Topological Inference via Meshing

Benoit Hudson, Gary Miller, Steve Oudot and Don Sheehy

SoCG 2010

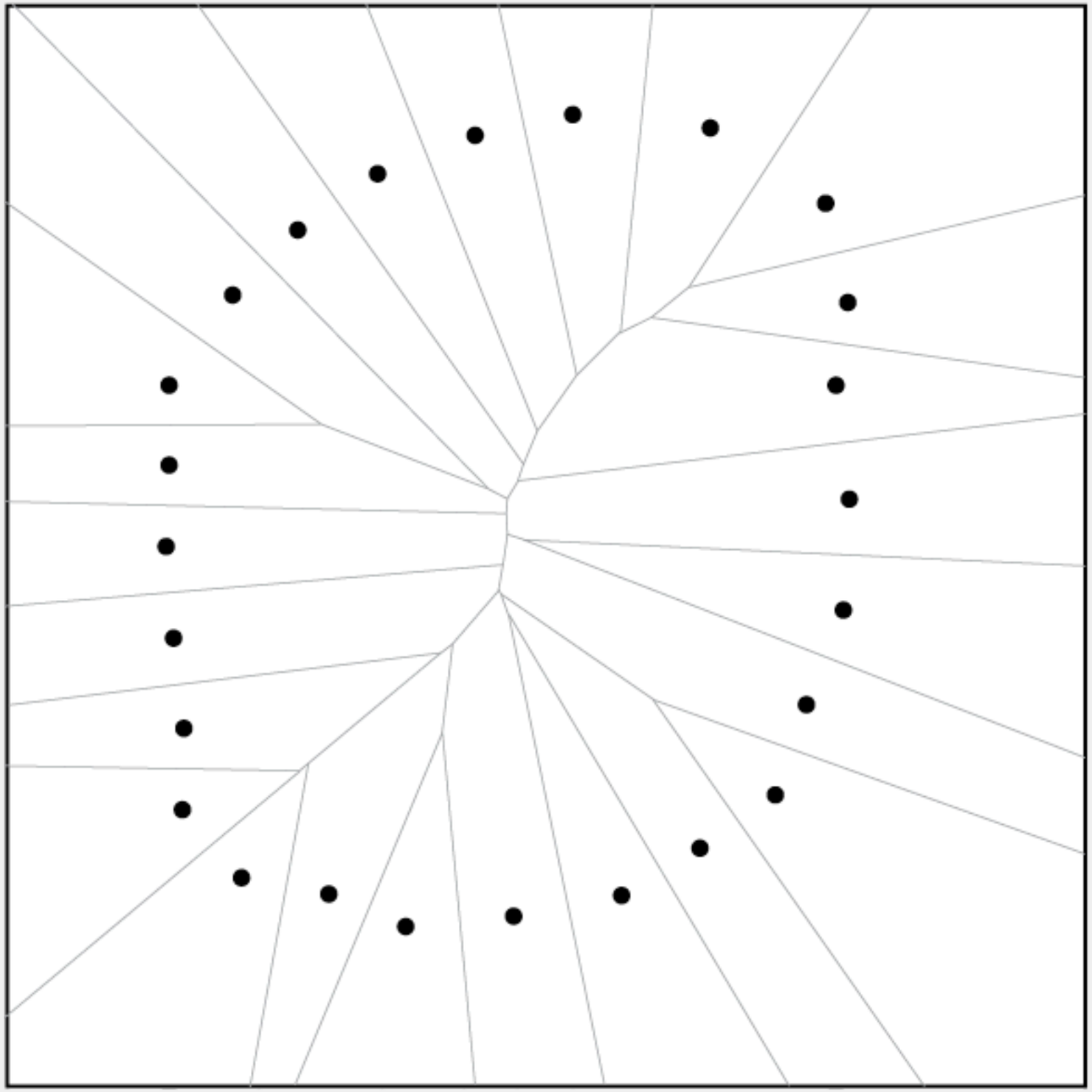
Mesh Generation and Persistent Homology

The Problem

Input: Points in Euclidean space sampled from some unknown object.

Output: Information about the topology of the unknown object.

Points, offsets, homology, and persistence.



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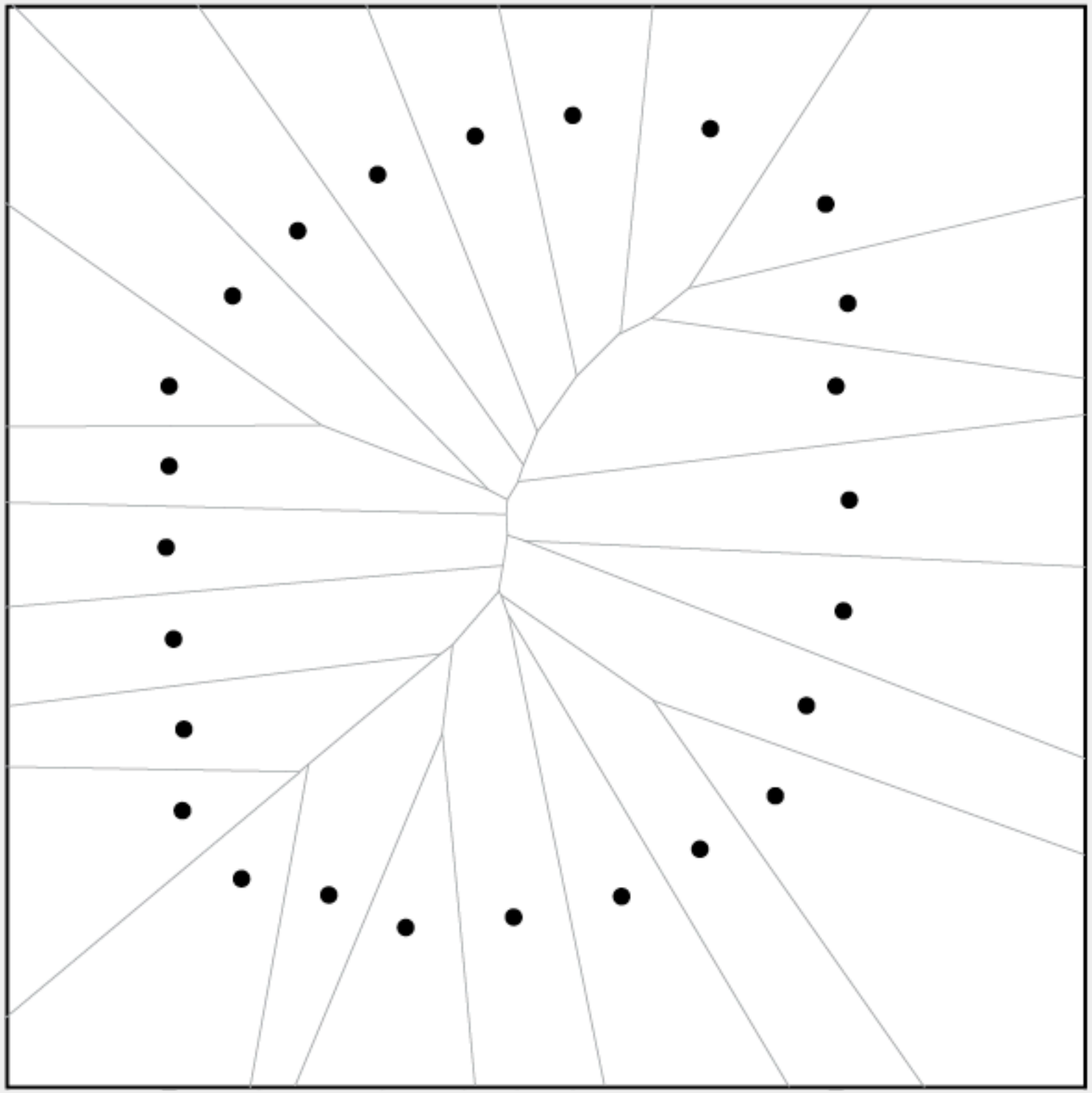

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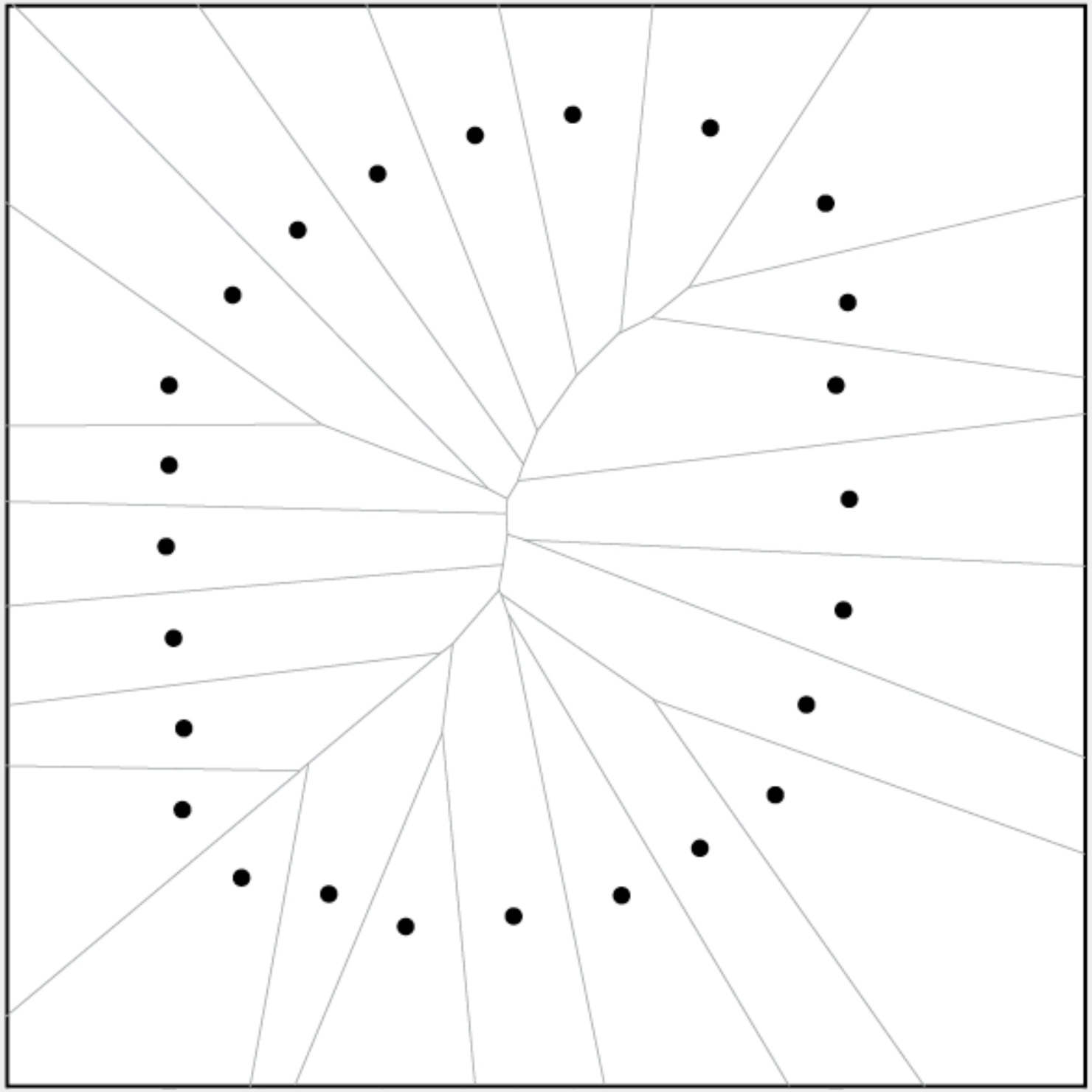

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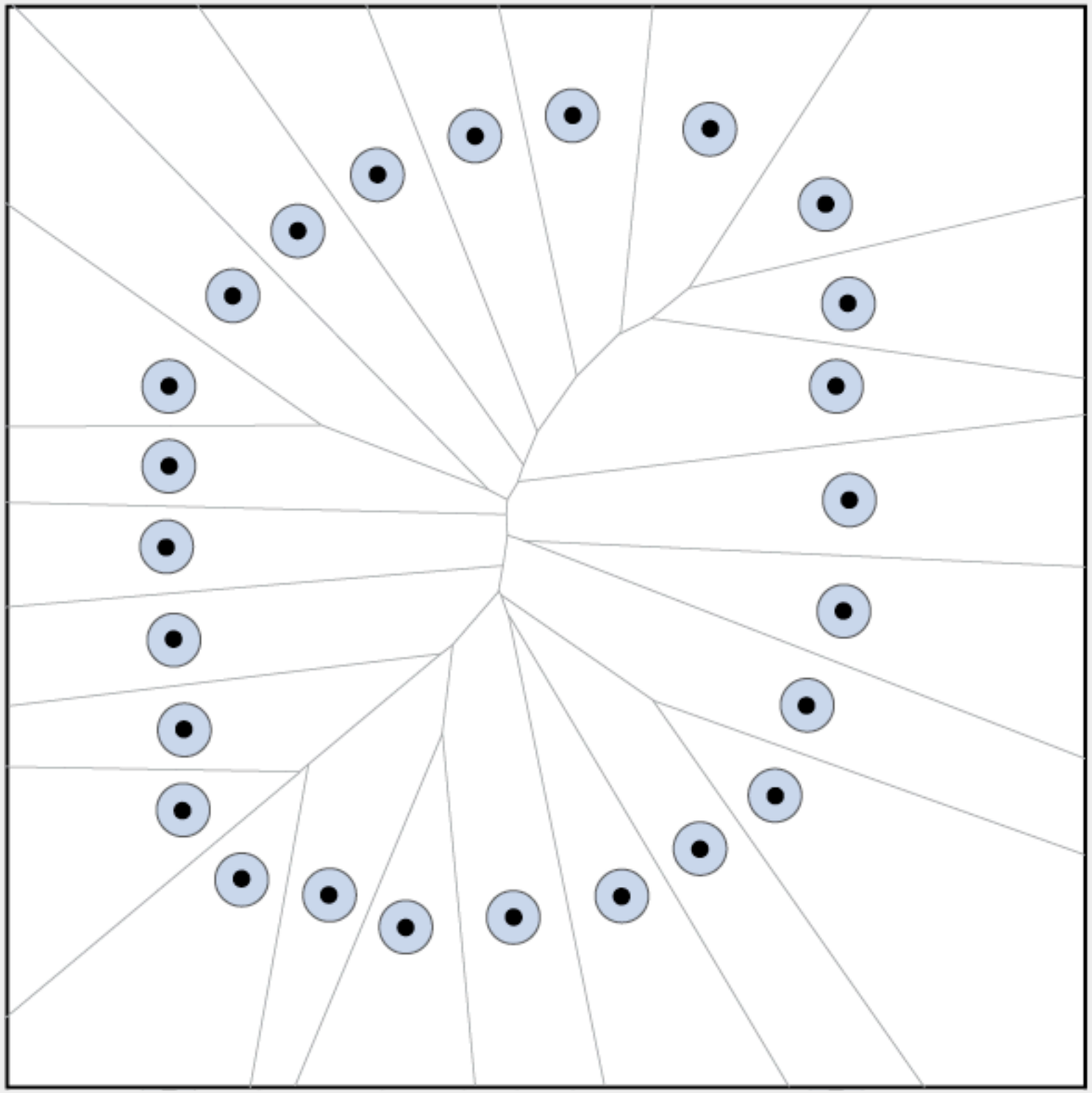

