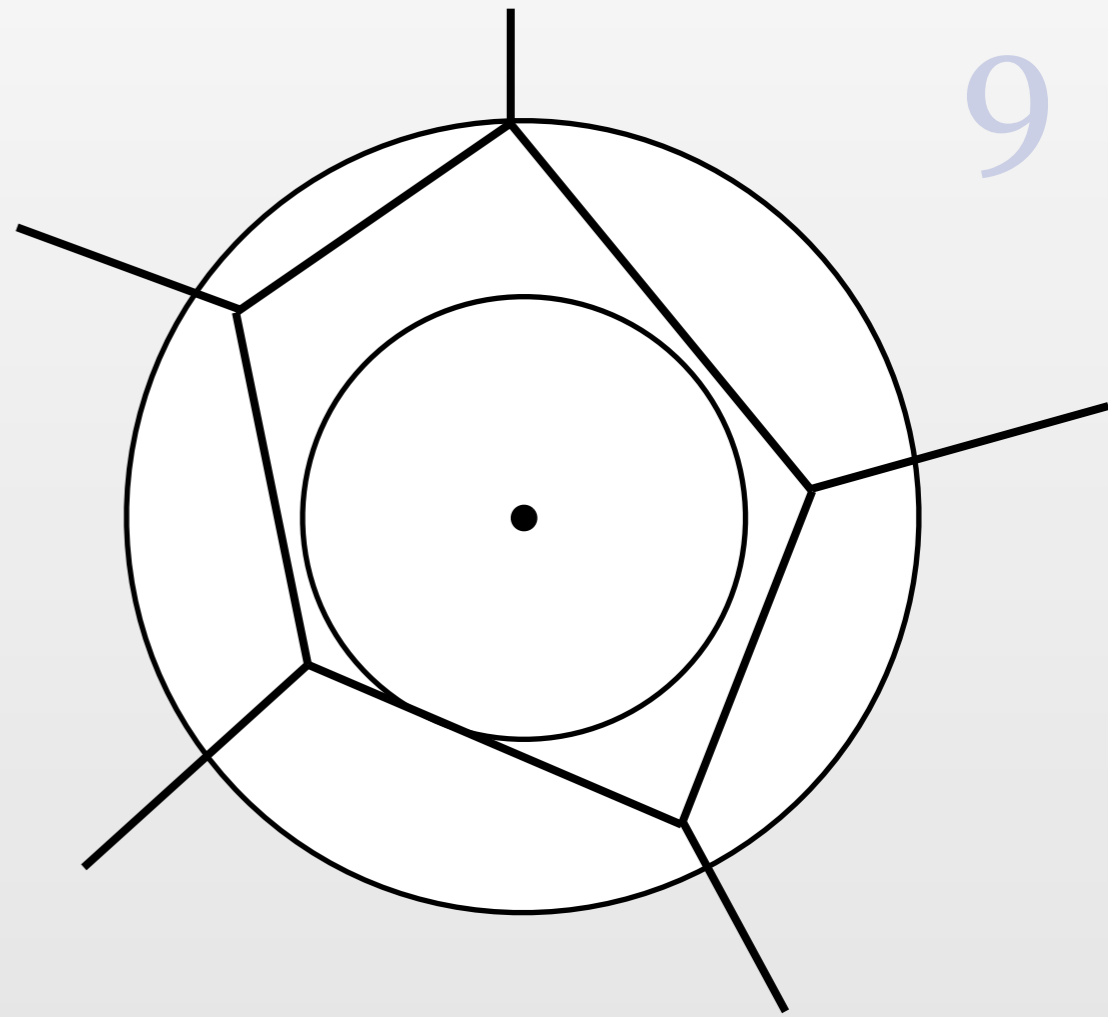
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Total number of faces is $O(m)$