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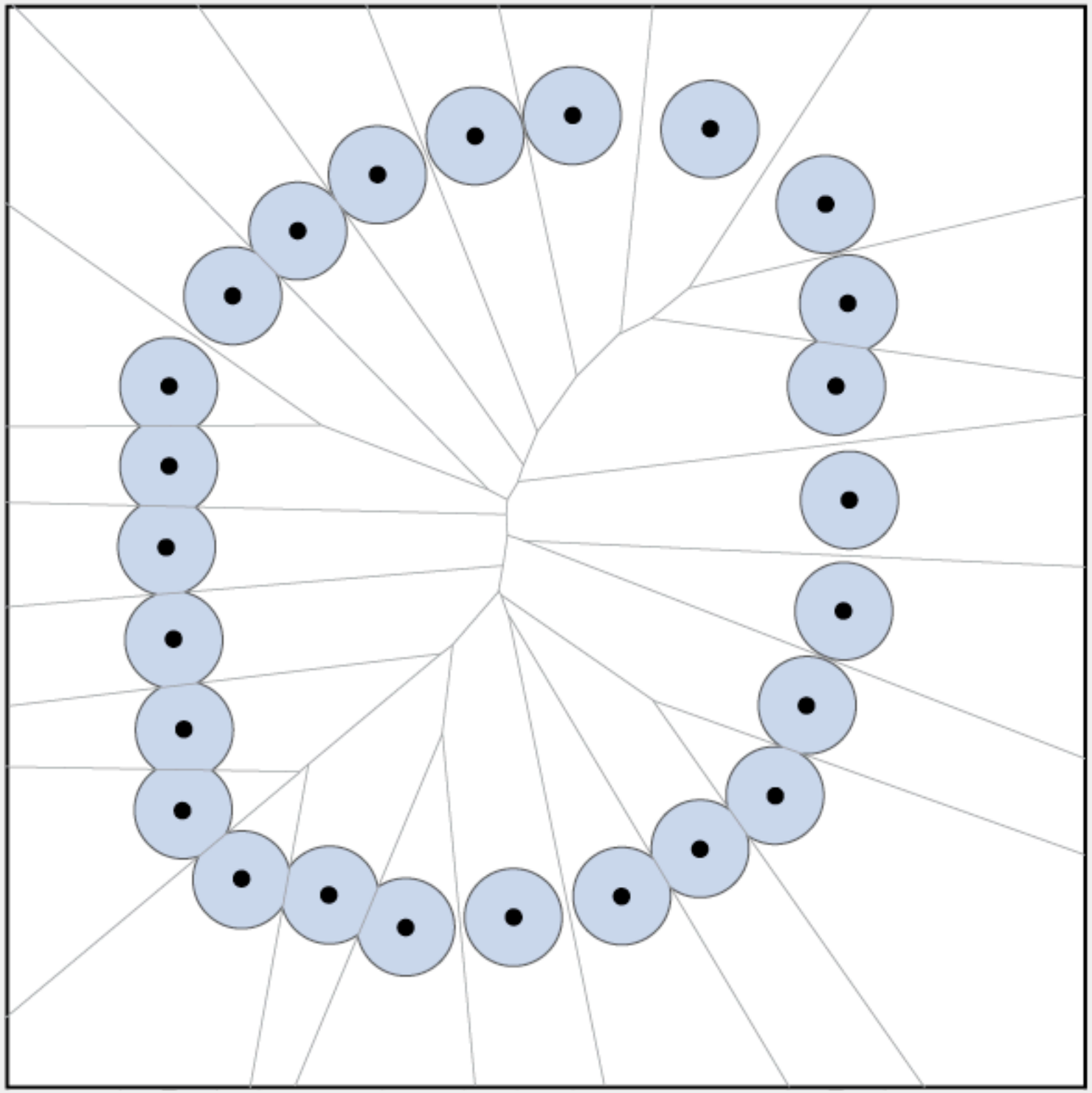

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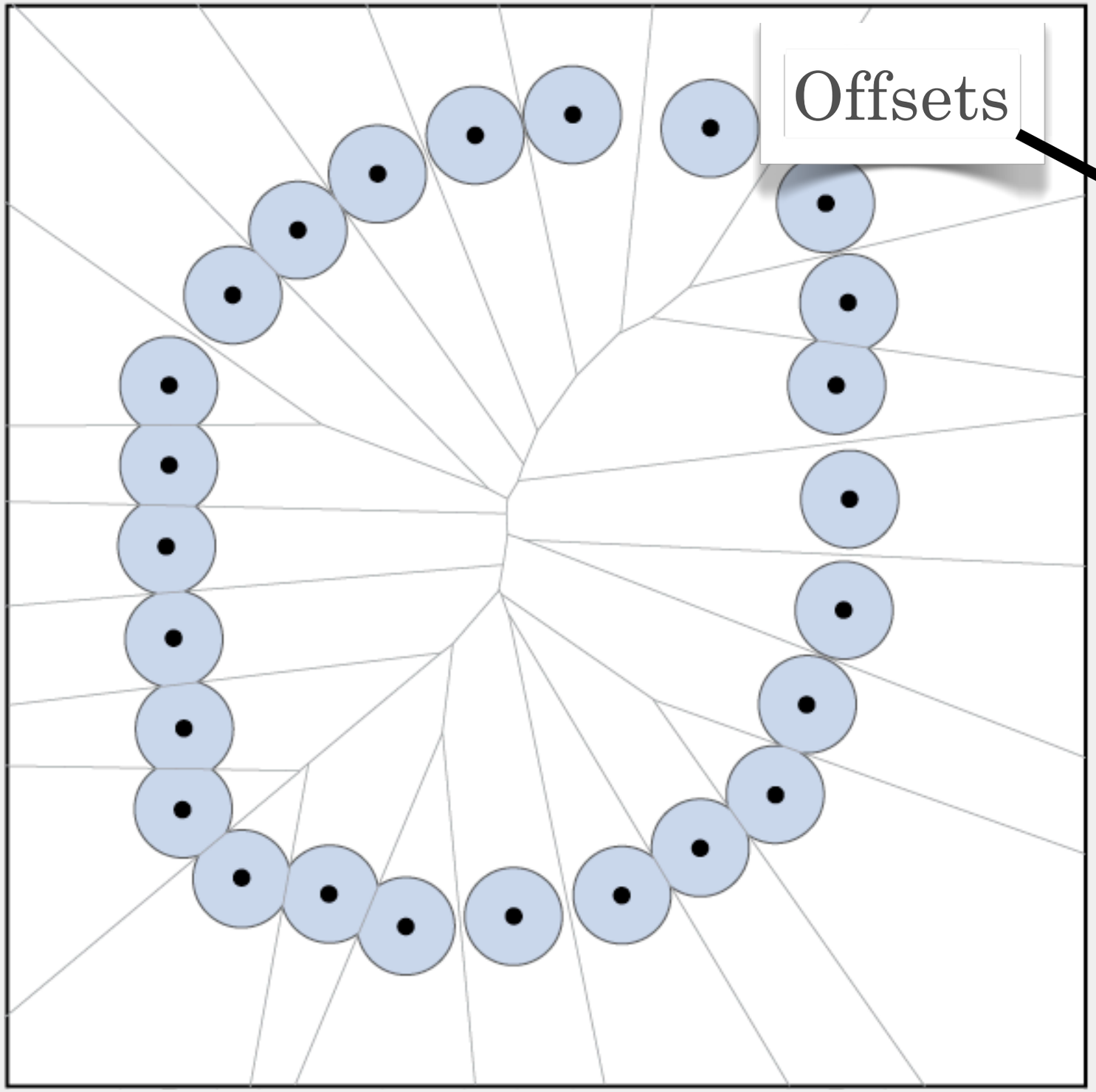

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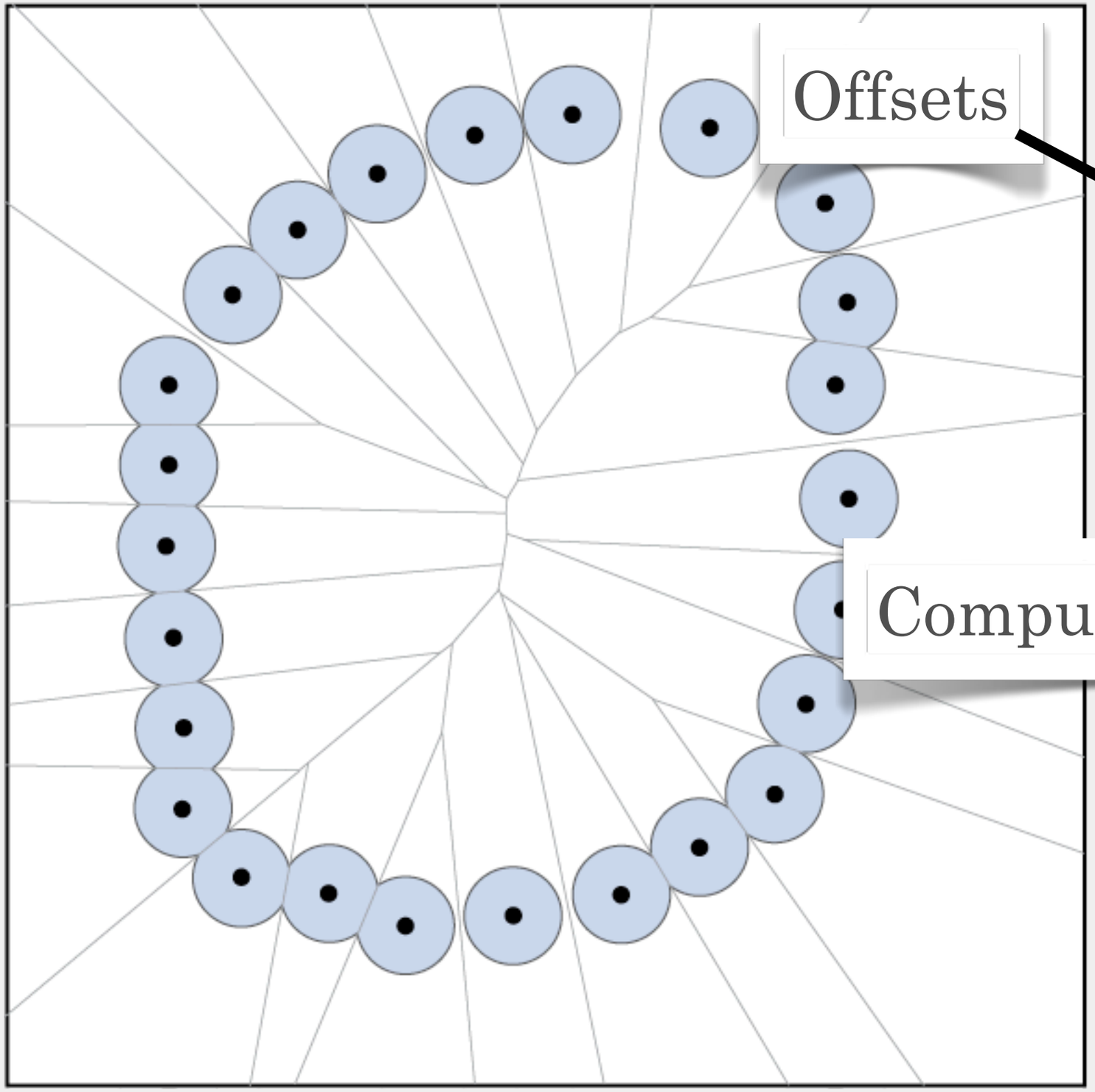

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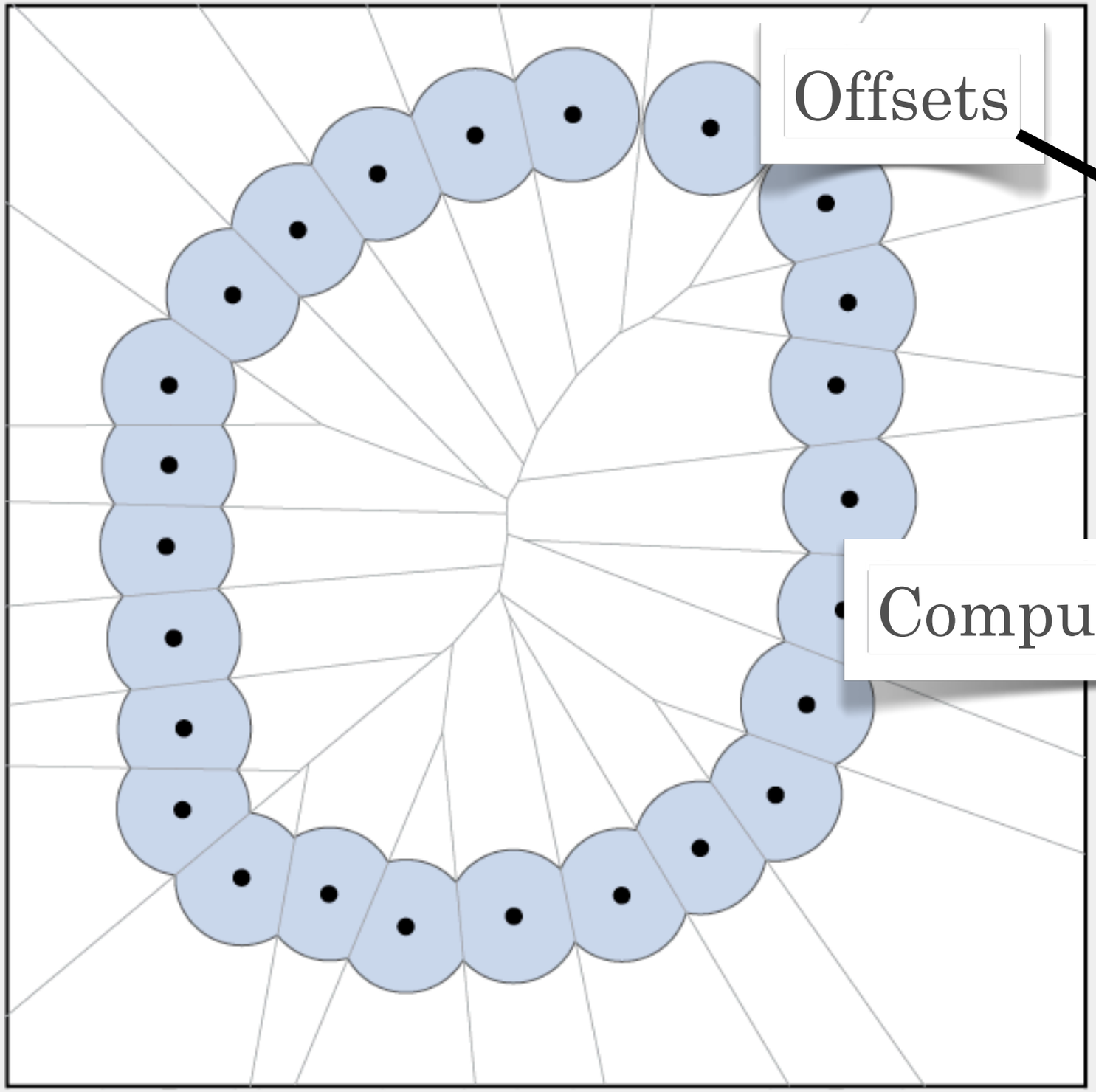

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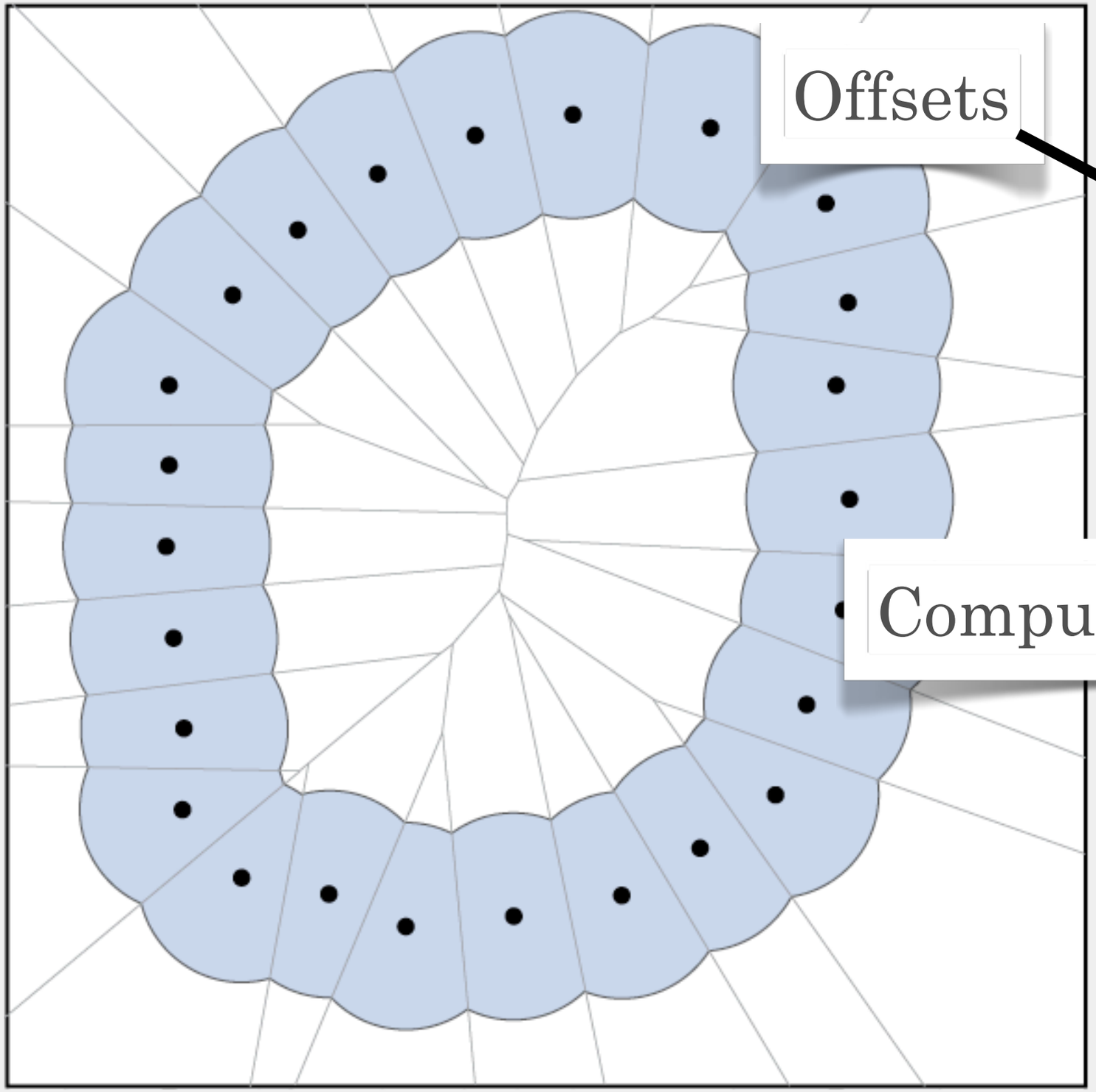