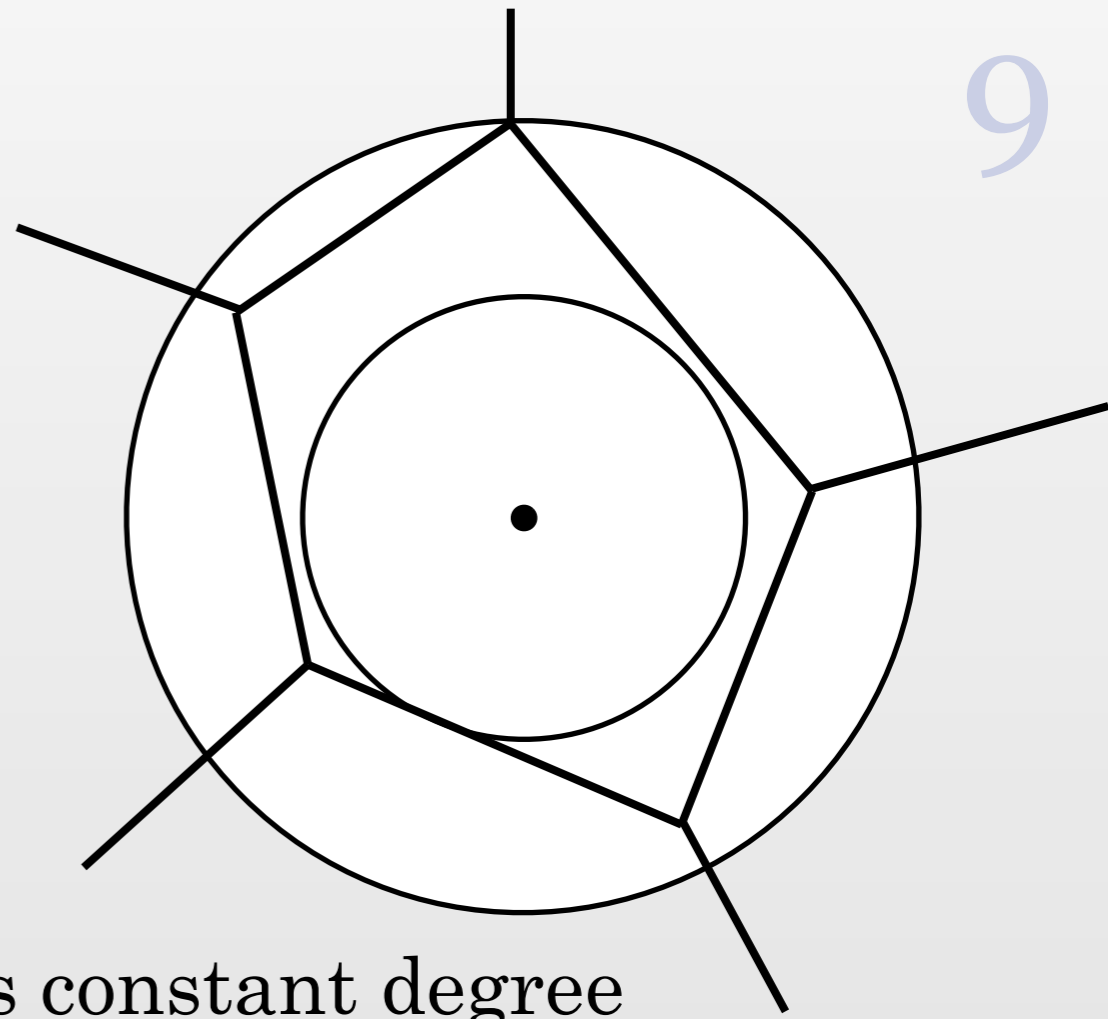


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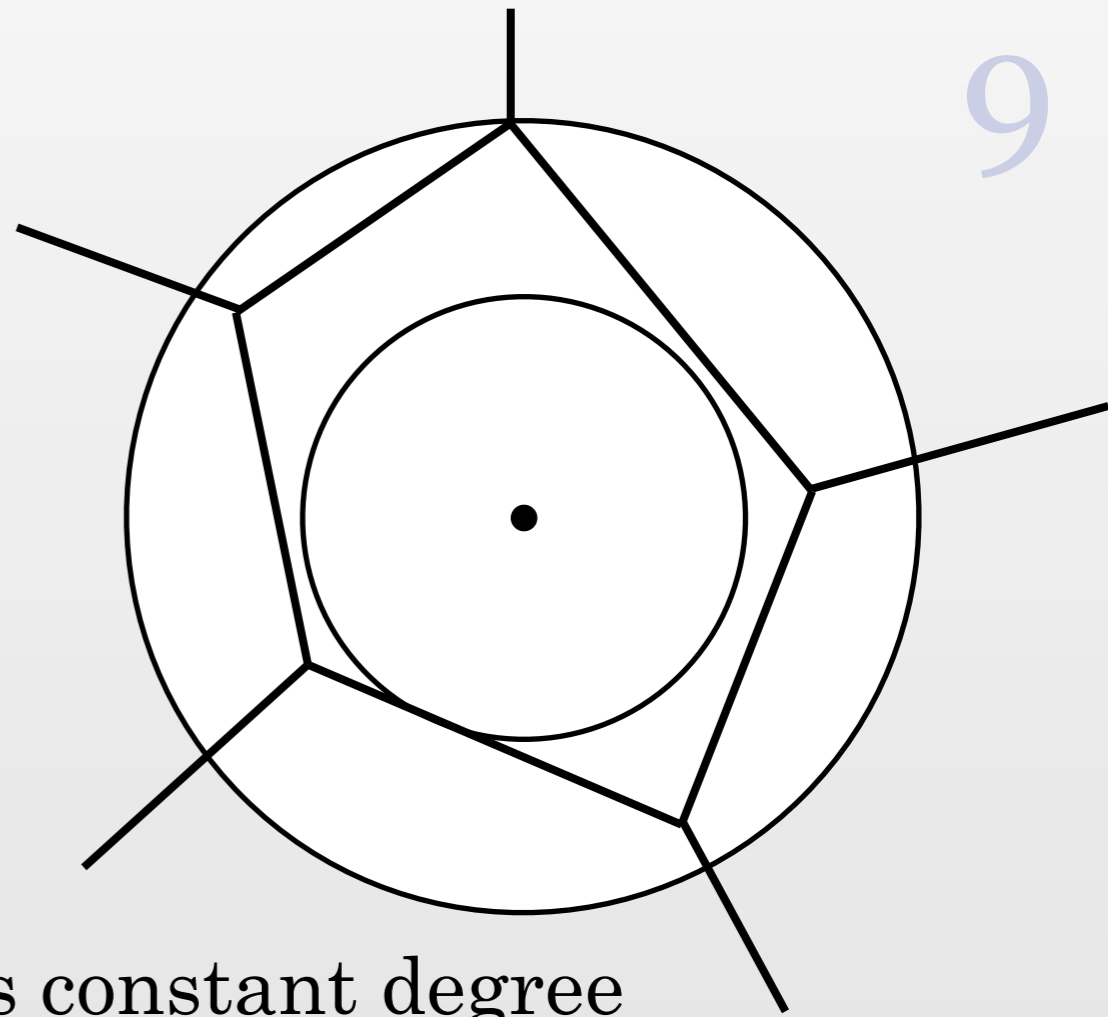
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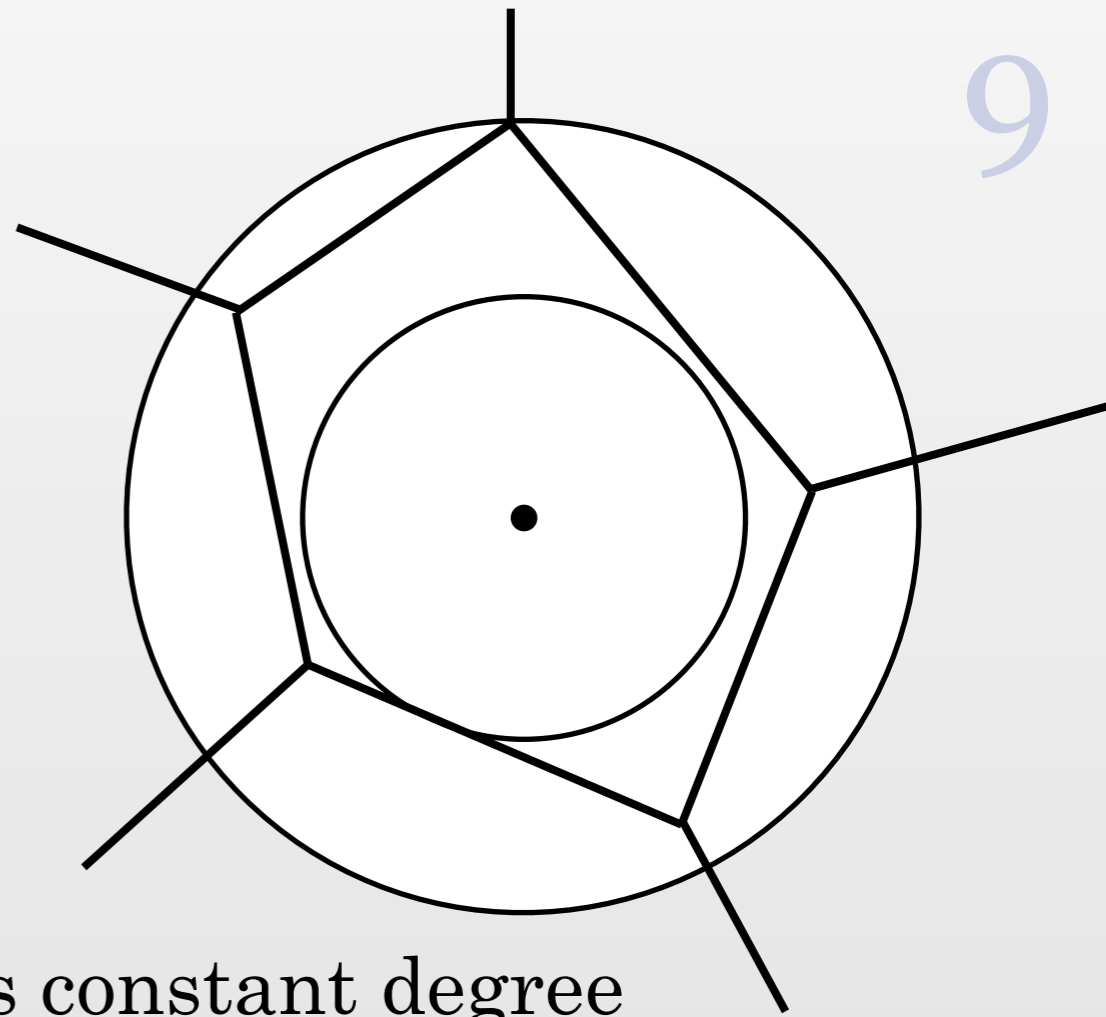
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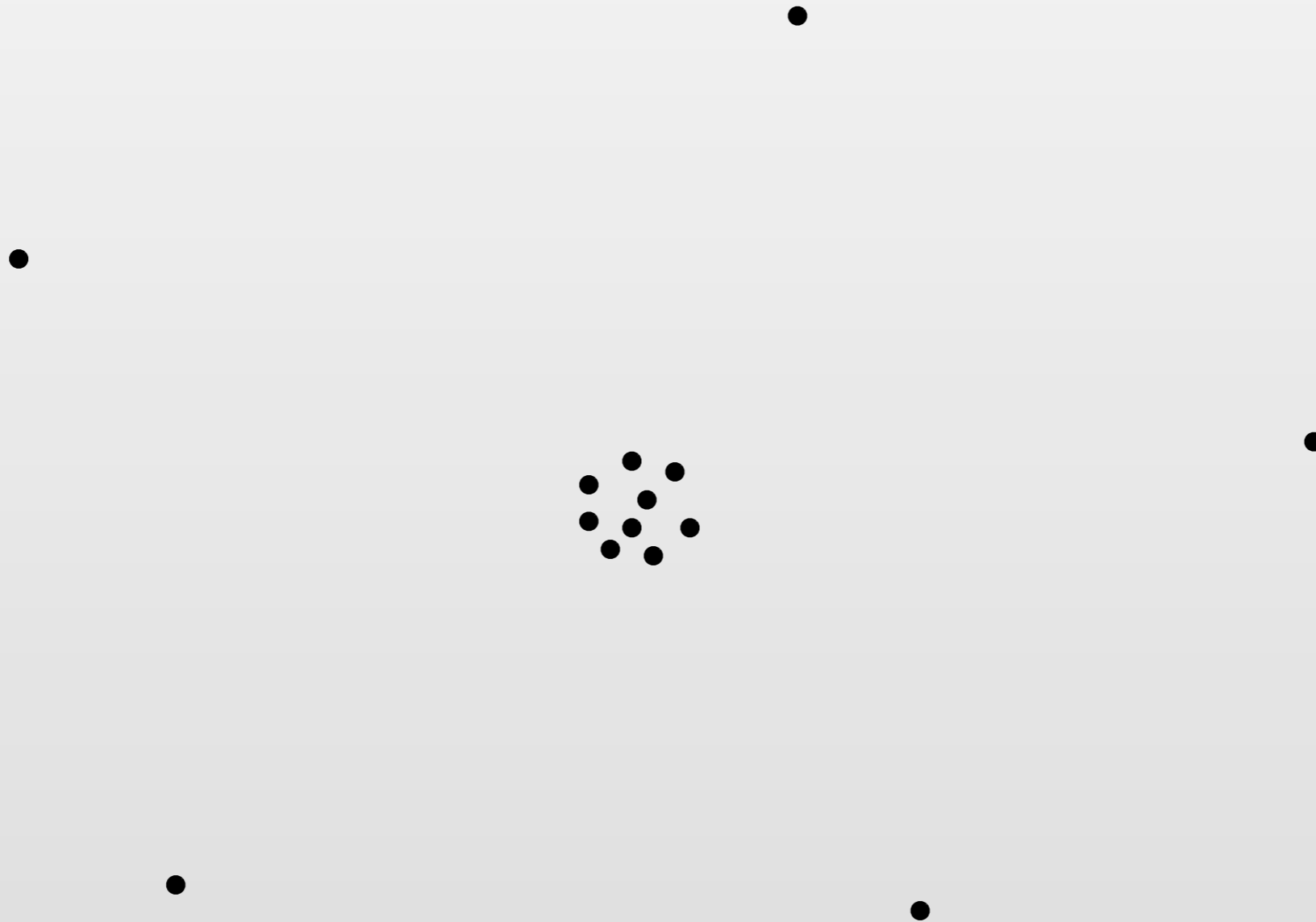
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Voronoi Refinement: If some cell is skinny, add a Steiner point at its farthest vertex.

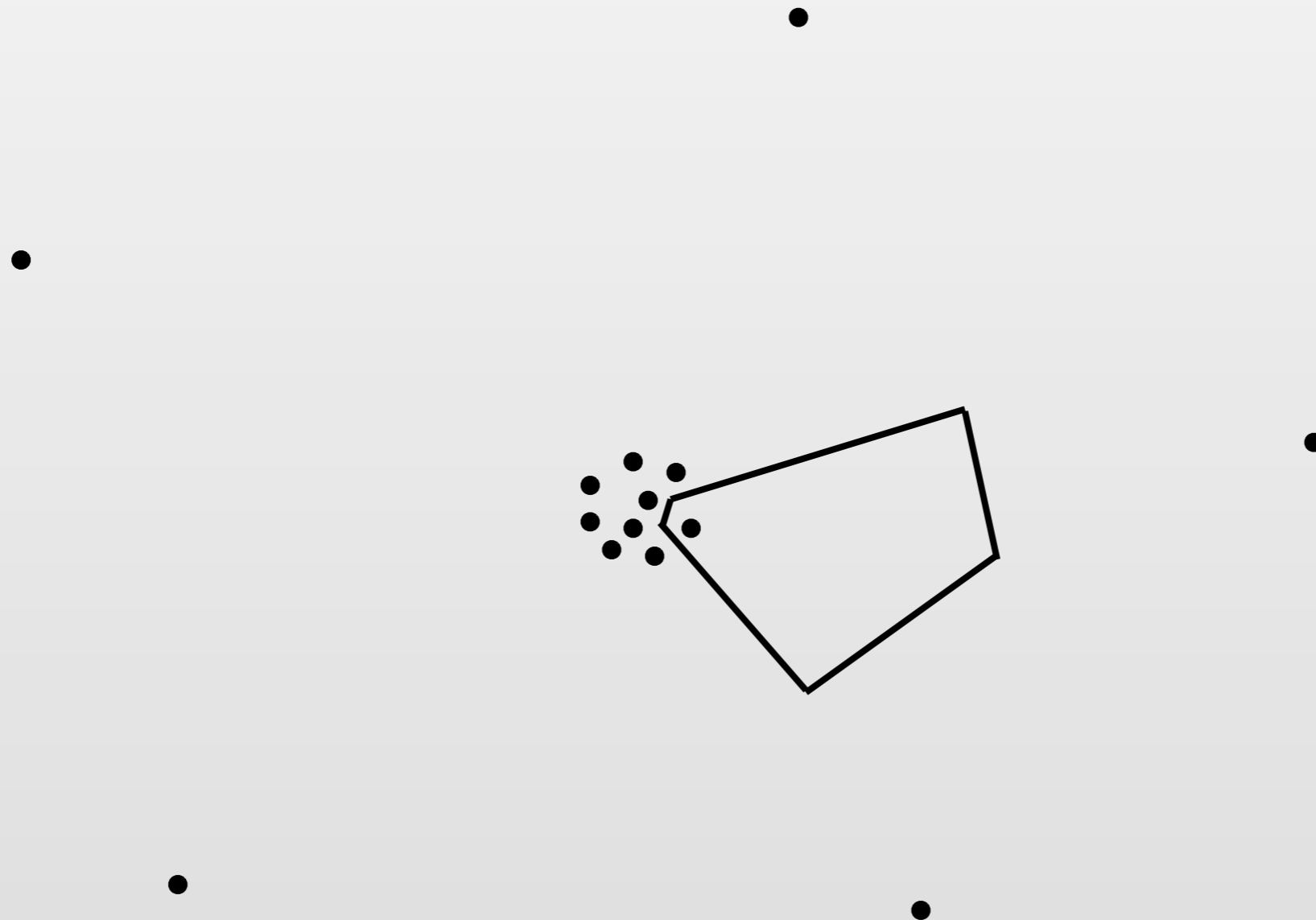


We replace *quality* with *hierarchical quality*. 10



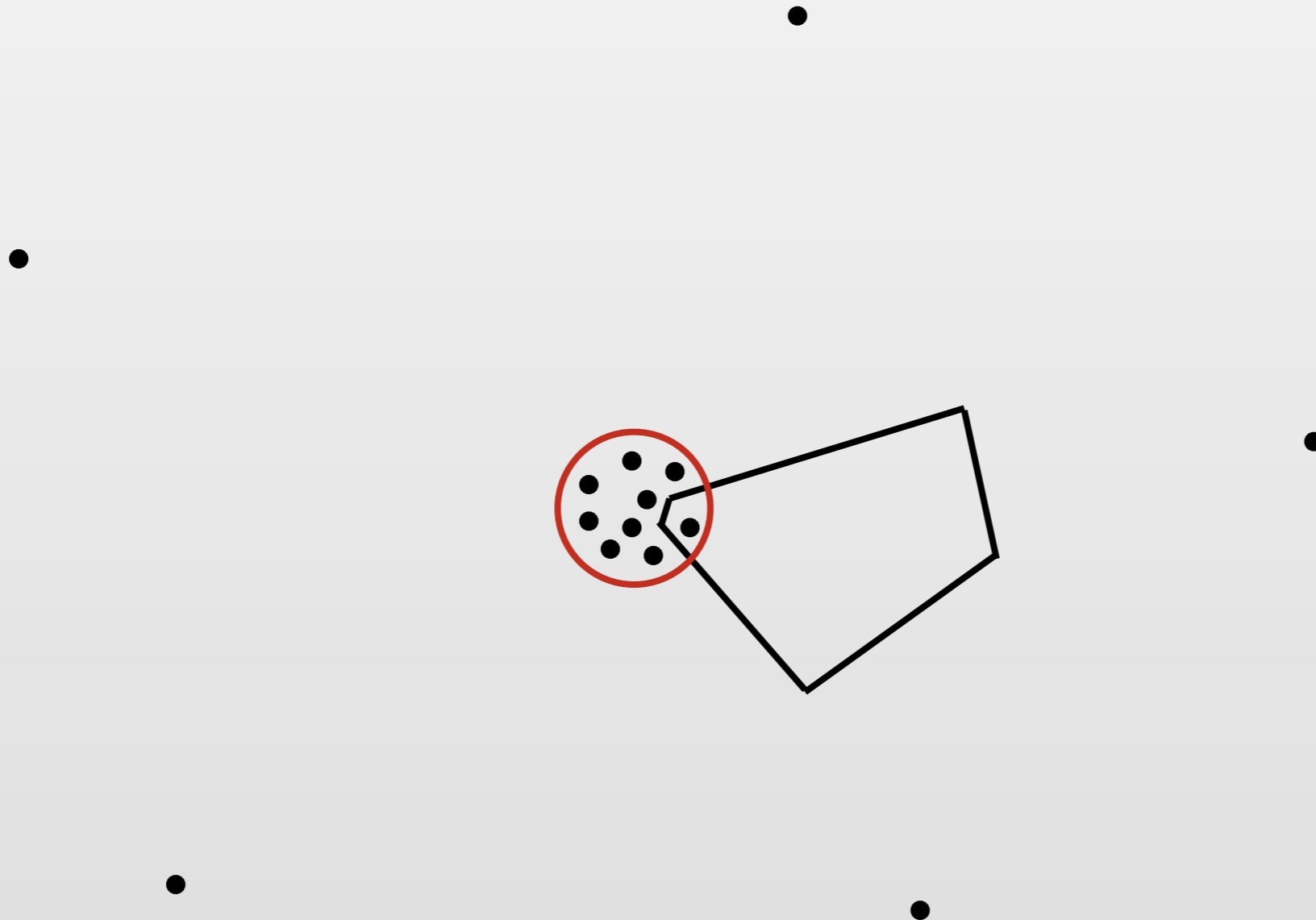
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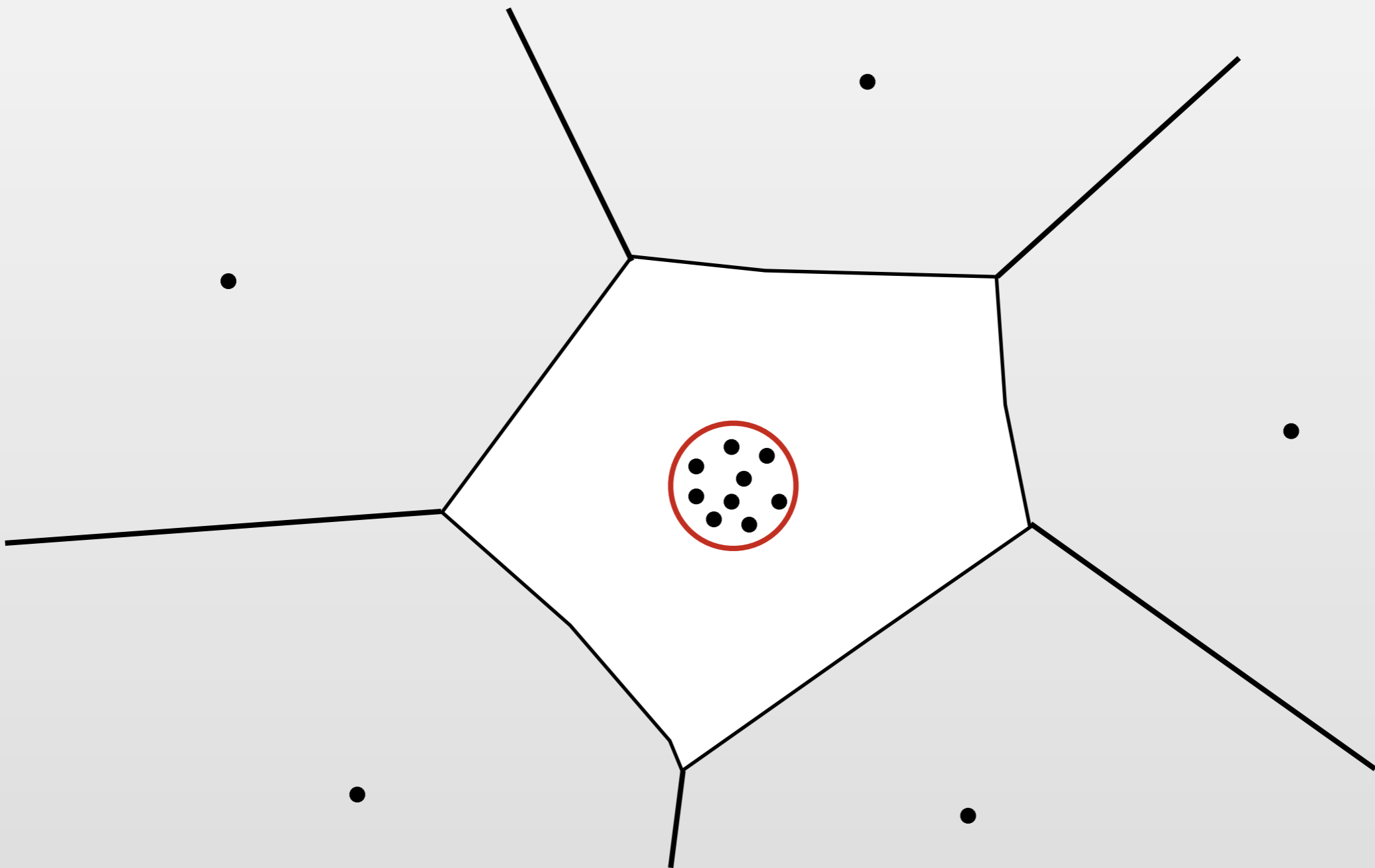


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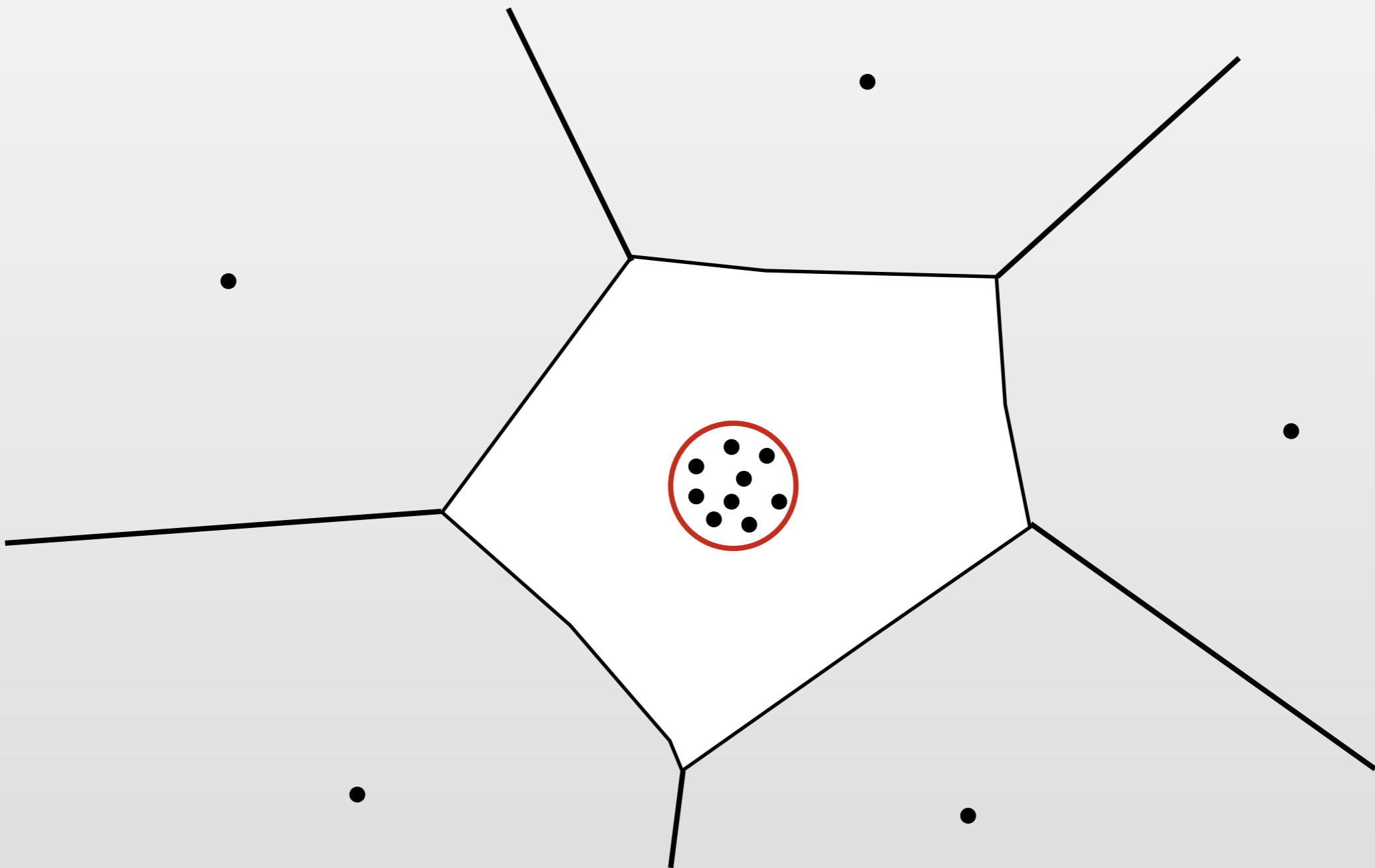
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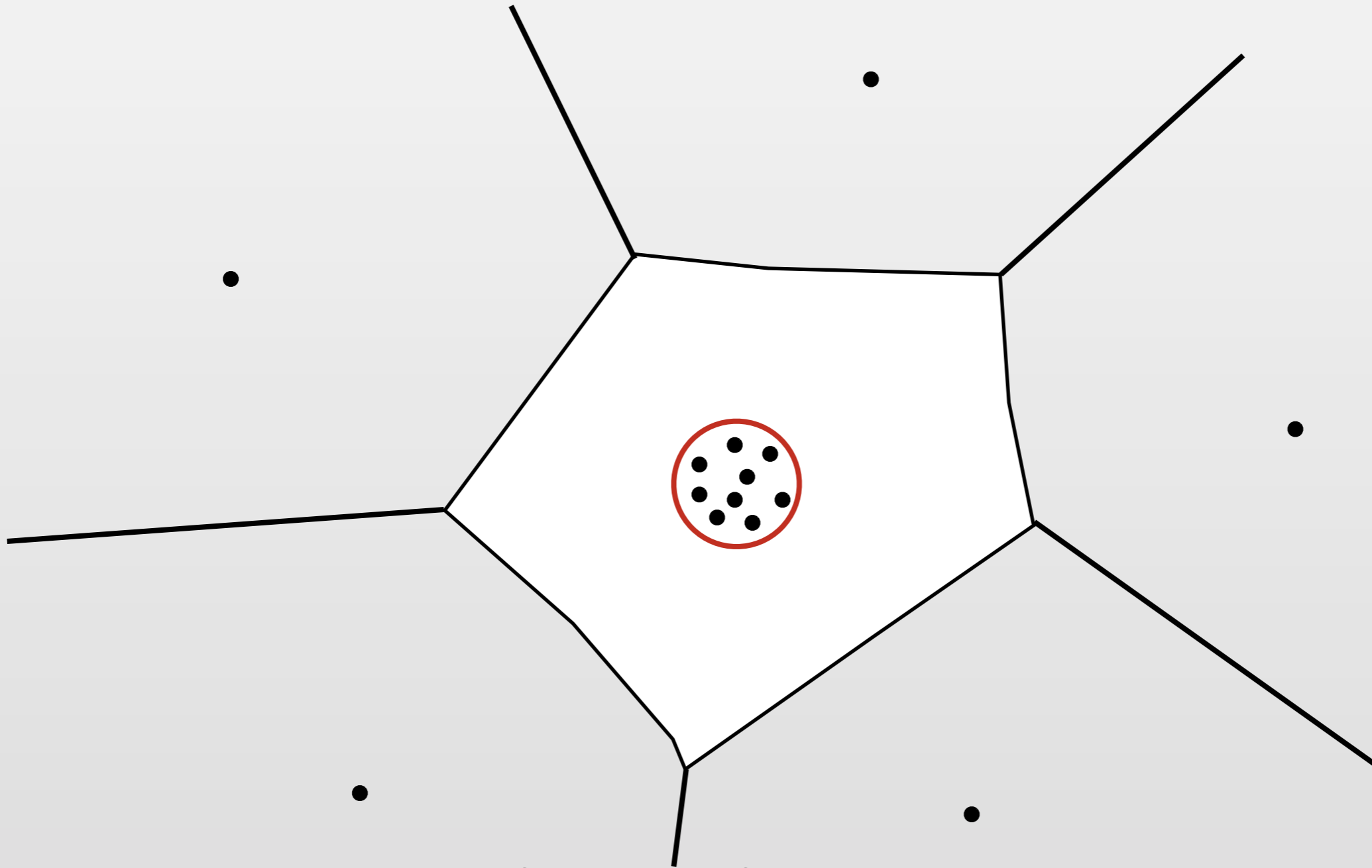
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Inside the cage: Old definition of quality.