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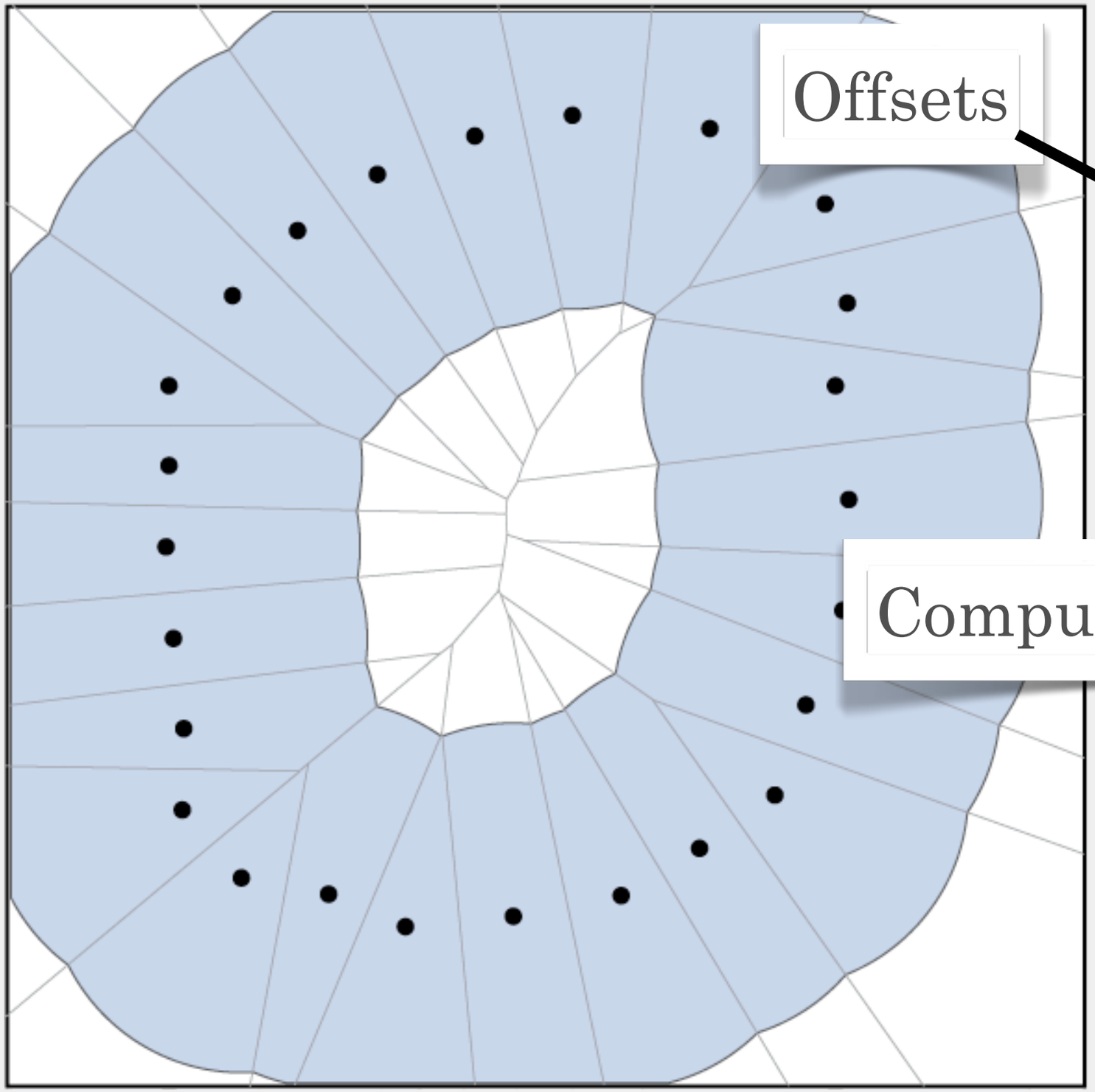

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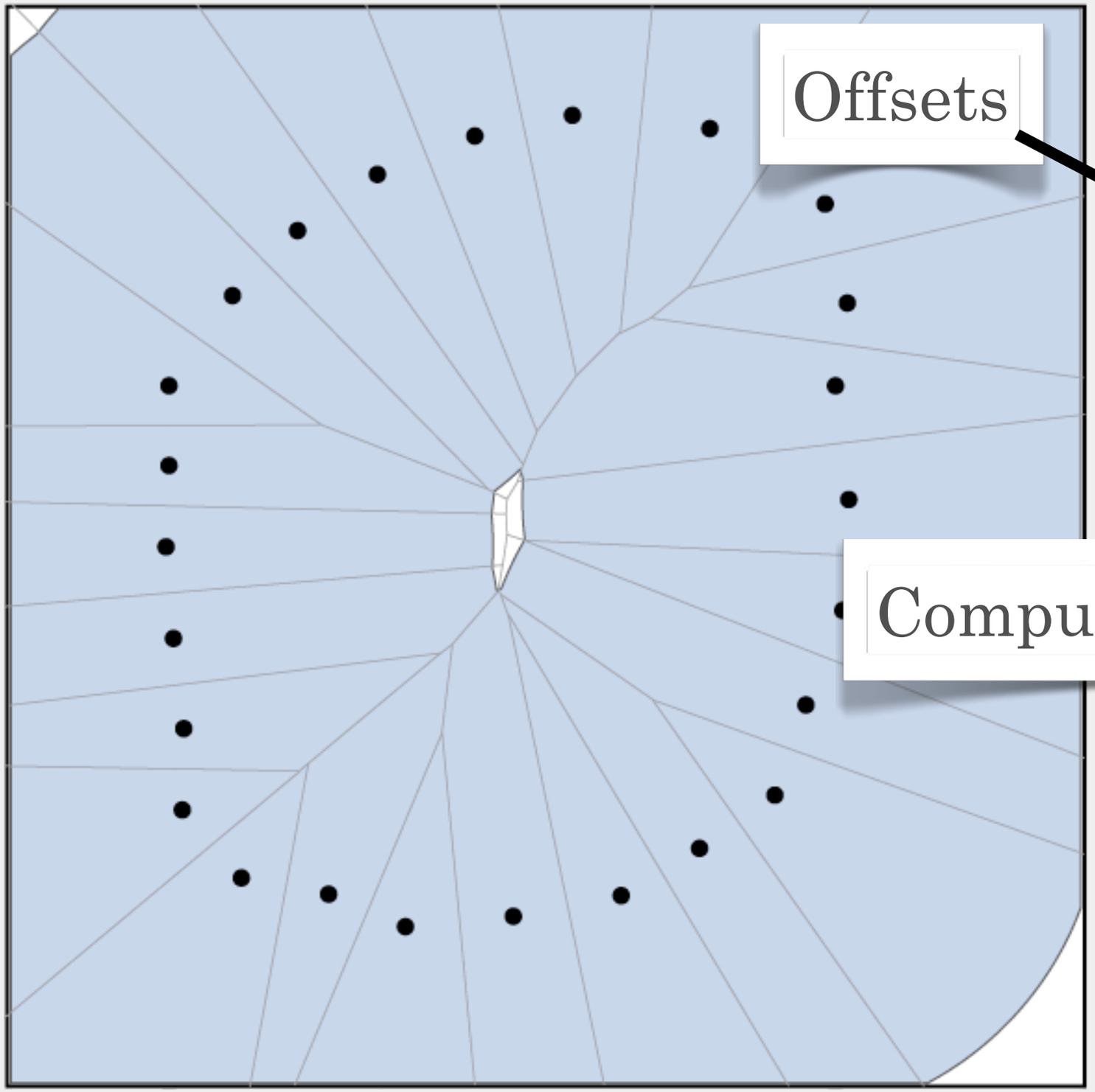

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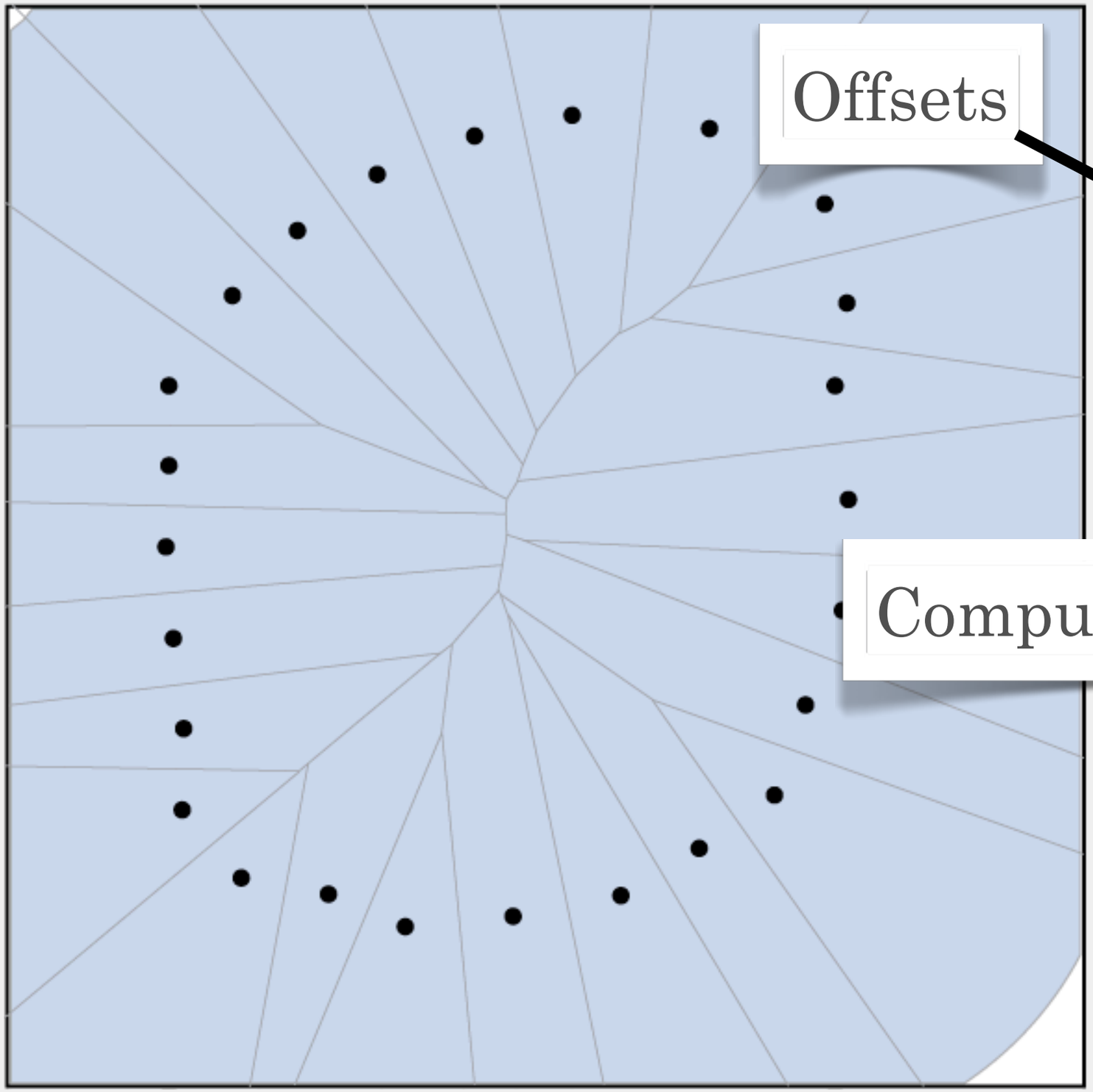

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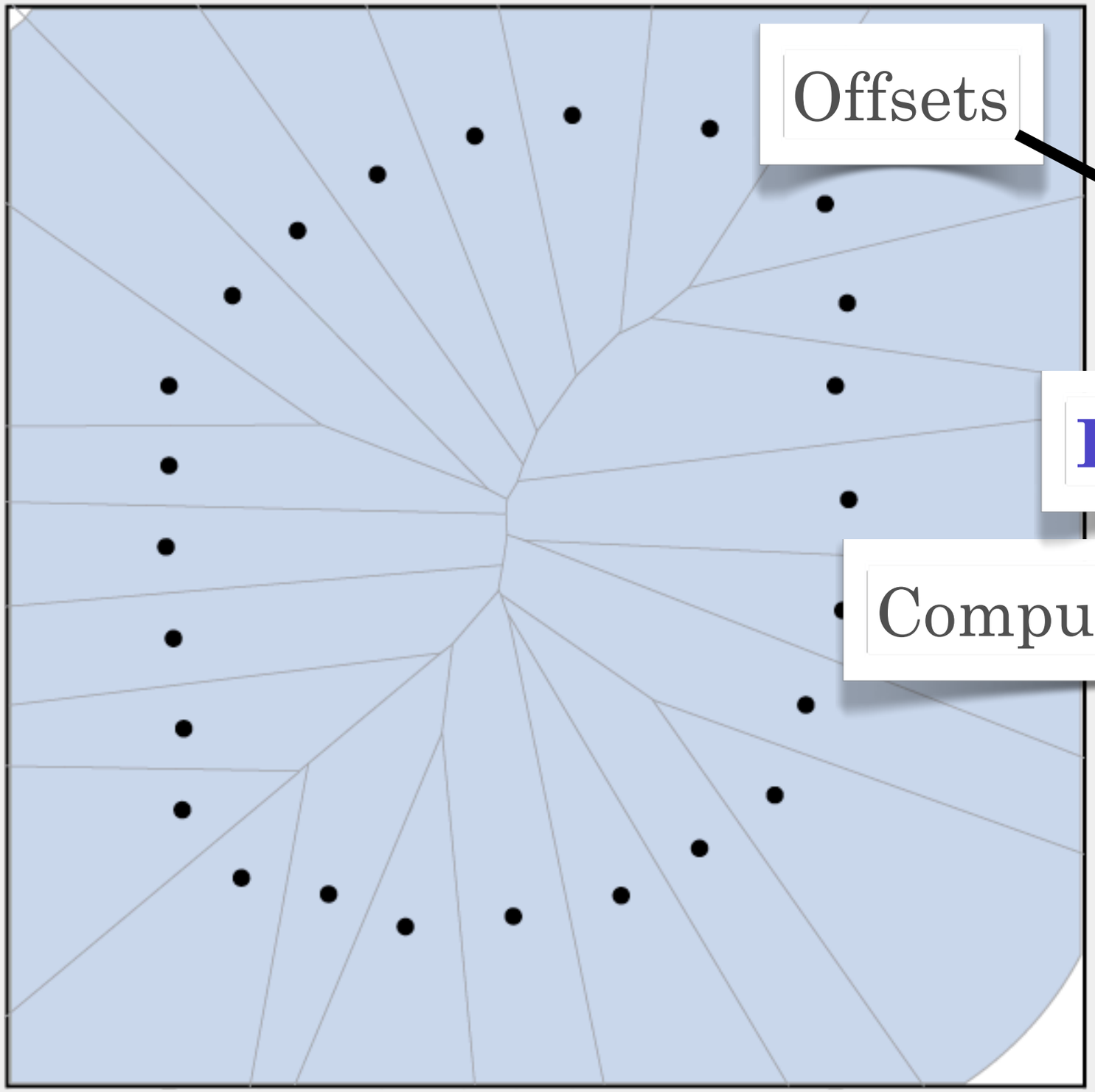