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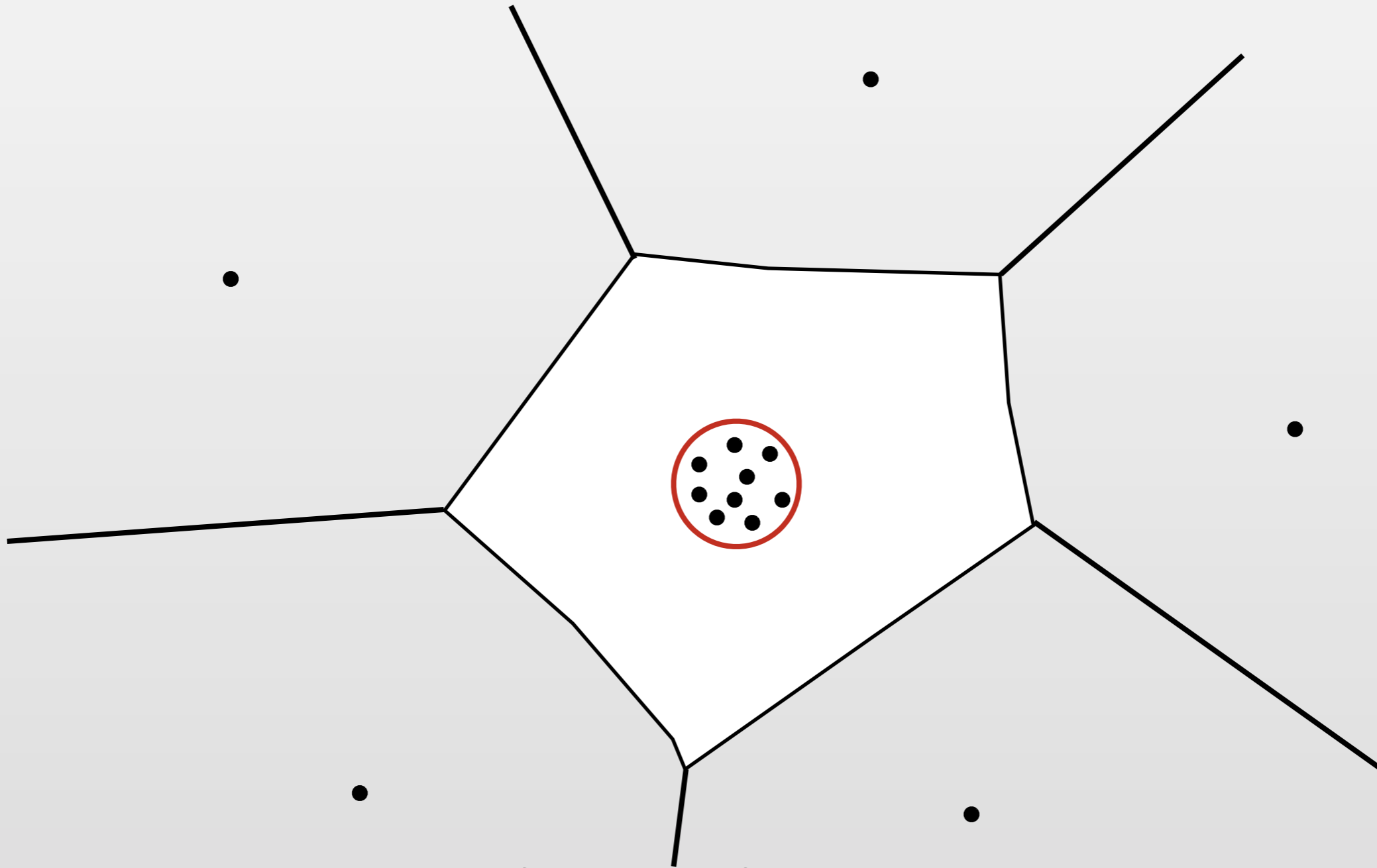


Inside the cage: Old definition of quality.

Outside: Treat the whole cage as a single object.

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Inside the cage: Old definition of quality.

Outside: Treat the whole cage as a single object.

Has the same important properties as quality meshes: ply, degree,...

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This is how we avoid the worst-case Voronoi bounds.

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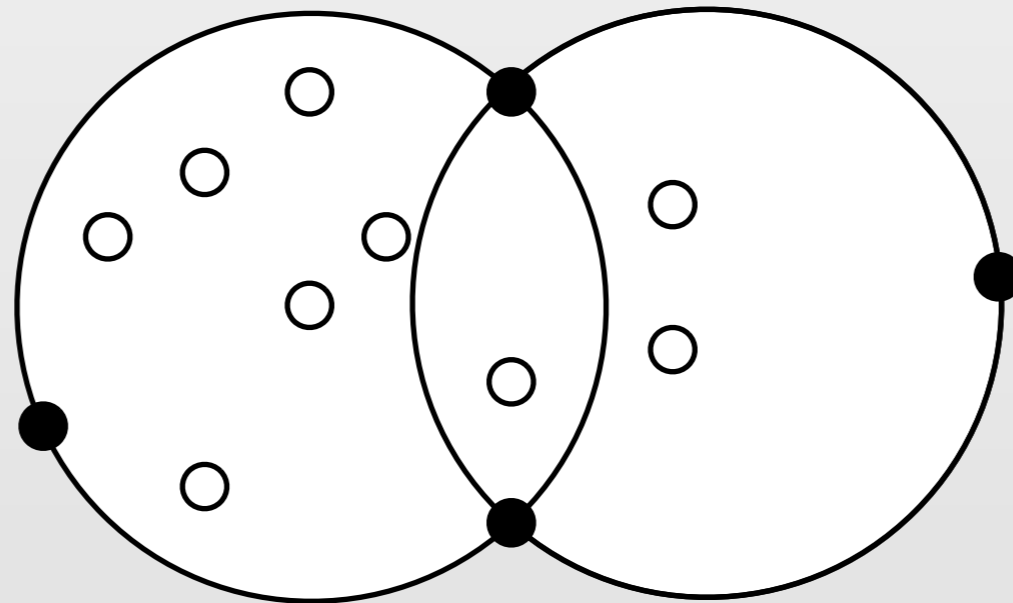
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- 5 Order the input points using range space nets.

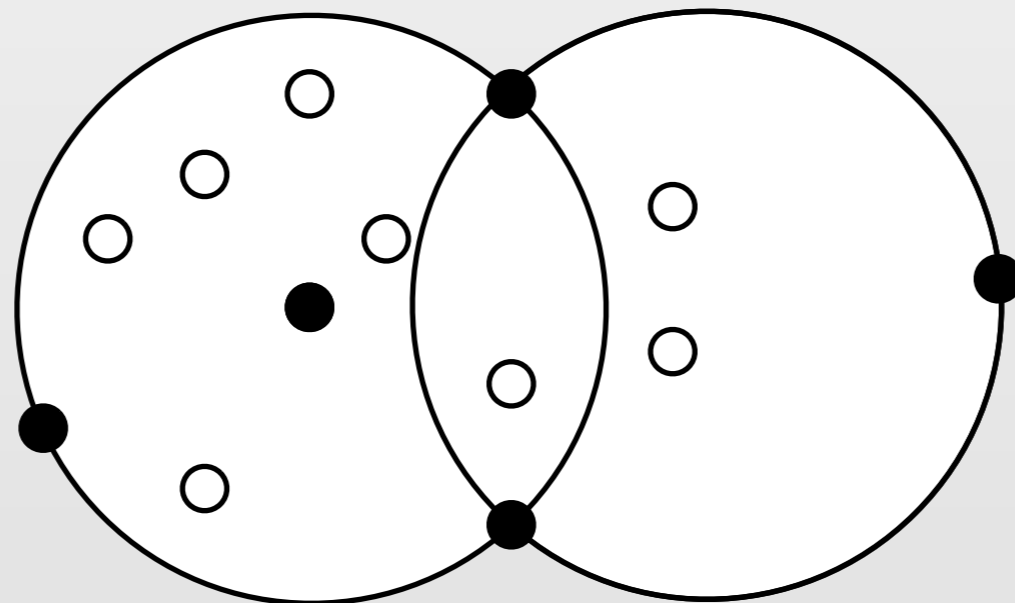
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Idea: Store the uninserted points in the D-balls.
When the balls change, make local updates.