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Persistent

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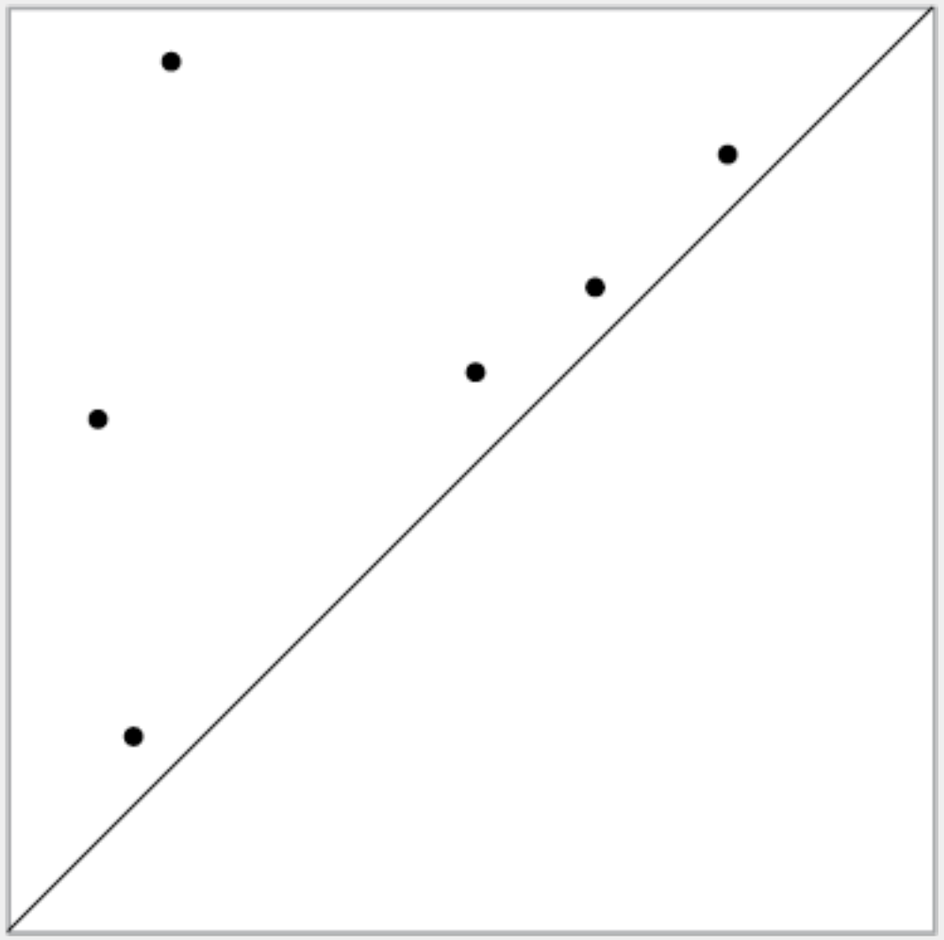
Persistence Diagrams



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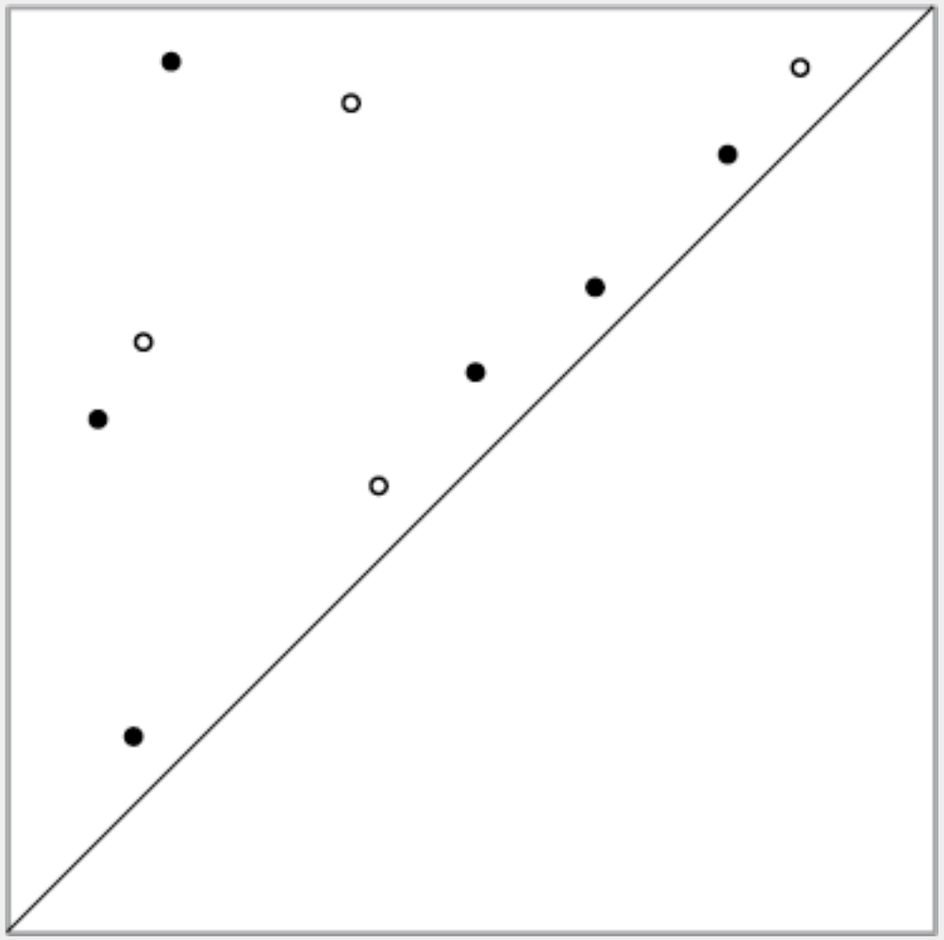

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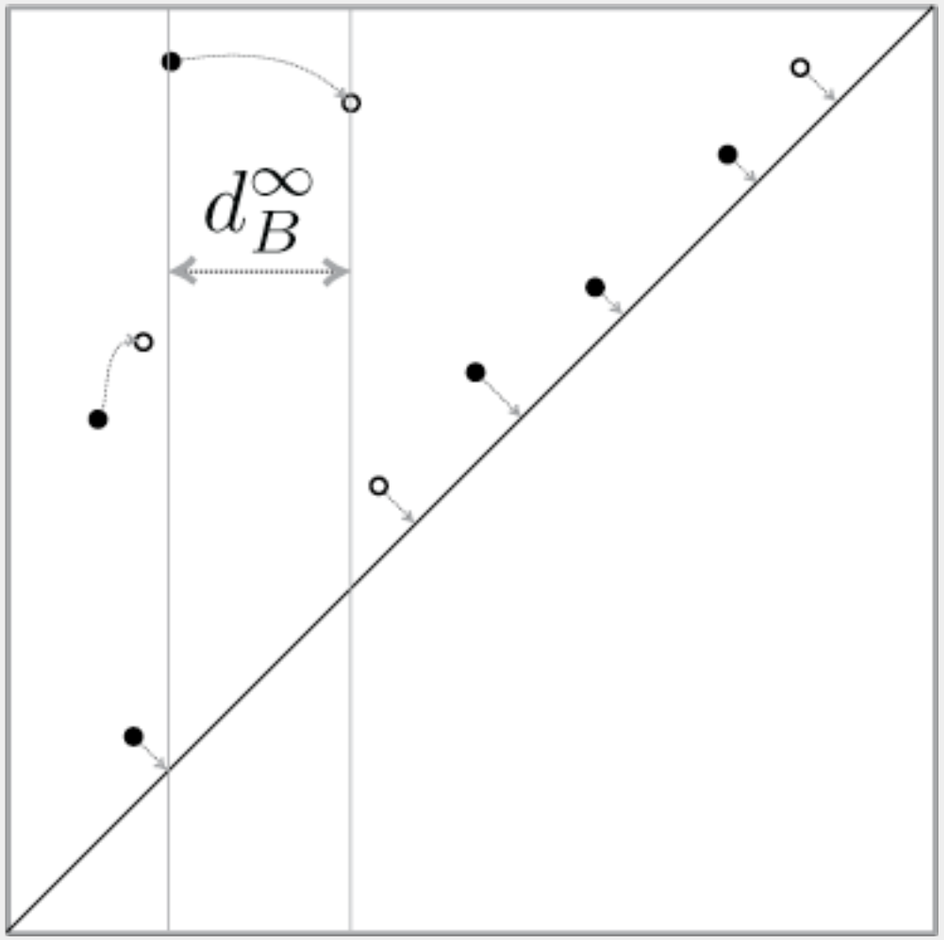

Bottleneck Distance

$$d_B^\infty = \max_i |p_i - q_i|_\infty$$

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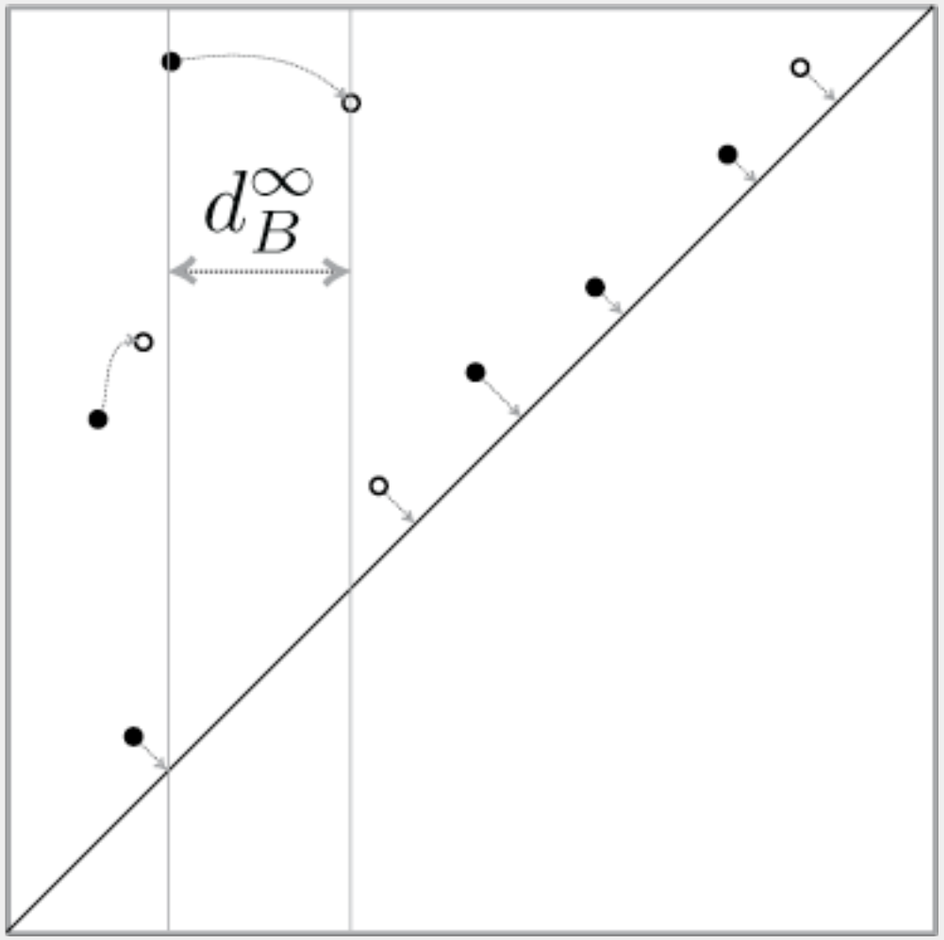

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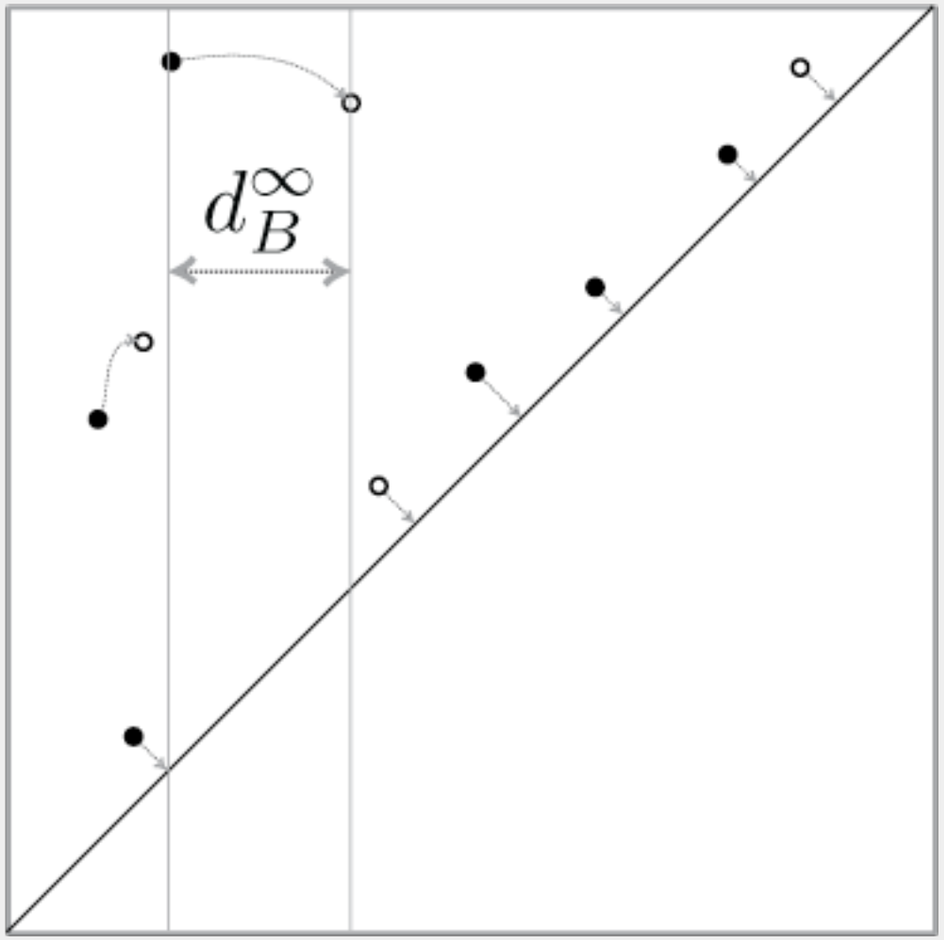

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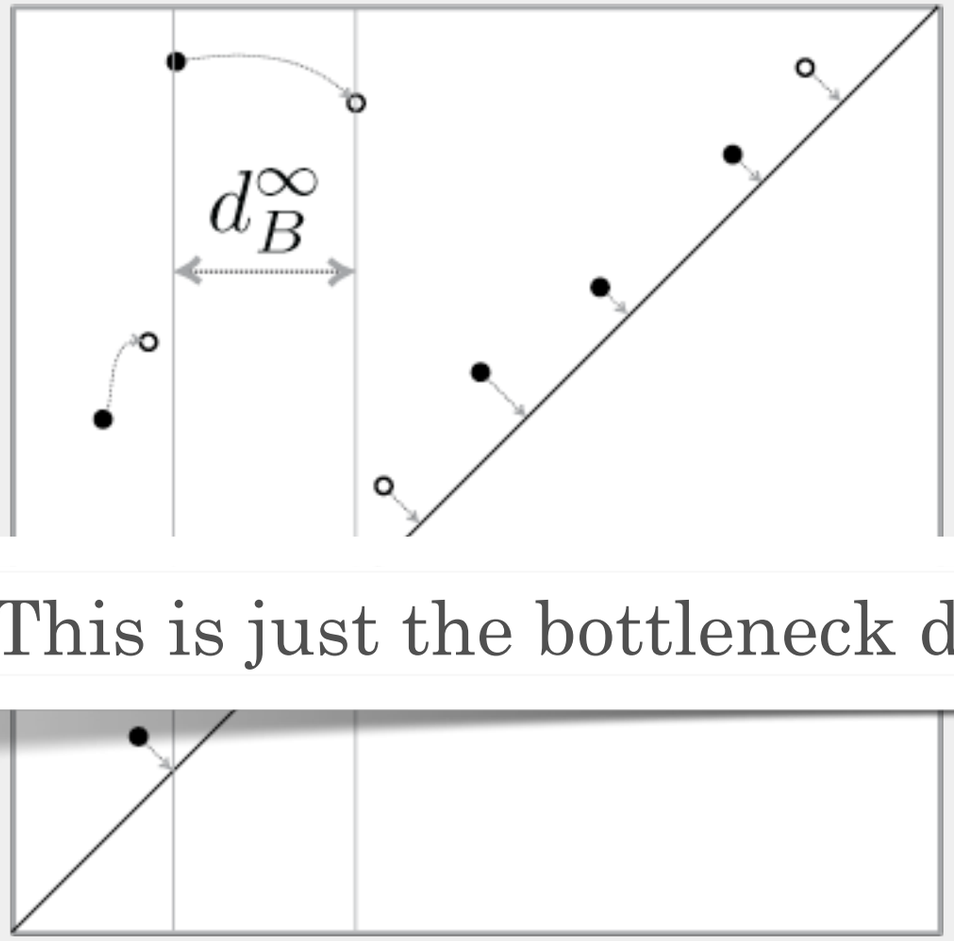