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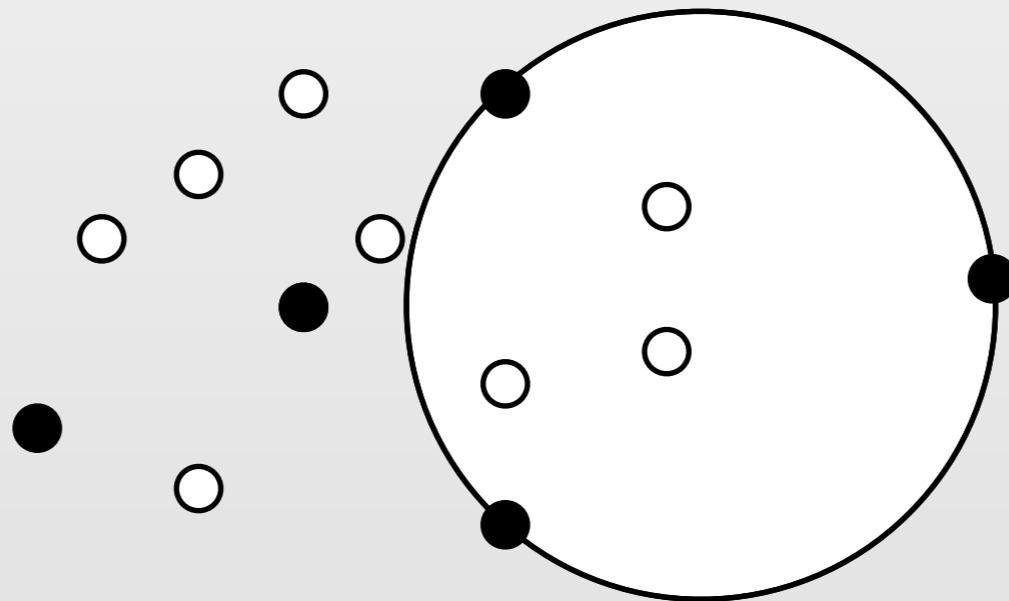
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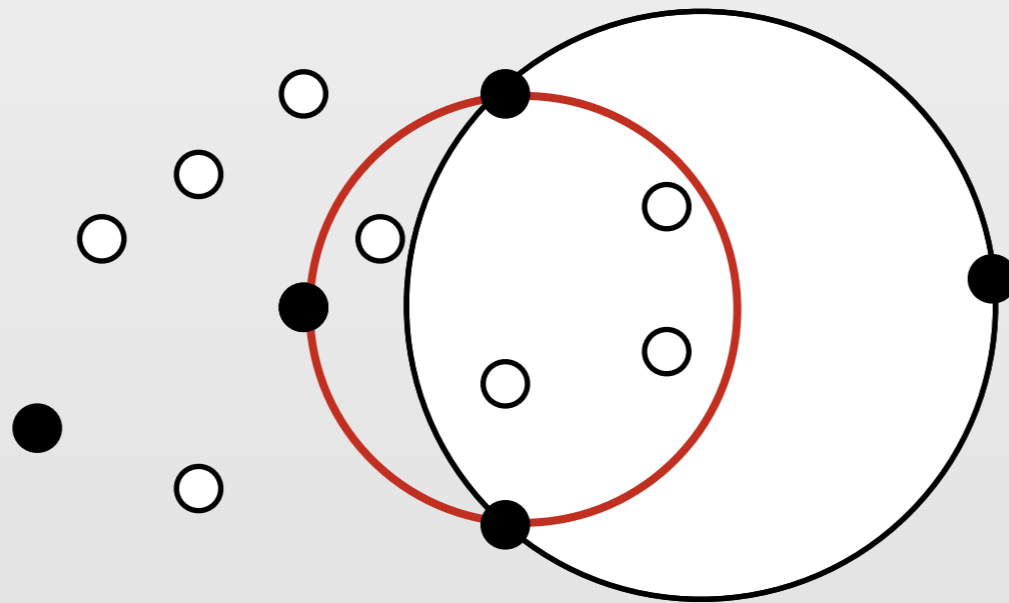
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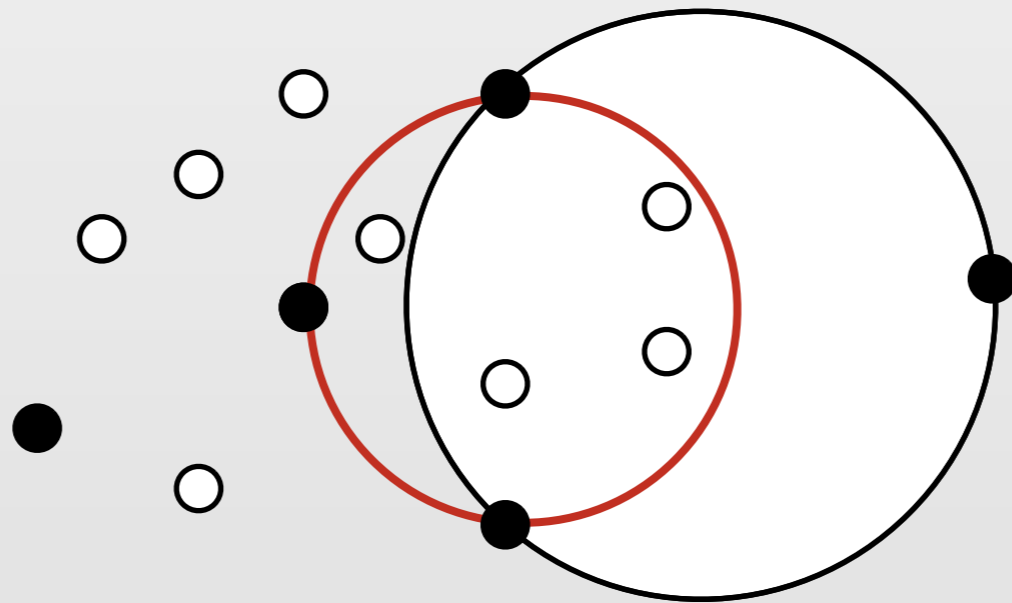
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It's a history DAG!

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Theorem: [Chazelle & Matousek 96] For ε, d fixed constants, ε -nets of size $O(1)$ can be computed in $O(n)$ deterministic time.

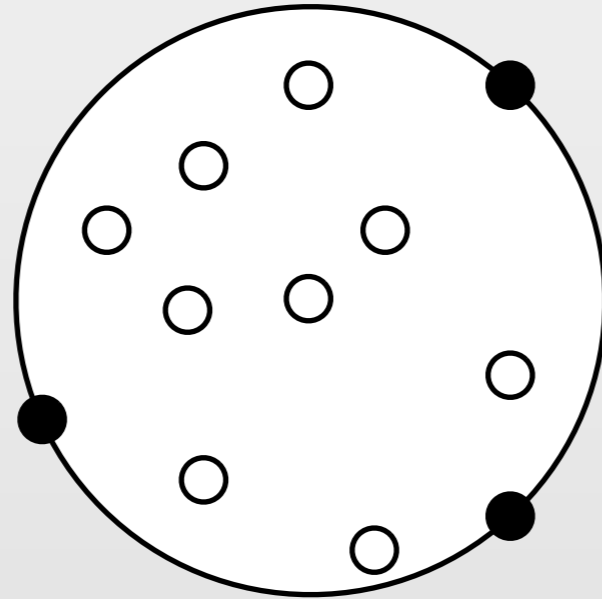
Ordering the inputs

For each D-Ball, select a $\frac{1}{2d}$ -net of the points it contains.

Take the union of these nets and call it a round.

Insert these.

Repeat.



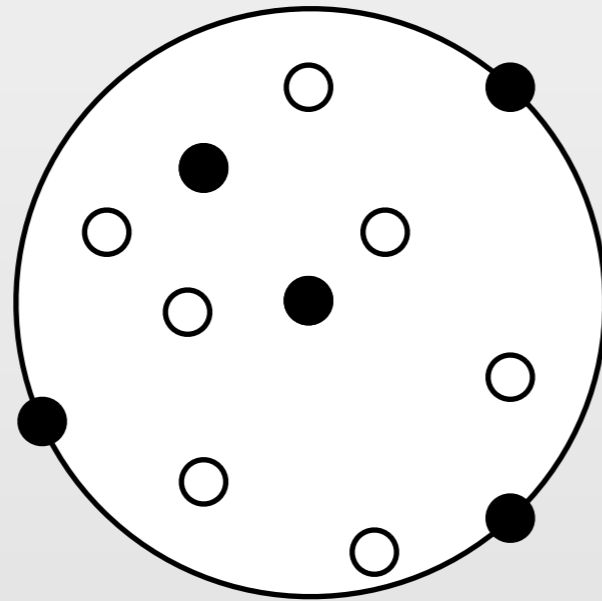
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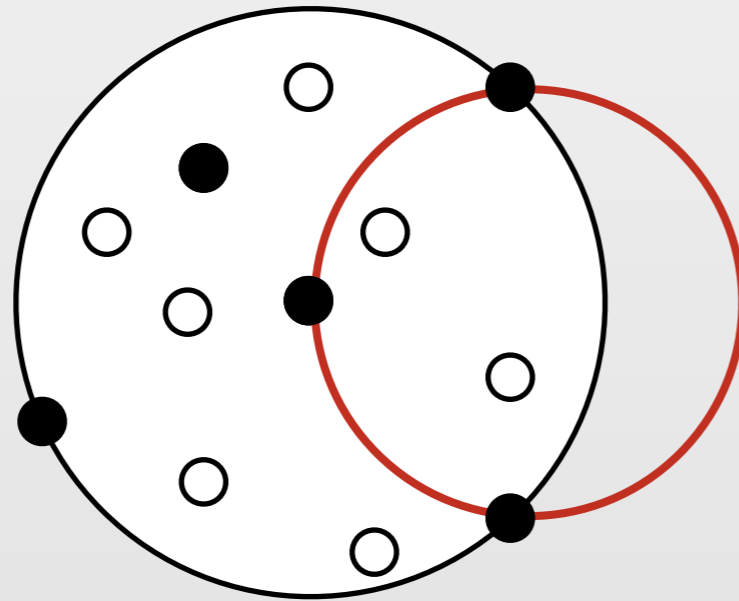


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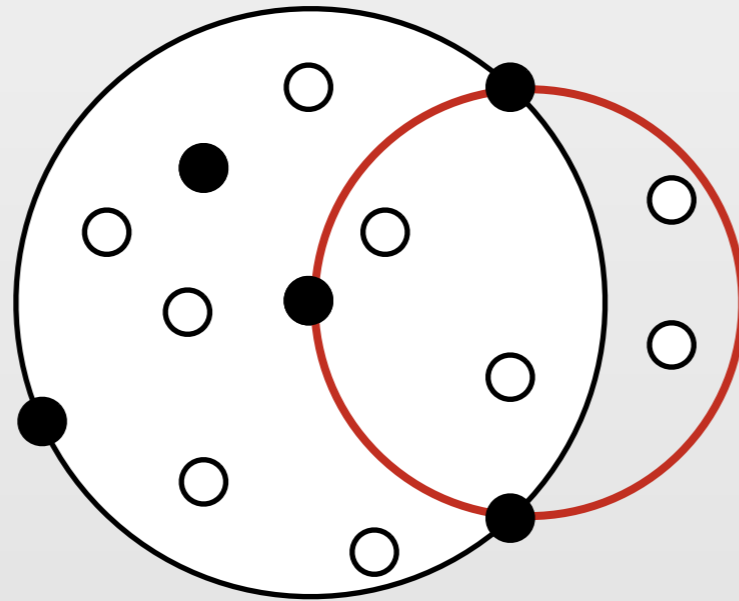


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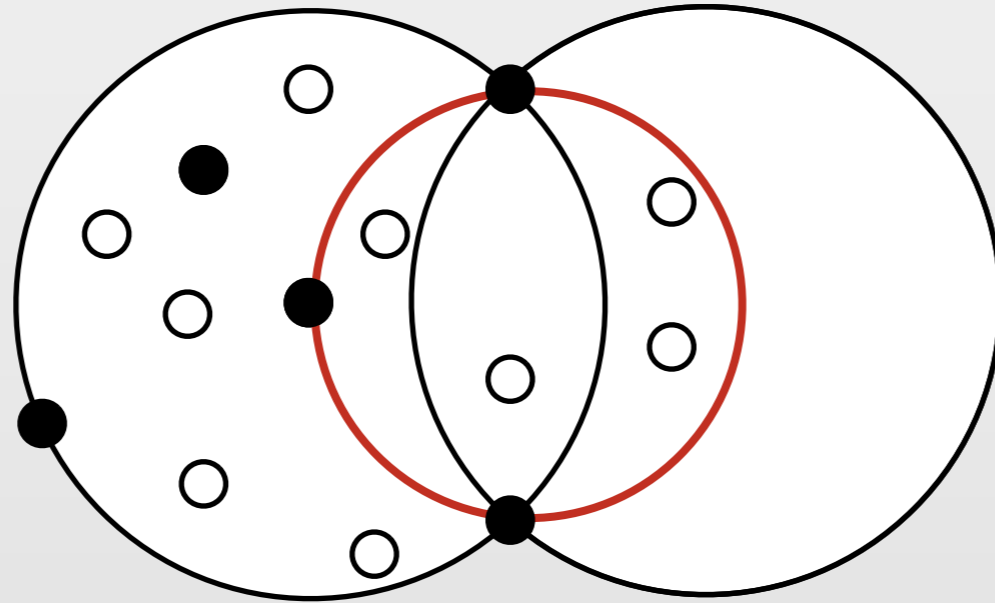