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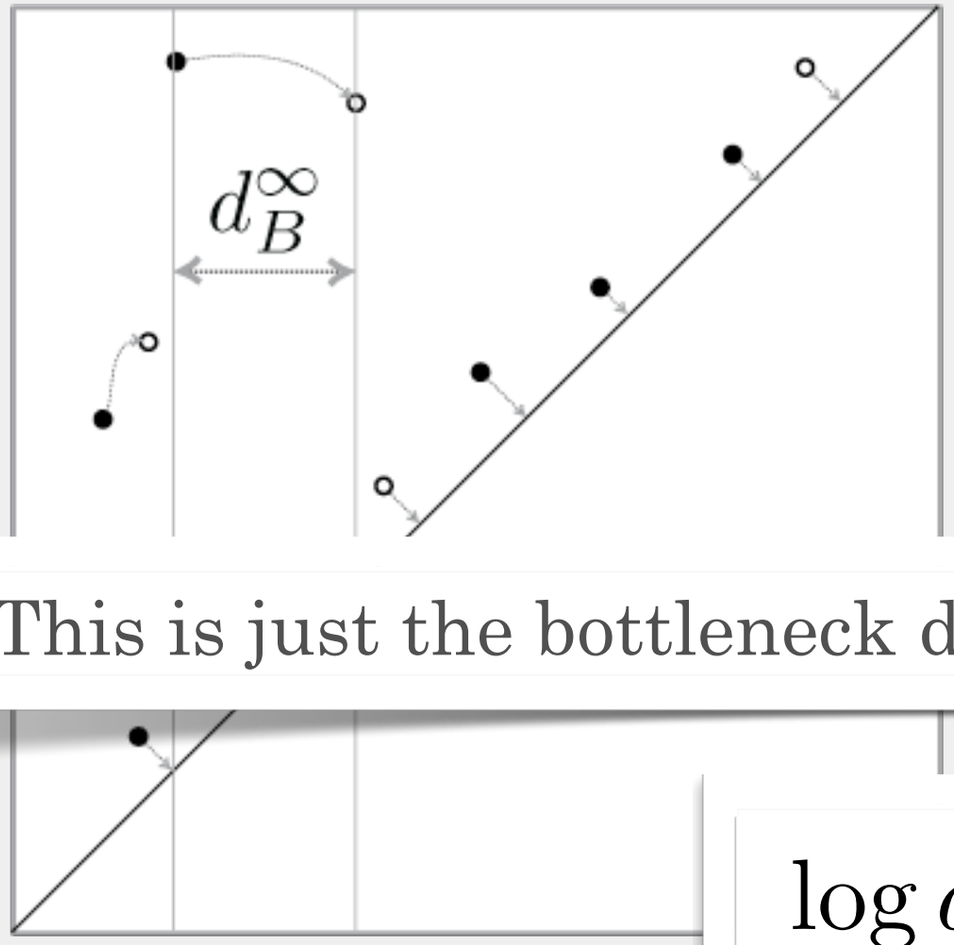

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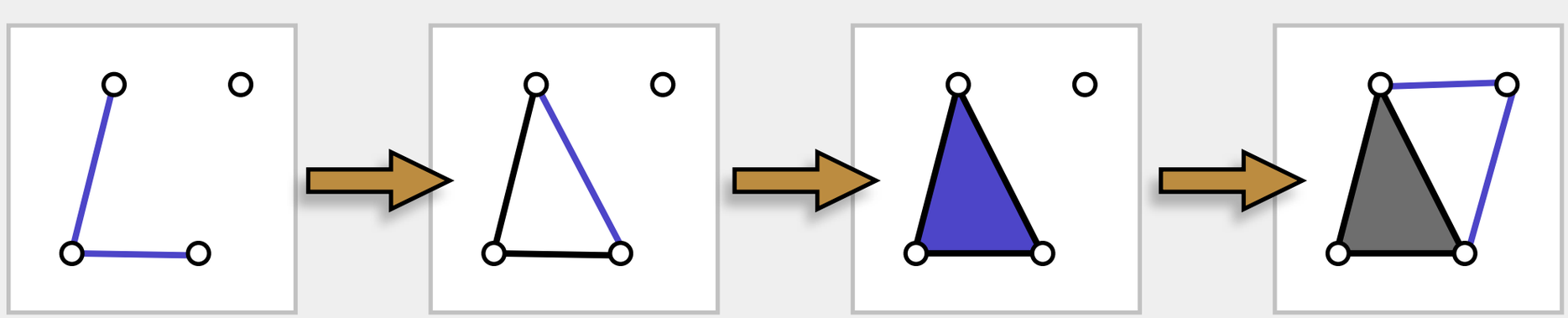
$$\begin{aligned} \log a - \log b &< \varepsilon \\ \log \tfrac{a}{b} &< \varepsilon \\ \tfrac{a}{b} &< 1 + \varepsilon \end{aligned}$$

We need to build a *filtered simplicial complex*.

Associate a *birth time* with each simplex in complex K.

At time α, we have a complex K_α consisting of all simplices born at or before time α.



time →

There are two phases, one is geometric the other is topological.

Geometry

Build a filtration, i.e. a filtered simplicial complex.

Topology
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Compute the persistence diagram

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Running time: **$O(N^3)$**.
N is the **size** of the complex.

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Still $\mathbf{n^d}$, but we can quit early.

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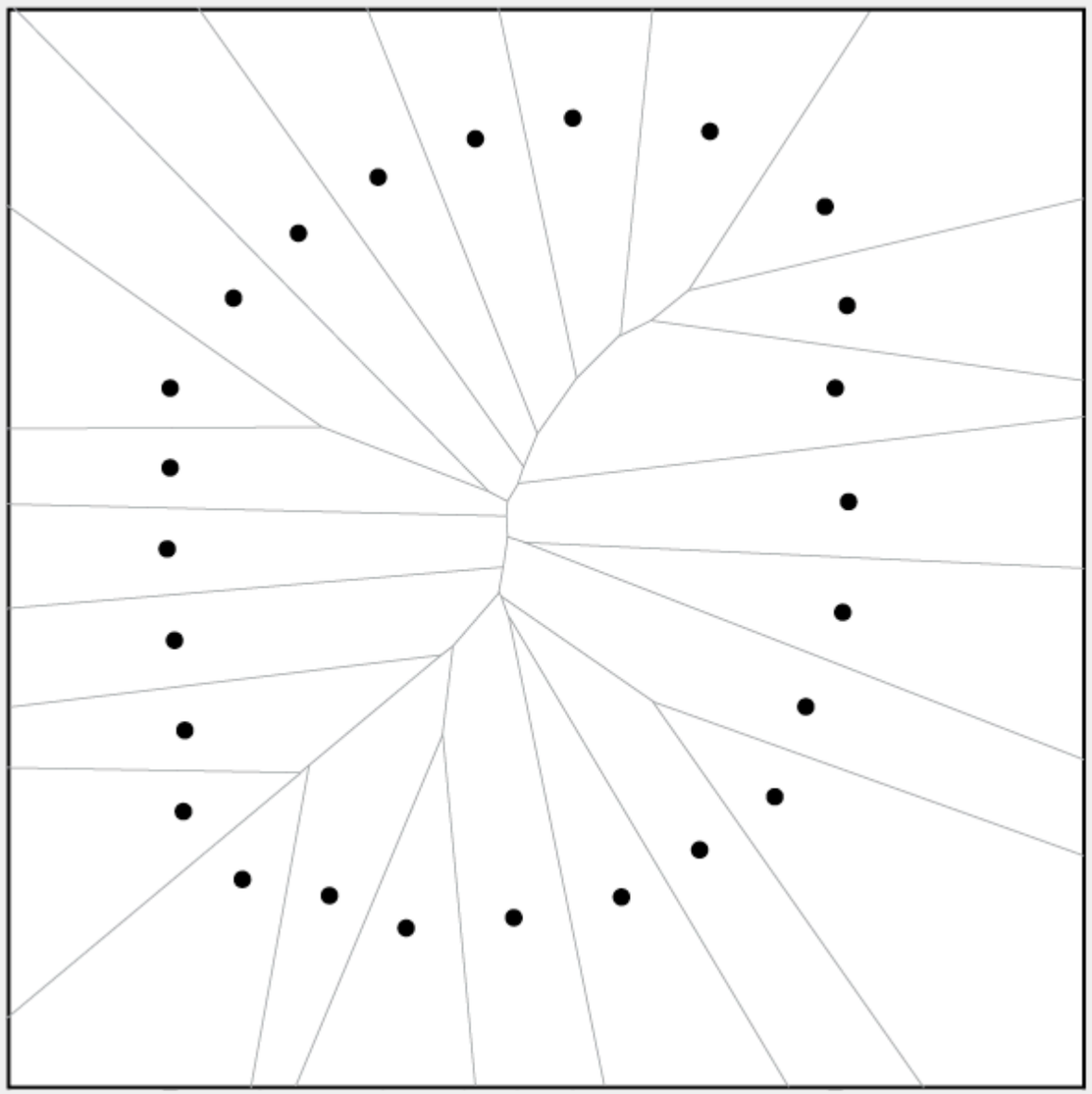

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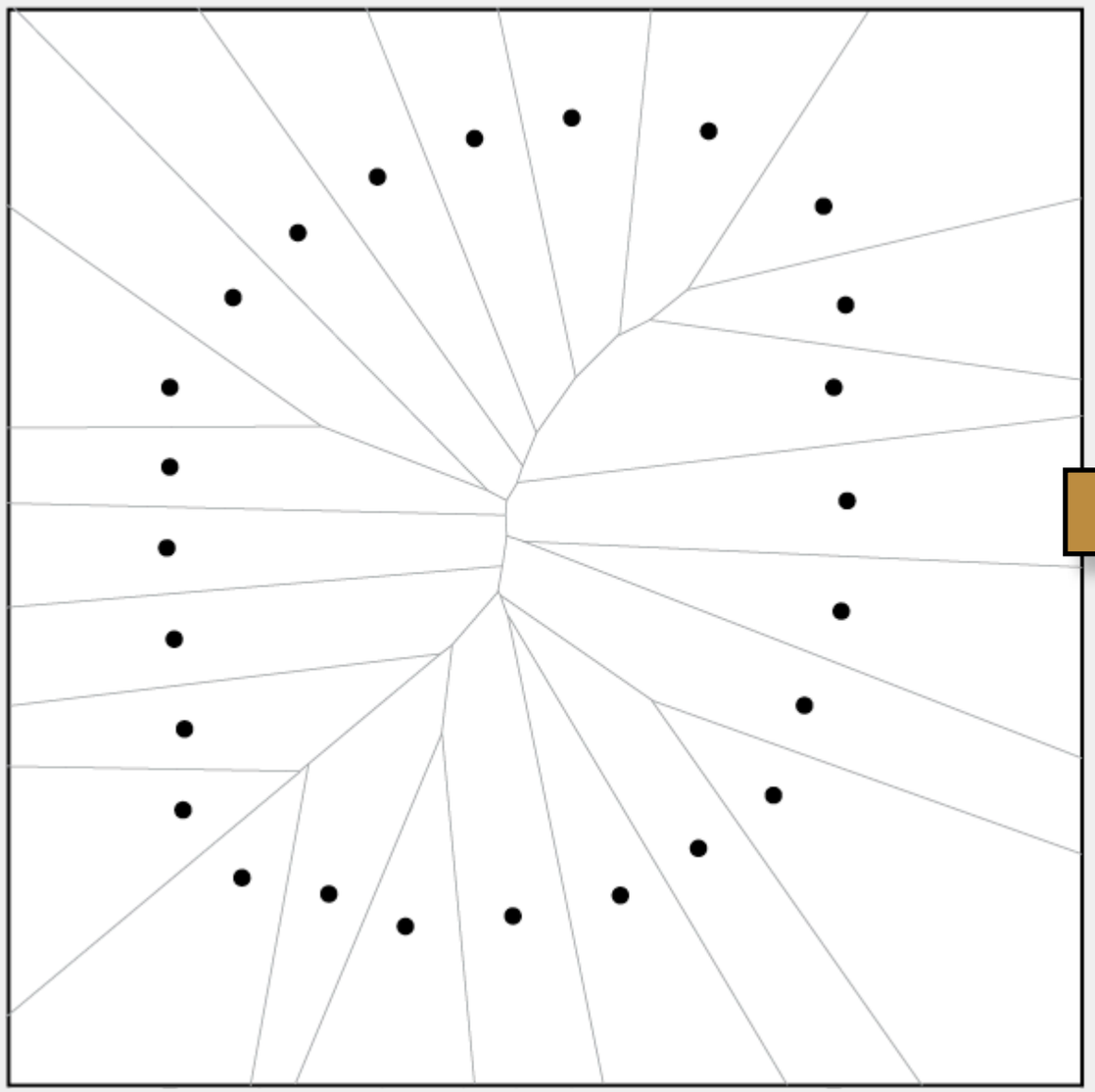