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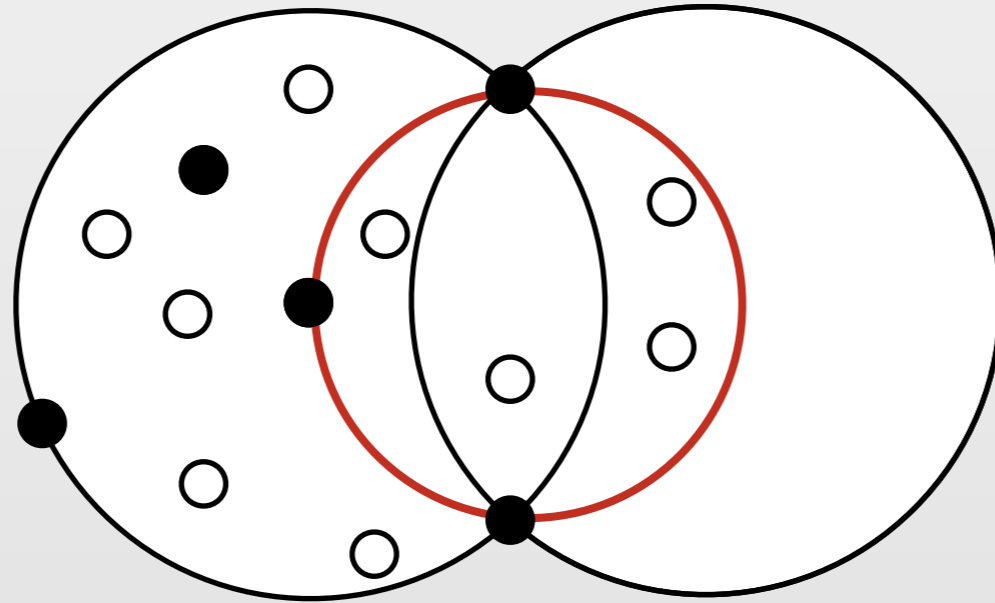


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Lemma. *Let M be a set of vertices. If an open ball B contains no points of M , then B is contained in the union of d D-balls of M .*

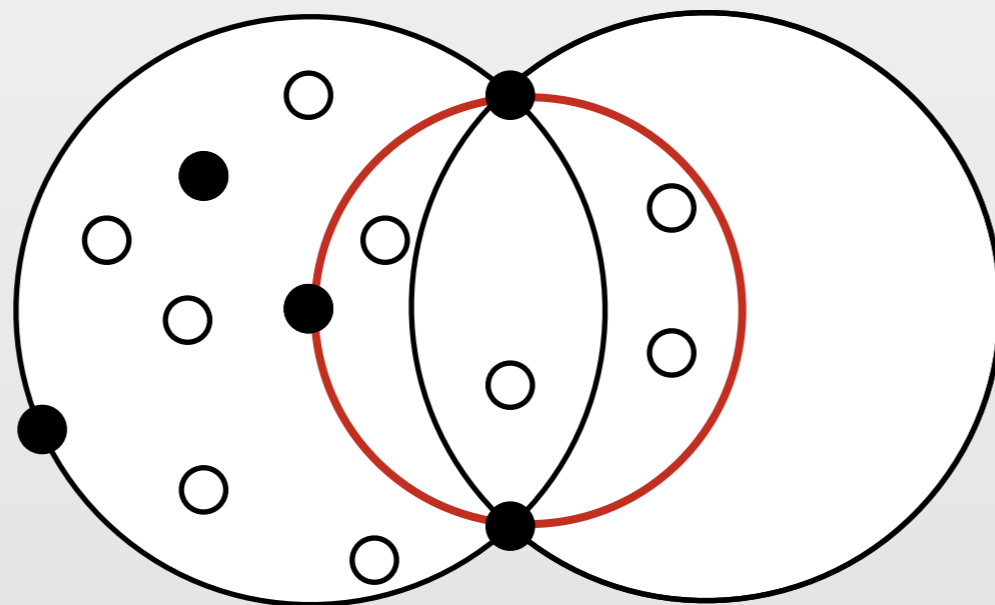
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$$\max_{\text{D-balls } B} |B \cap P|$$

Goes down by a factor
of 2 each round.

Lemma. *Let M be a set of vertices. If an open ball B contains no points of M , then B is contained in the union of d D-balls of M .*

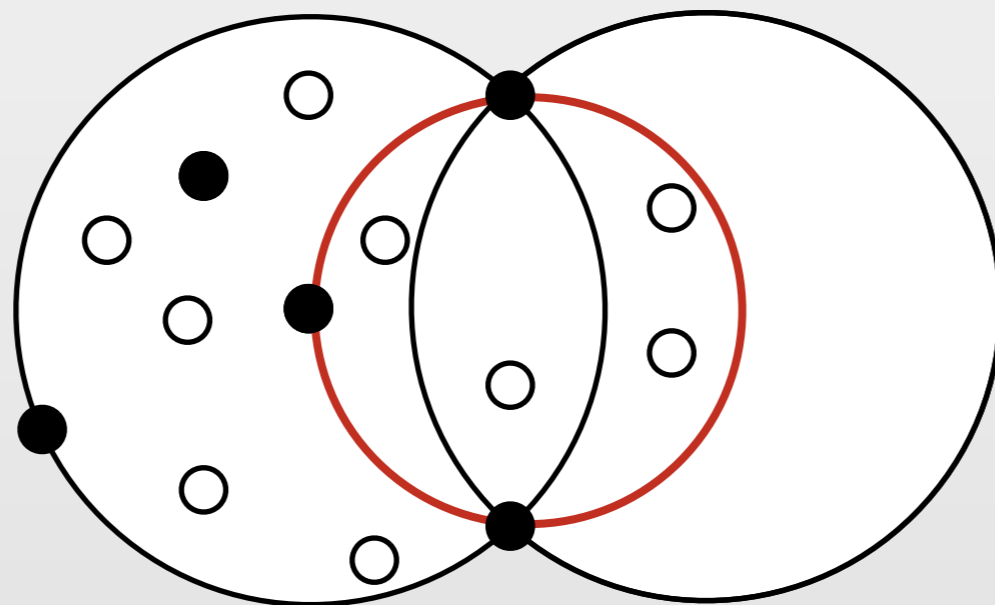
Ordering the inputs

For each D-Ball, select a $\frac{1}{2d}$ -net of the points it contains.

Take the union of these nets and call it a round.

Insert these.

Repeat.



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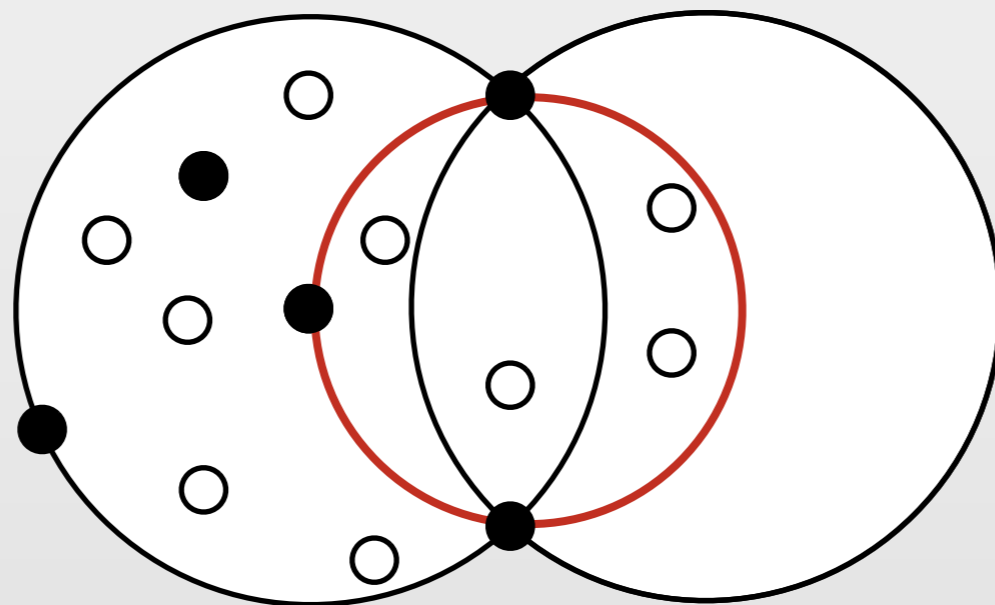
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Note: After $k = \log \frac{1}{\epsilon}$ rounds, the intermediate mesh is a **weak ϵ -net** for the range space of Euclidean balls.

Size: $O\left(\frac{1}{\epsilon}\right)$, Time: $O(nk) = O\left(n \log \frac{1}{\epsilon}\right)$.

To complete the analysis, we must show that the cost of a Round is $O(n)$.

$$\log n \text{ rounds} \times O(n) \text{ time/round} = O(n \log n)$$

Watch an uninserted point x .

Claim: x only gets touched $O(1)$ times per round.

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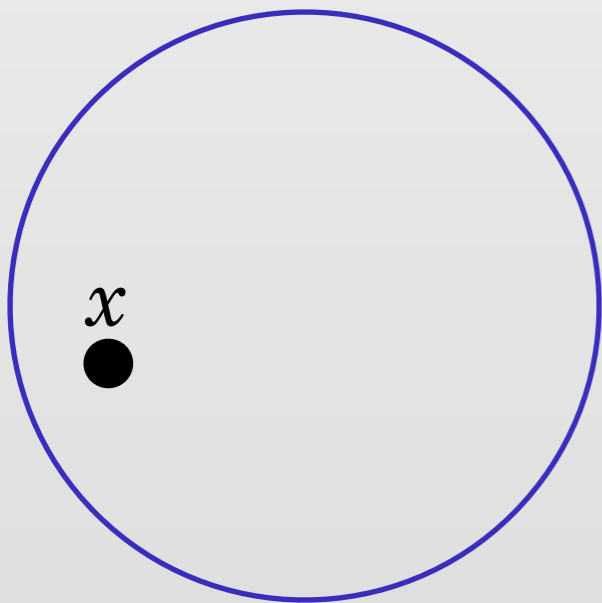
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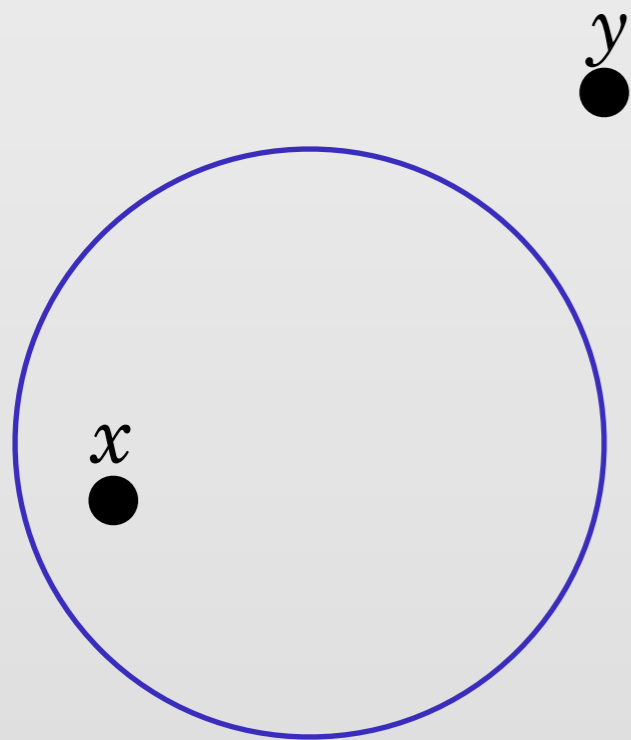