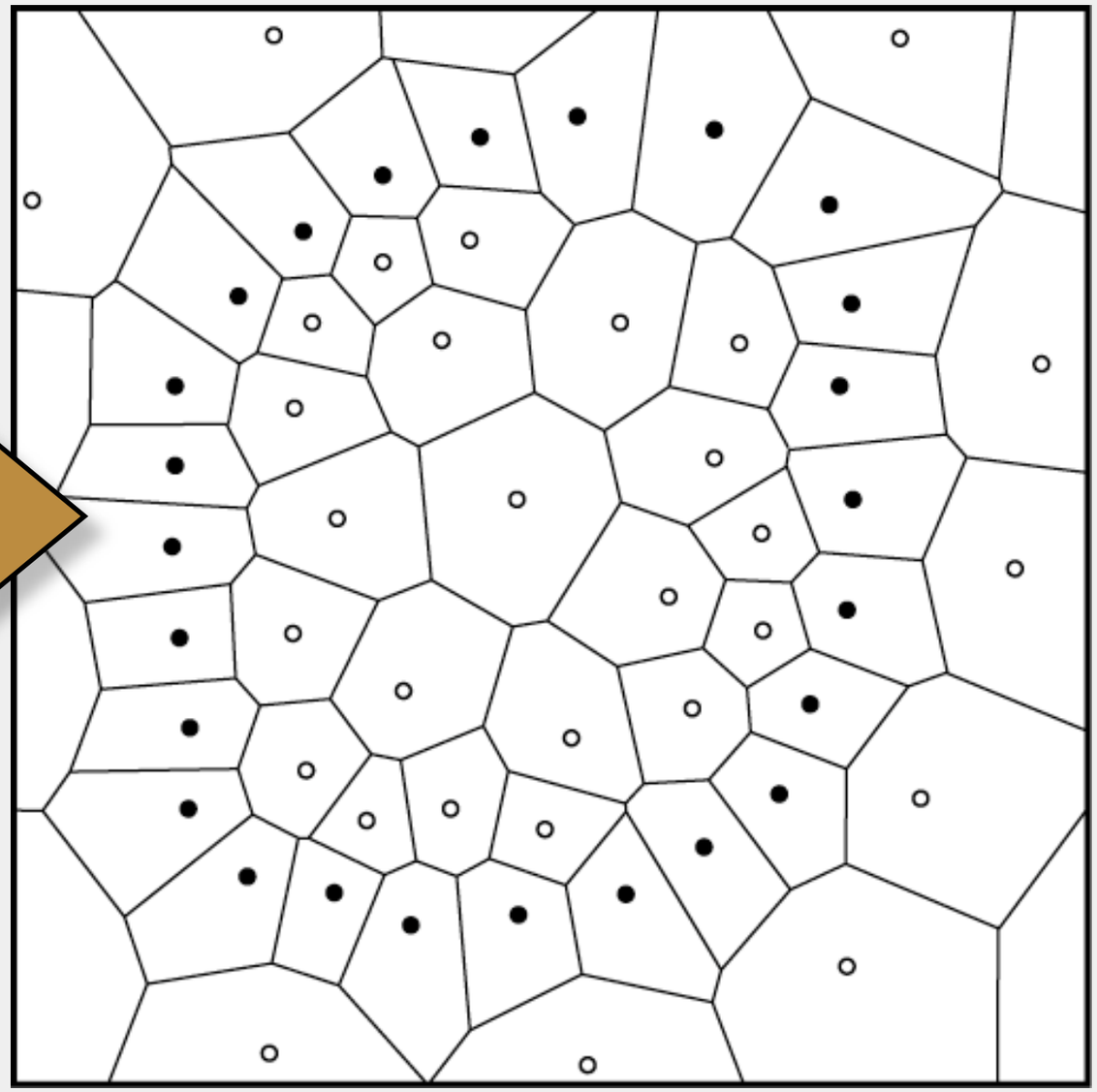

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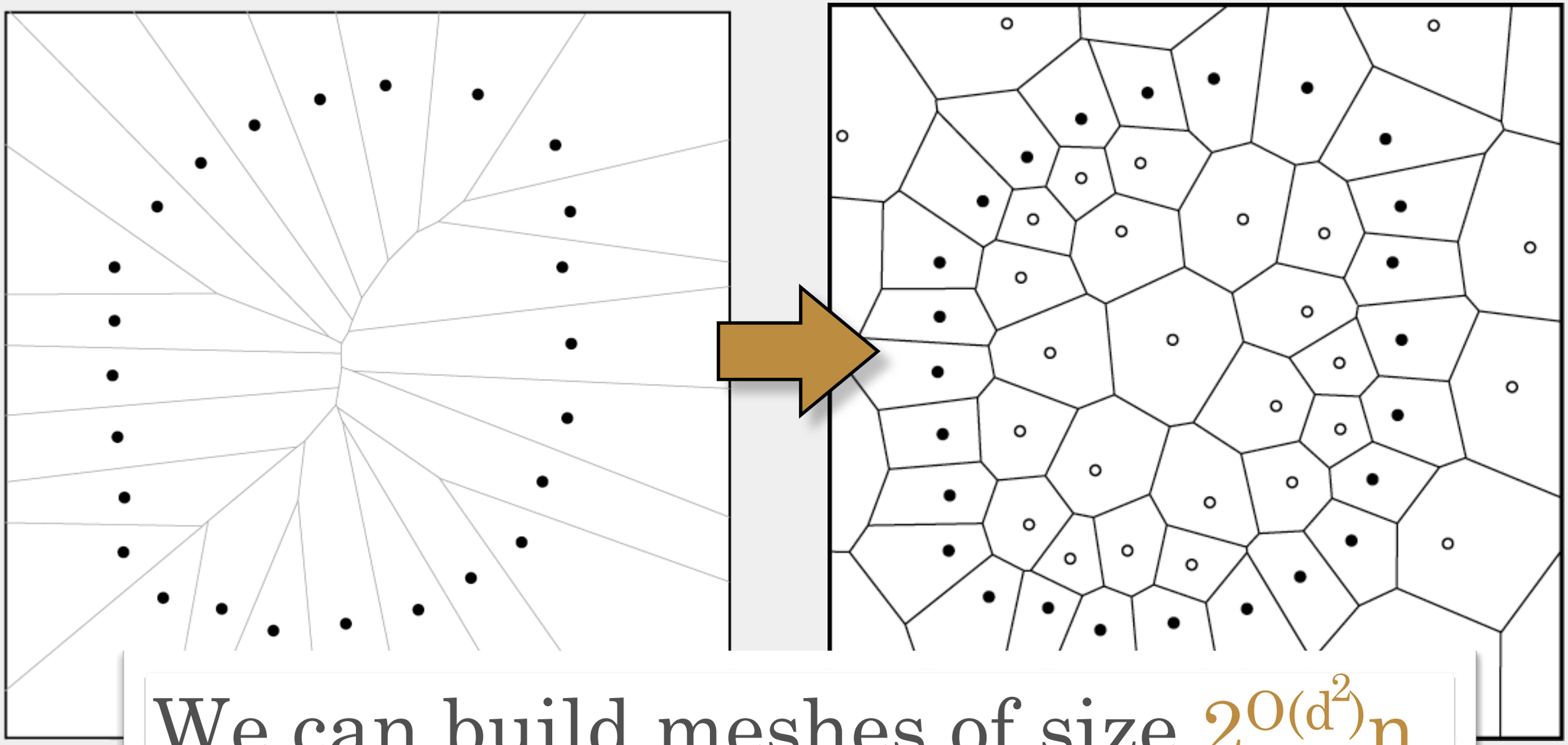

We can build meshes of size $2^{O(d^2)}n$.

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Theorem [Miller, Phillips, Sheehy, '08]:

A quality mesh of a *well-paced* point set has size $O(n)$.

The α-mesh filtration

1. Build a mesh M.

2. Assign birth times to vertices based on distance to P (special case points very close to P).

3. For each simplex s of Del(M), let birth(s) be the min birth time of its vertices.

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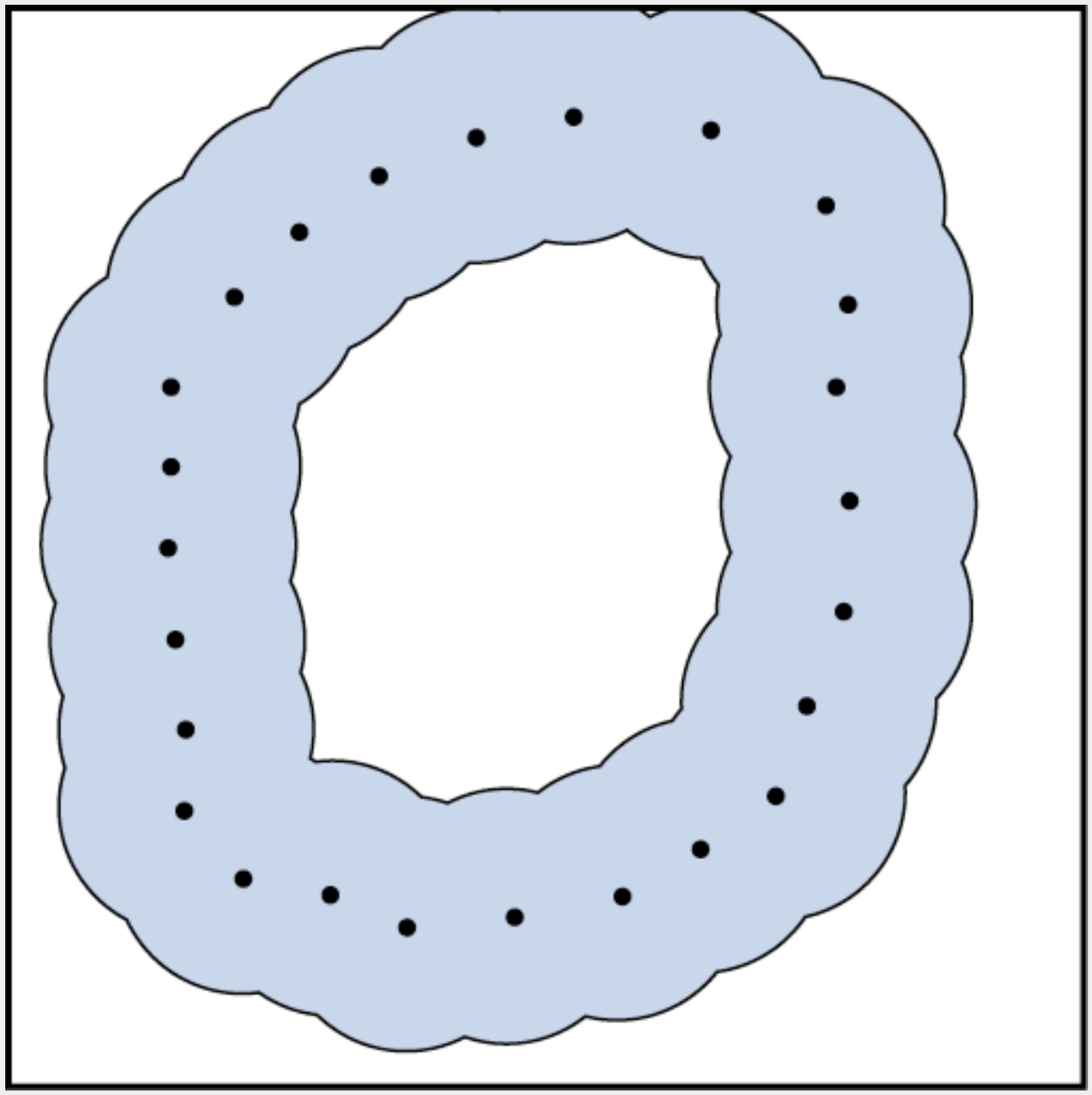

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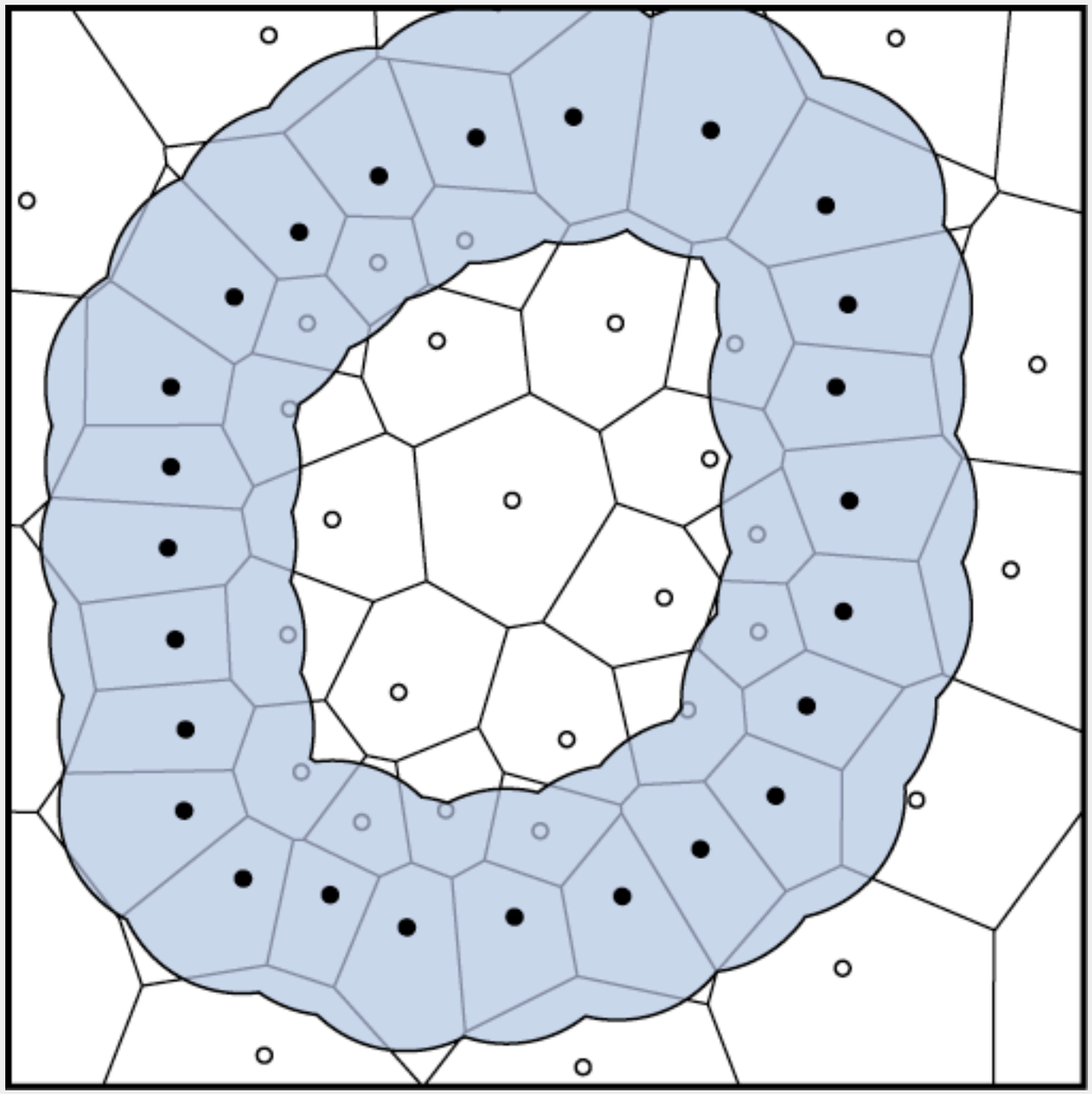

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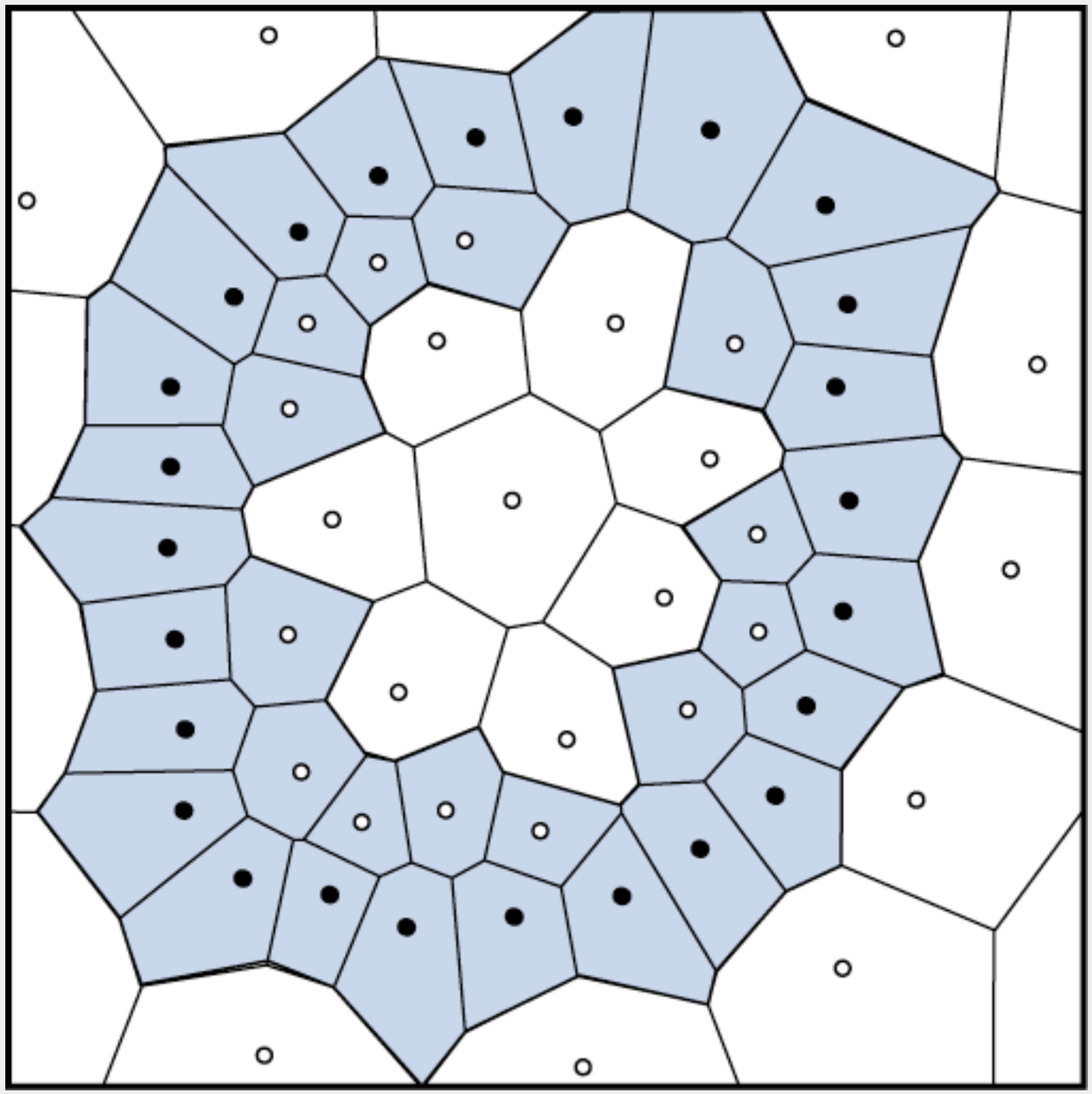

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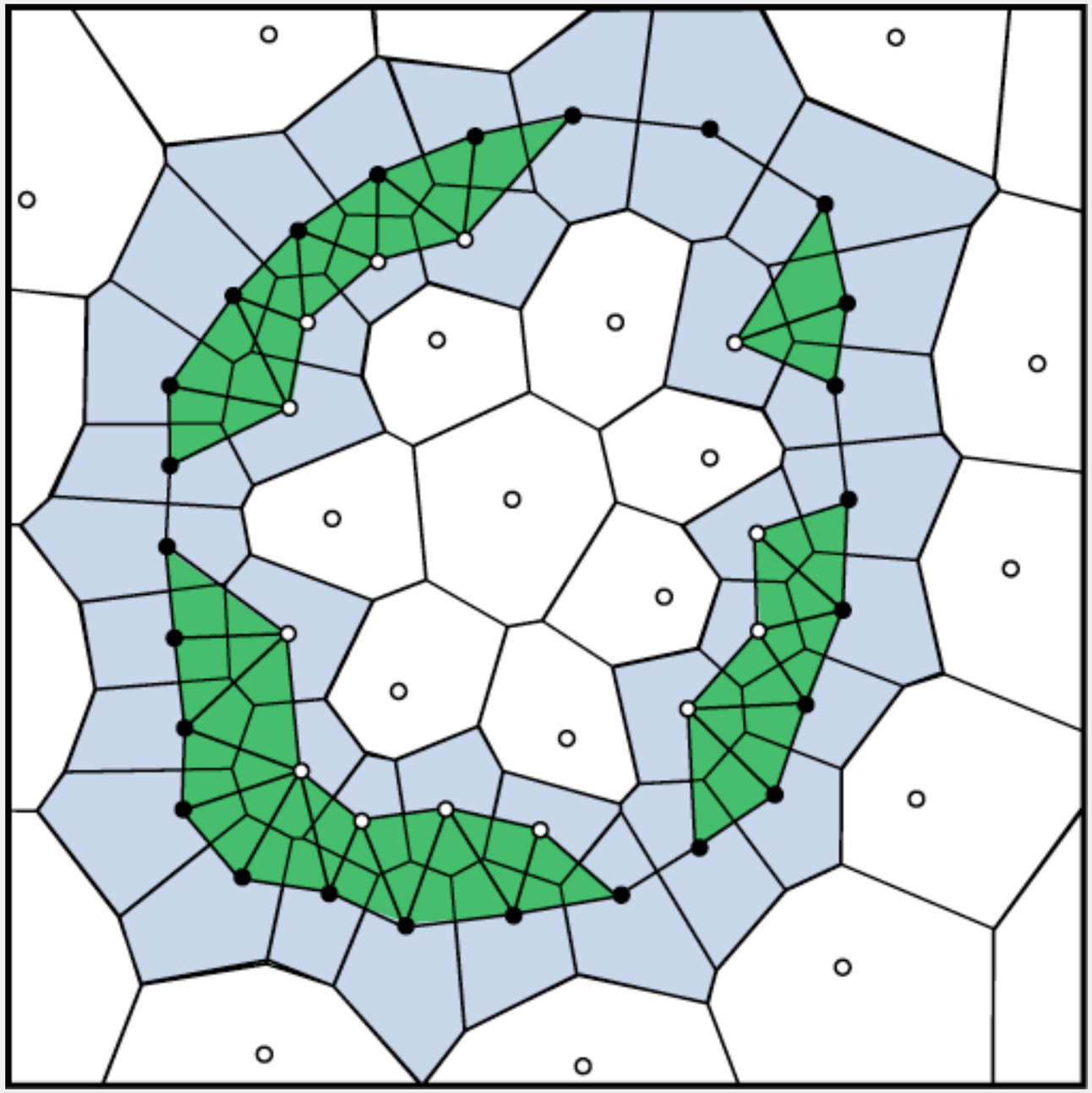

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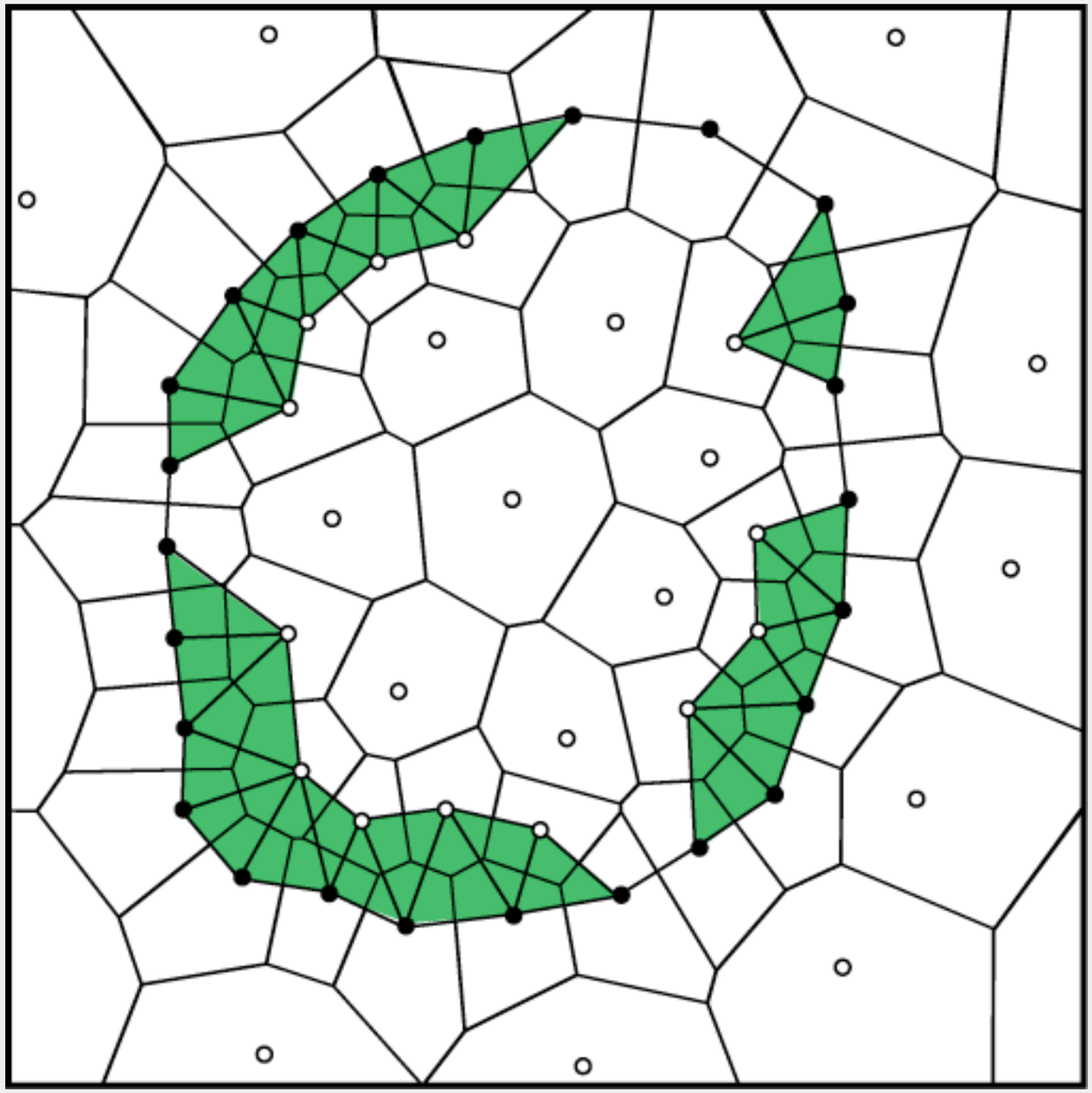

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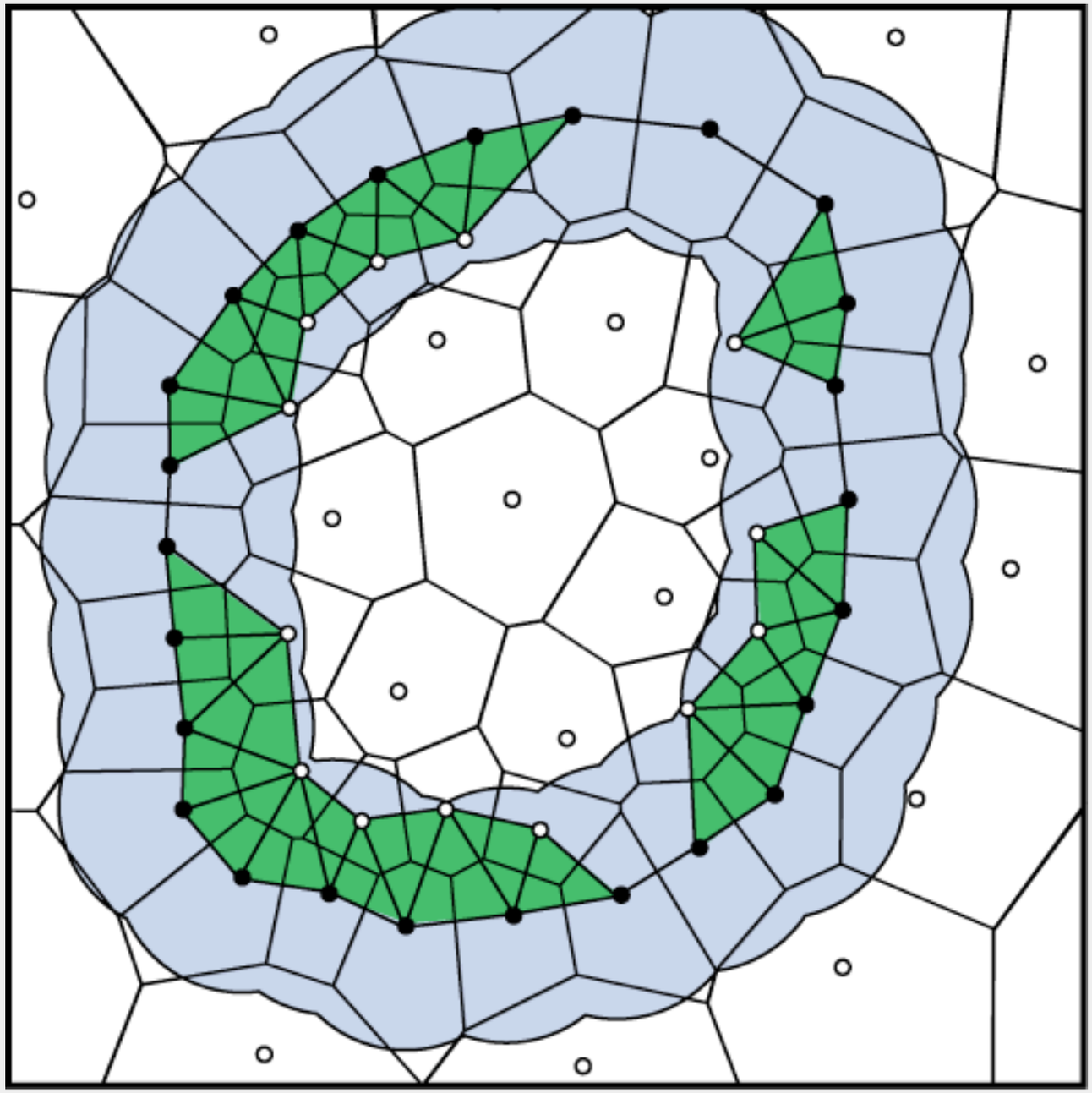

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Theorem [Chazal et al, '09]:

If $\{P_\alpha\}$ and $\{Q_\alpha\}$ are ε-interleaved then their persistence diagrams are ε-close in the bottleneck distance.

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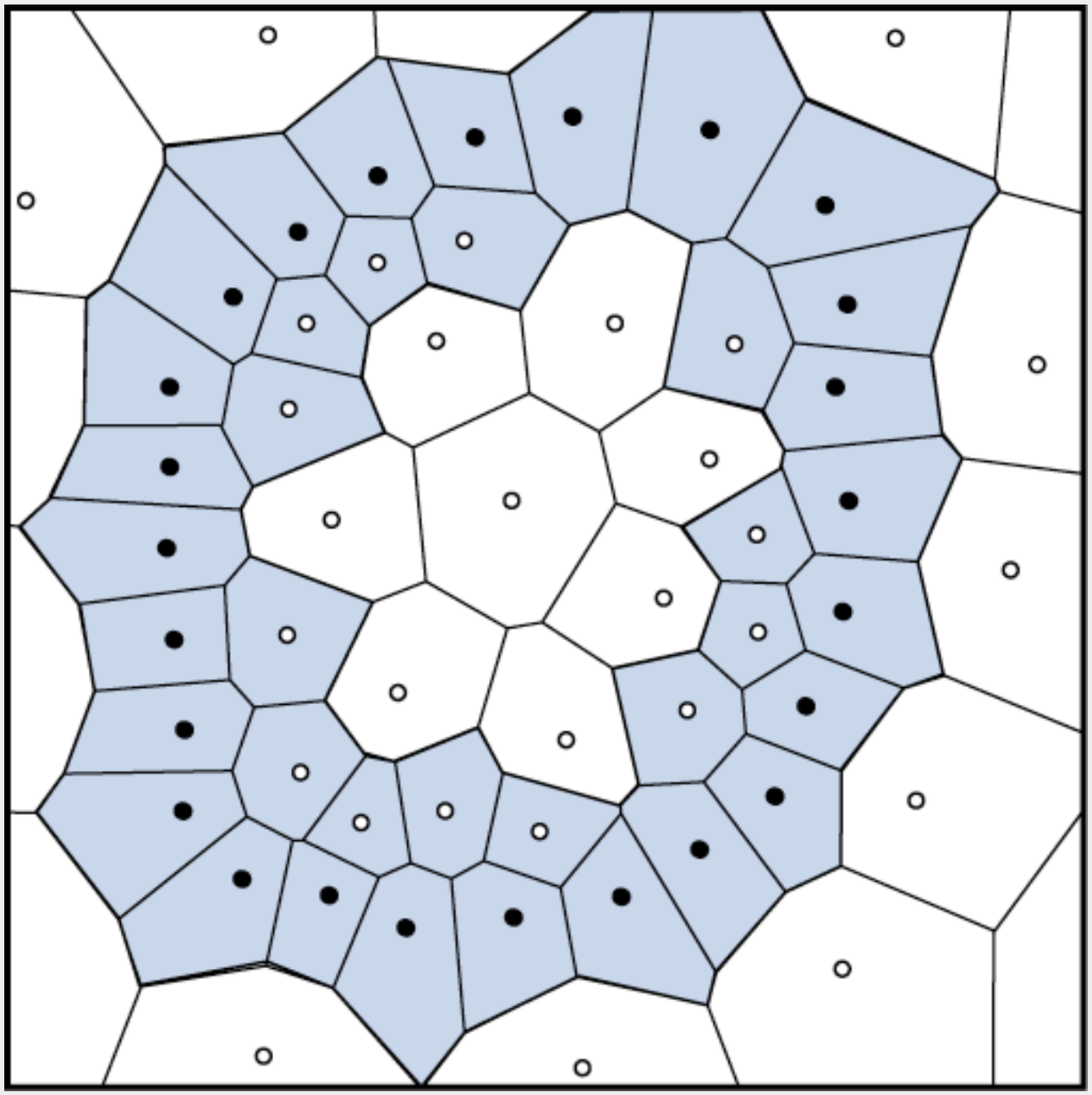

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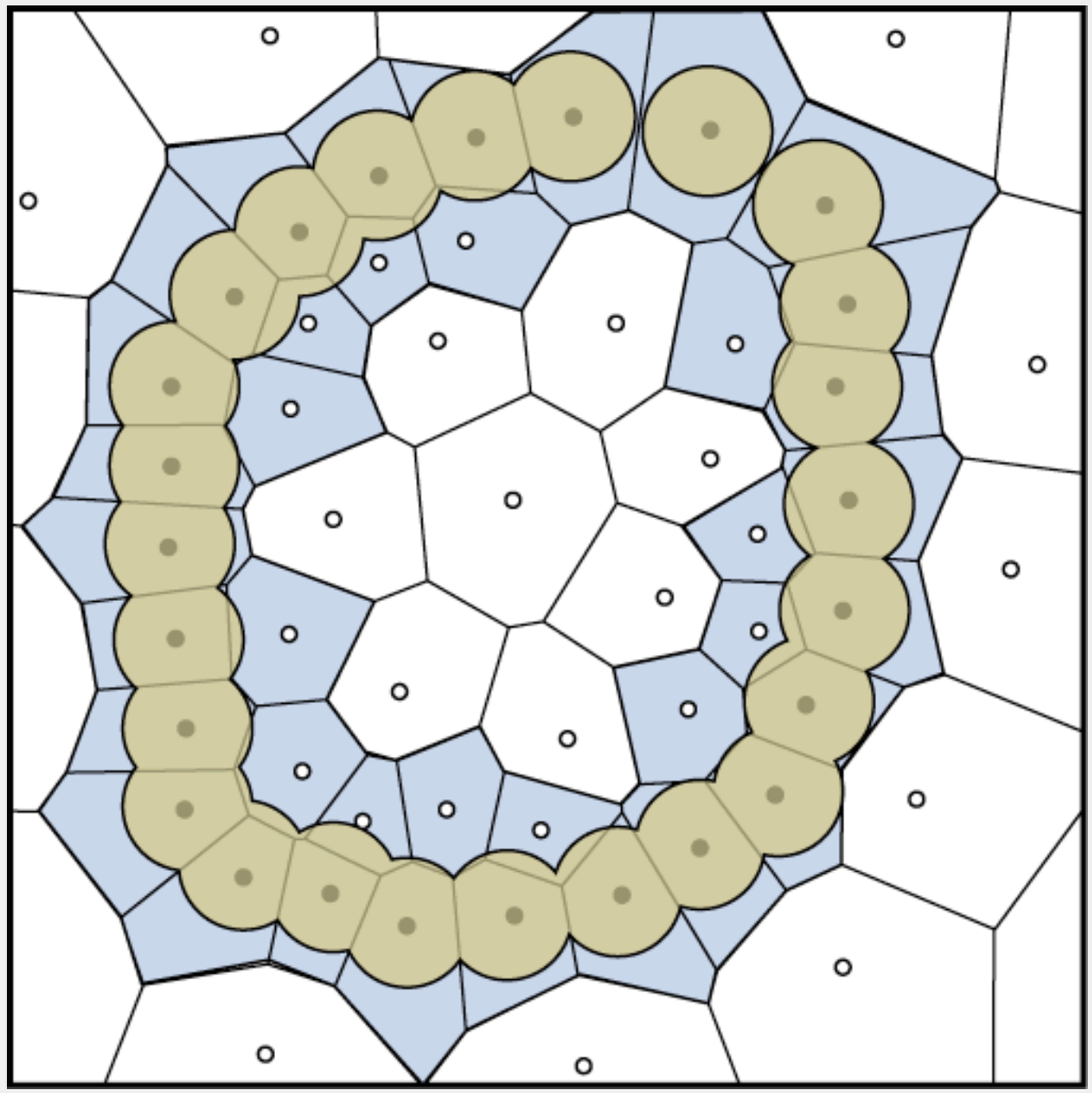