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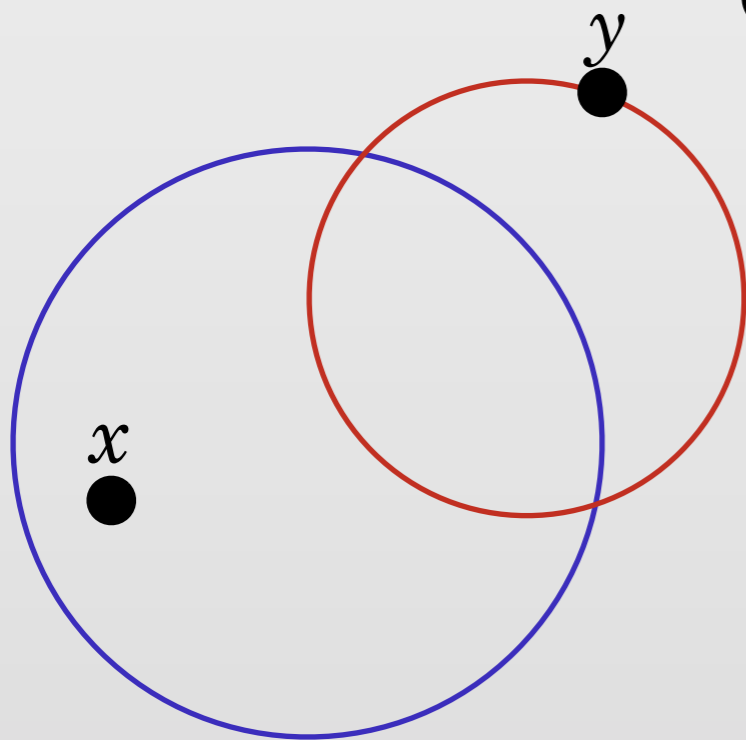


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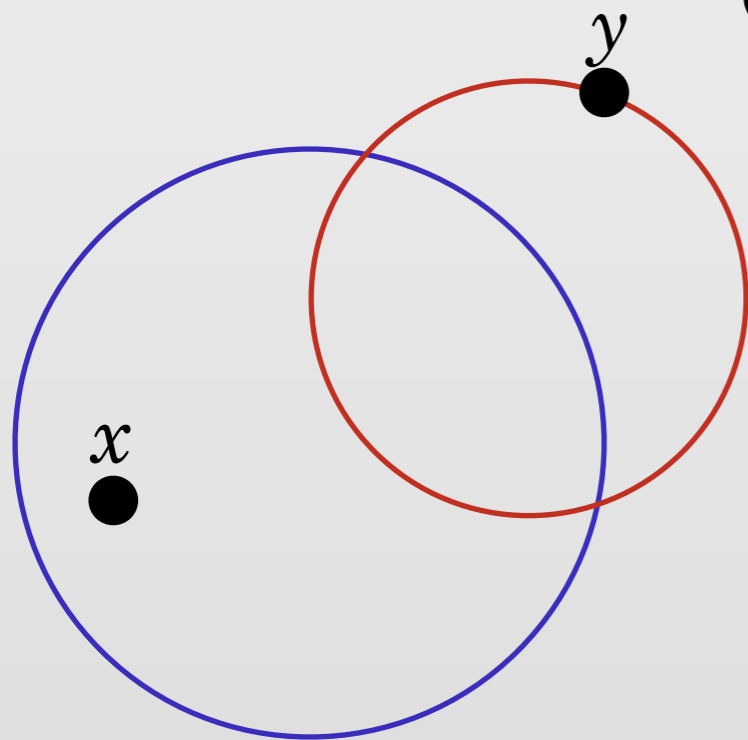


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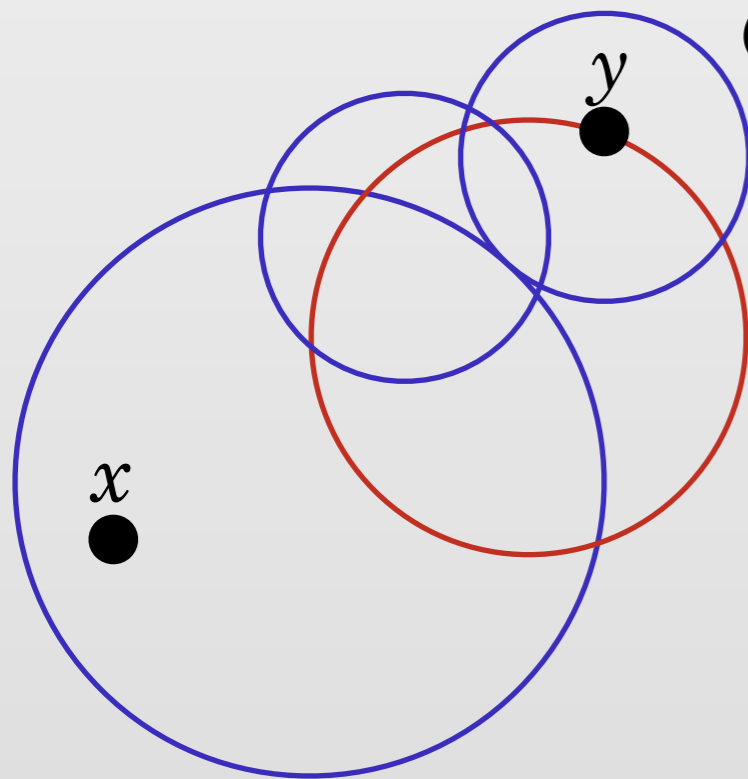
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close = 2 hops among D-balls

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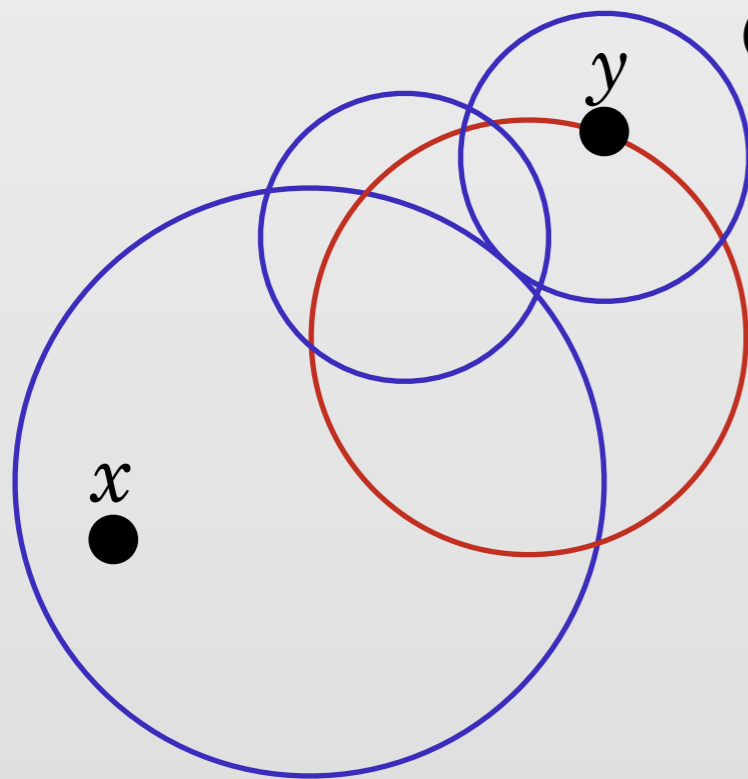
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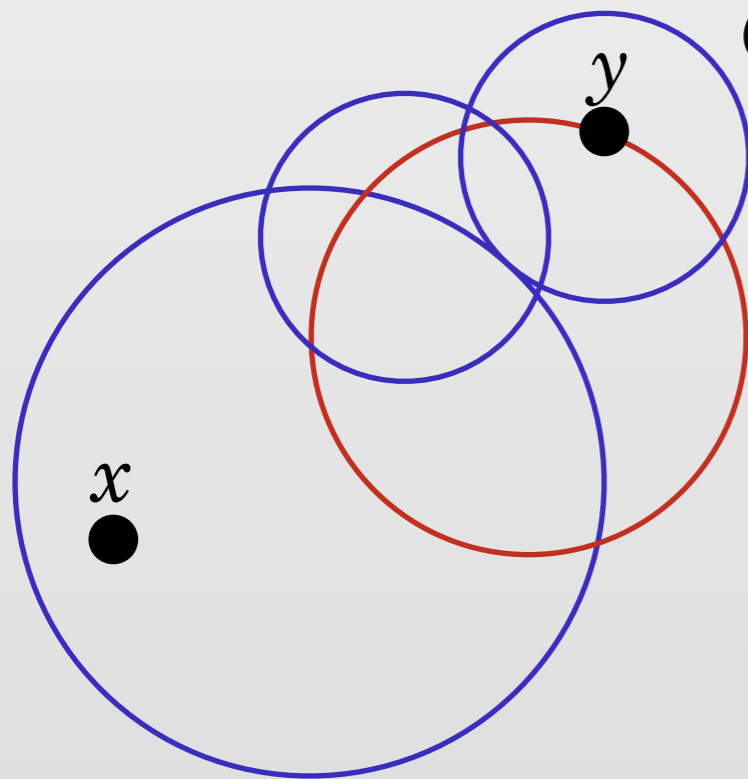
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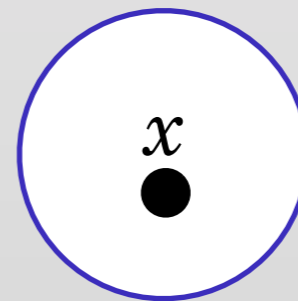
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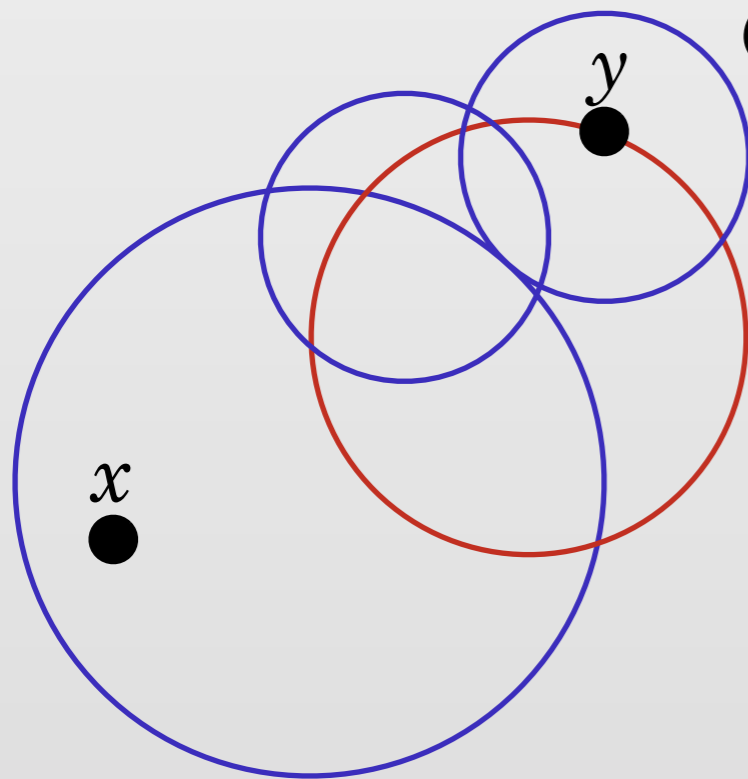


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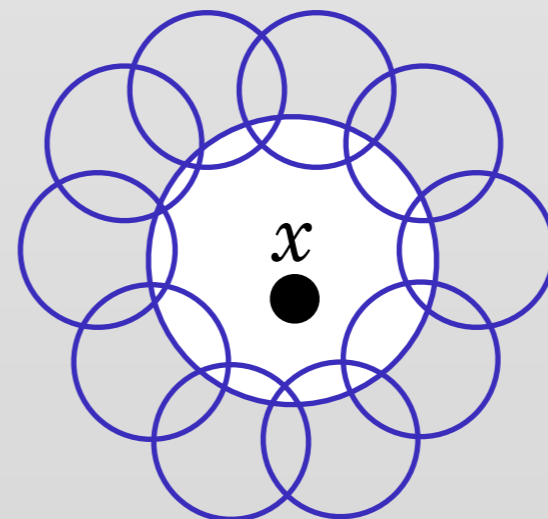
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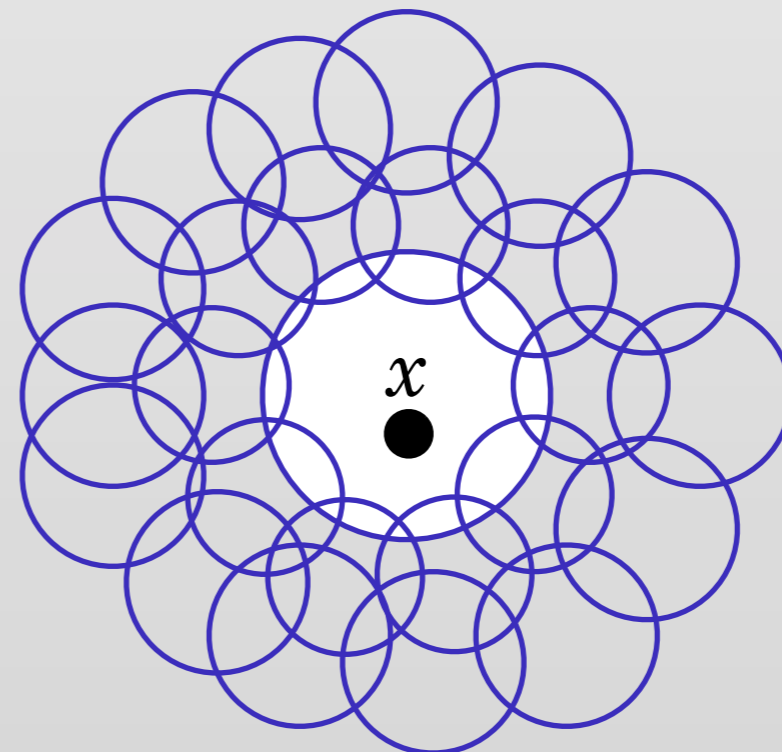
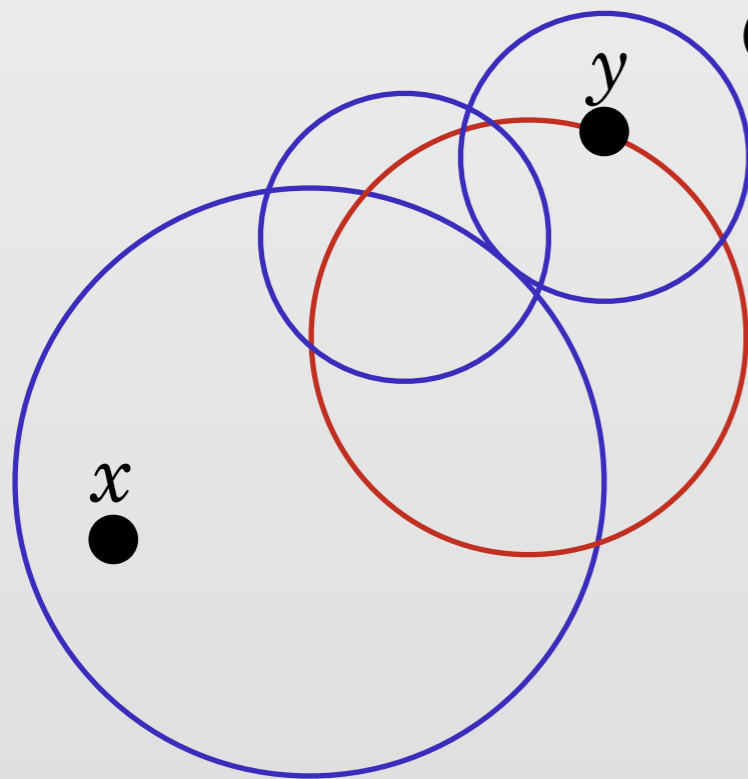
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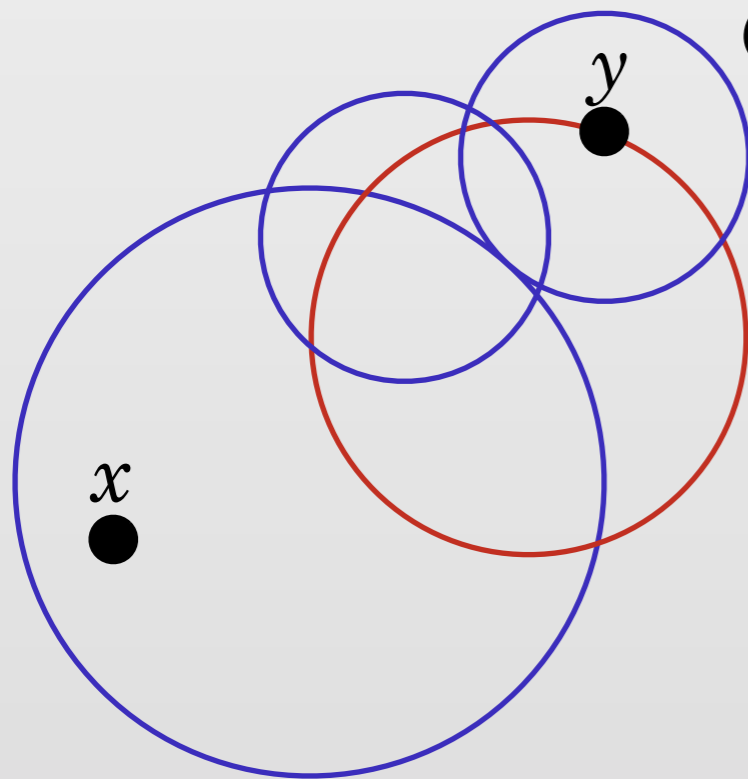
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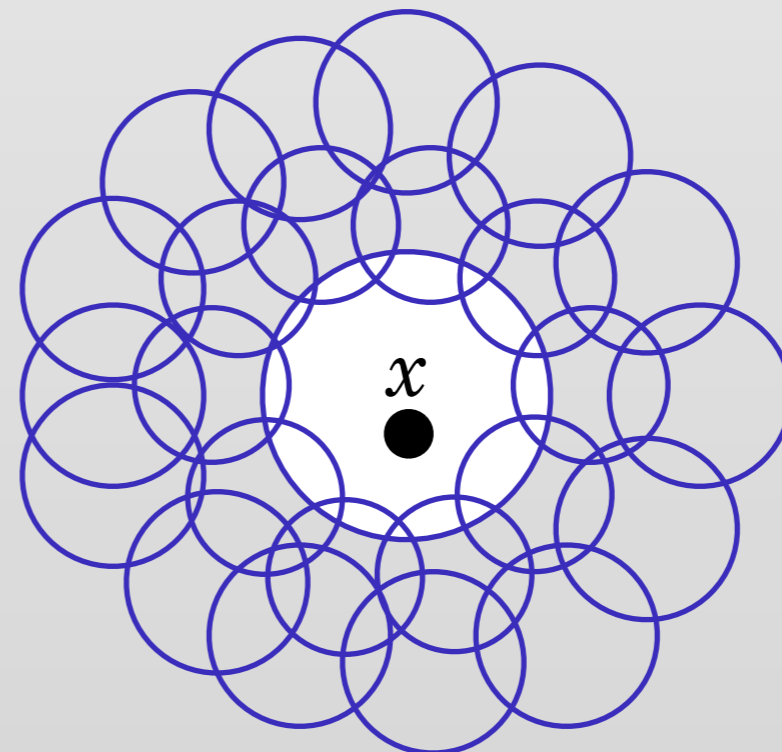
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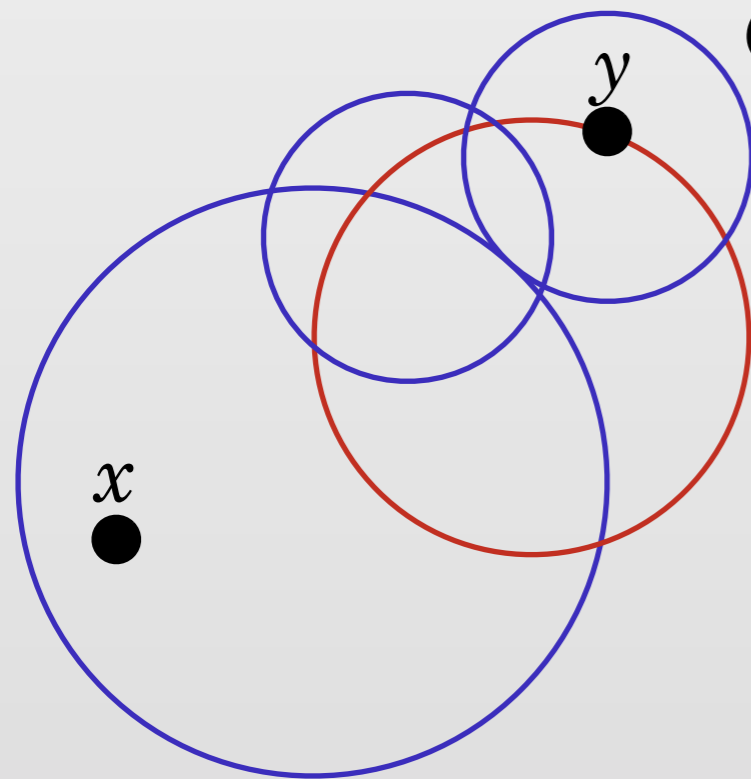


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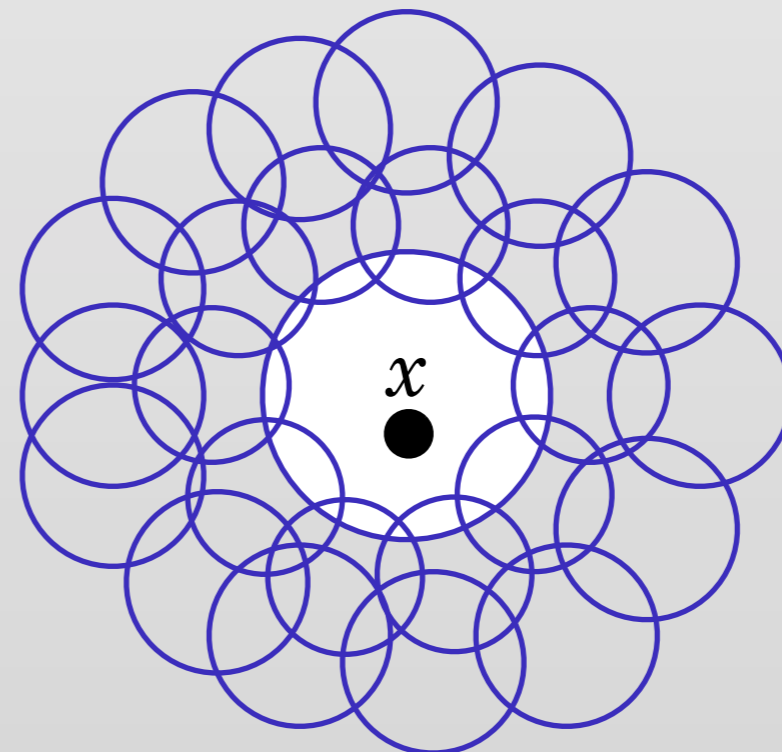


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$O(n)$ total work per round.



Meshing Points in (optimal)
 $O(n \log n + m)$ time.

Thank you.