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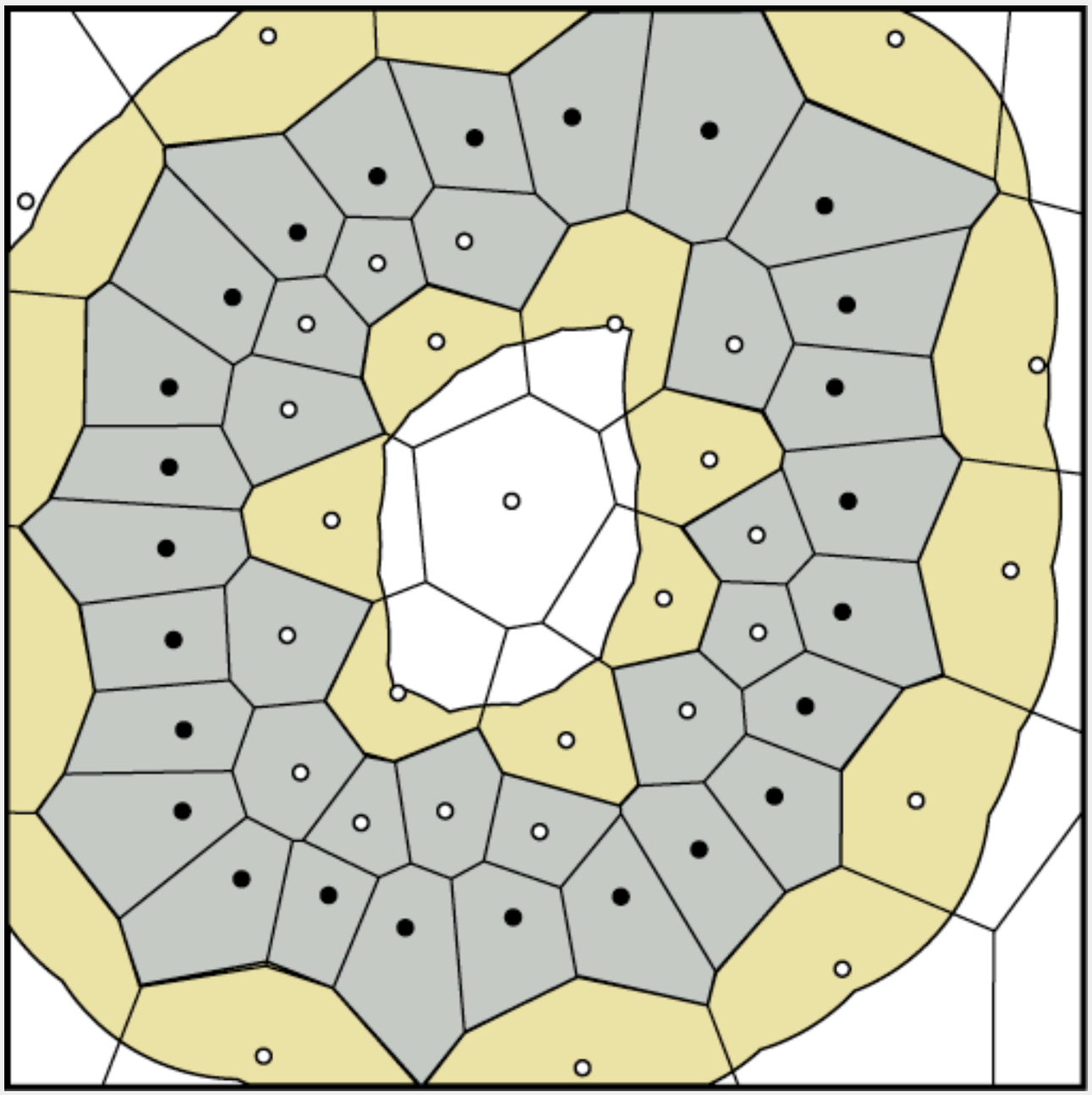

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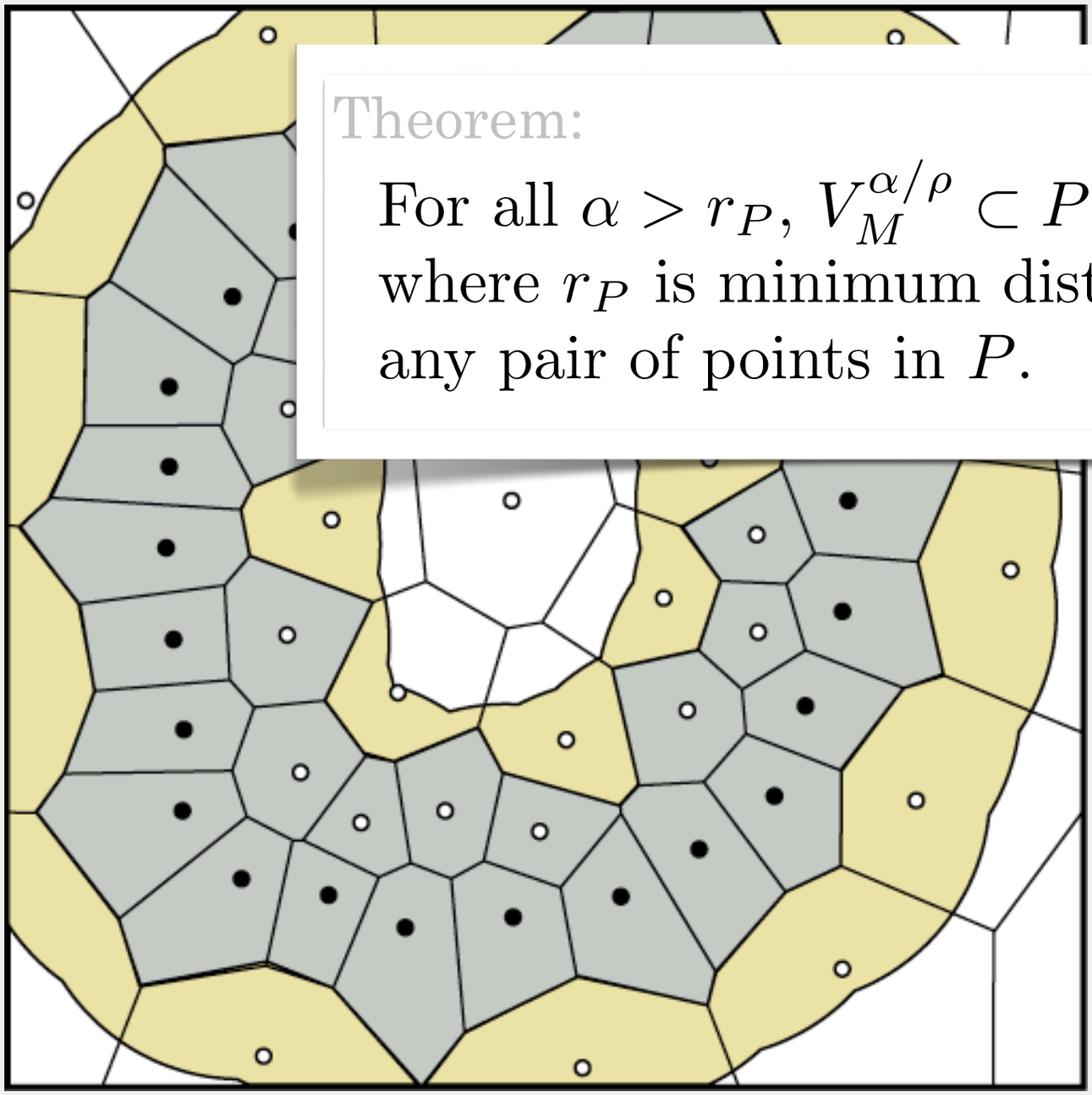

Theorem:

For all $\alpha > r_P$, $V_M^{\alpha/\rho} \subset P^\alpha \subset V_M^{\alpha\rho}$, where r_P is minimum distance between any pair of points in P.

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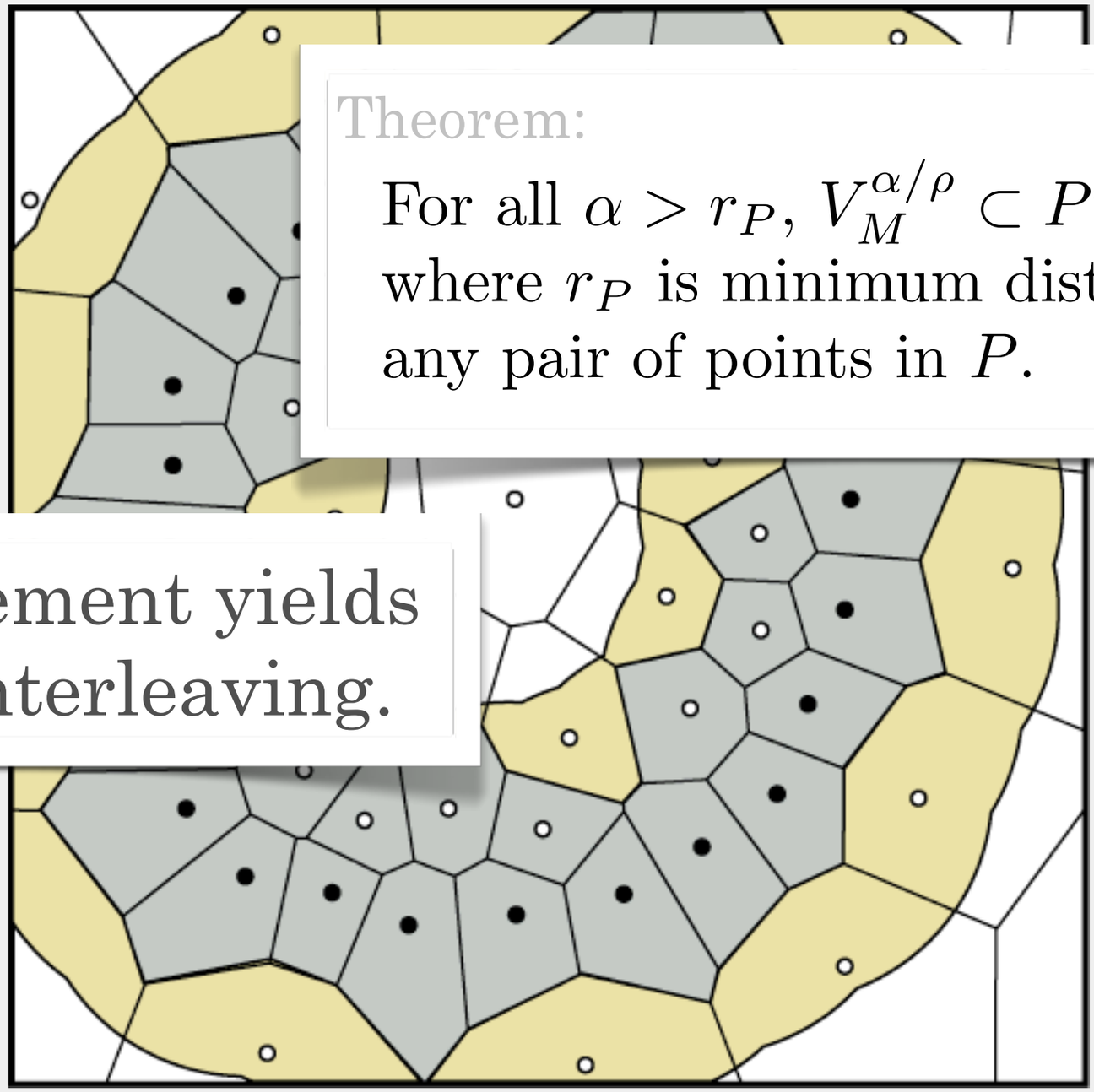

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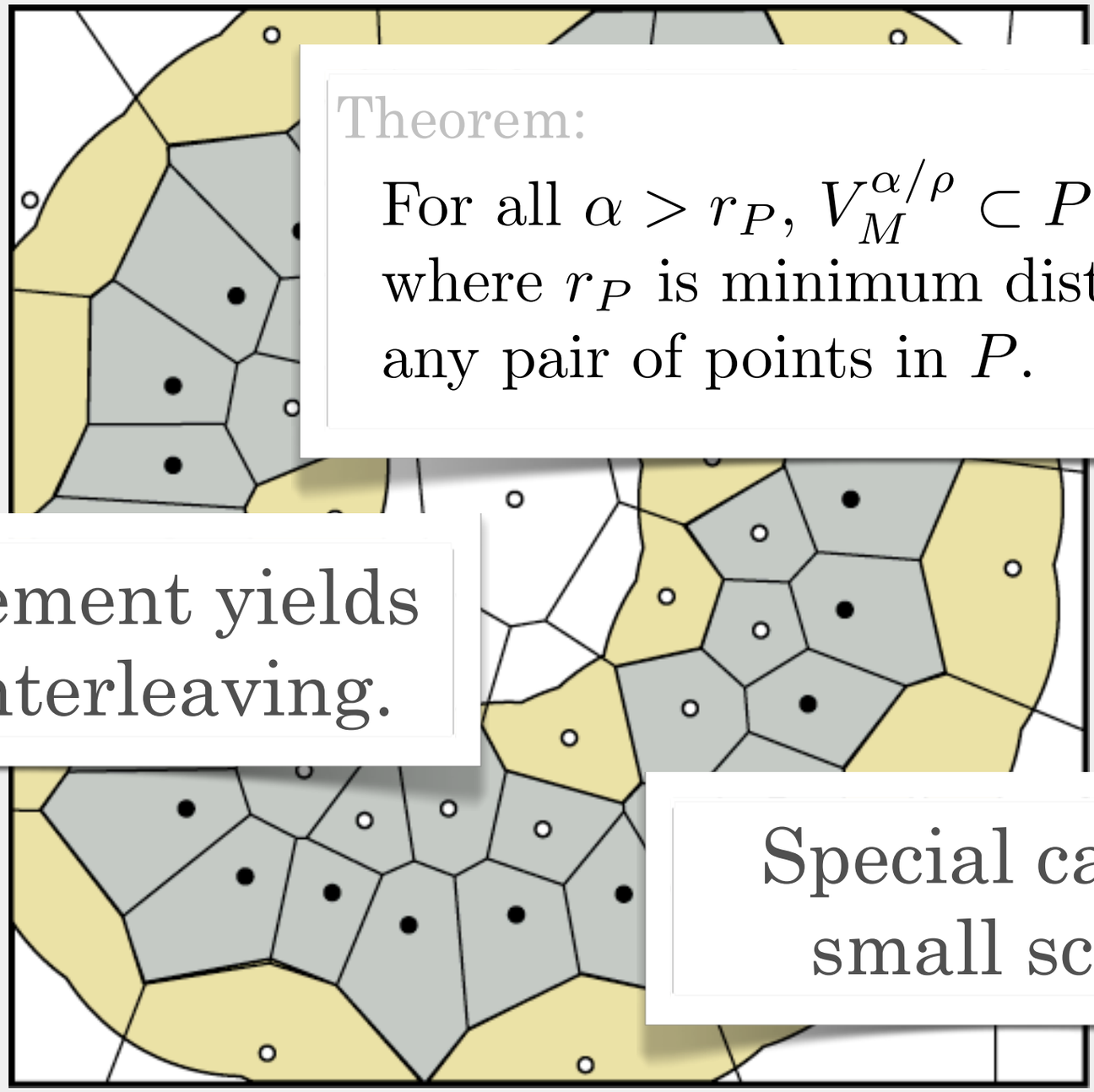
For all $\alpha > r_P$, $V_M^{\alpha/\rho} \subset P^\alpha \subset V_M^{\alpha\rho}$, where r_P is minimum distance between any pair of points in P.

Finer refinement yields a tighter interleaving.

The Voronoi filtration interleaves with the offset filtration.



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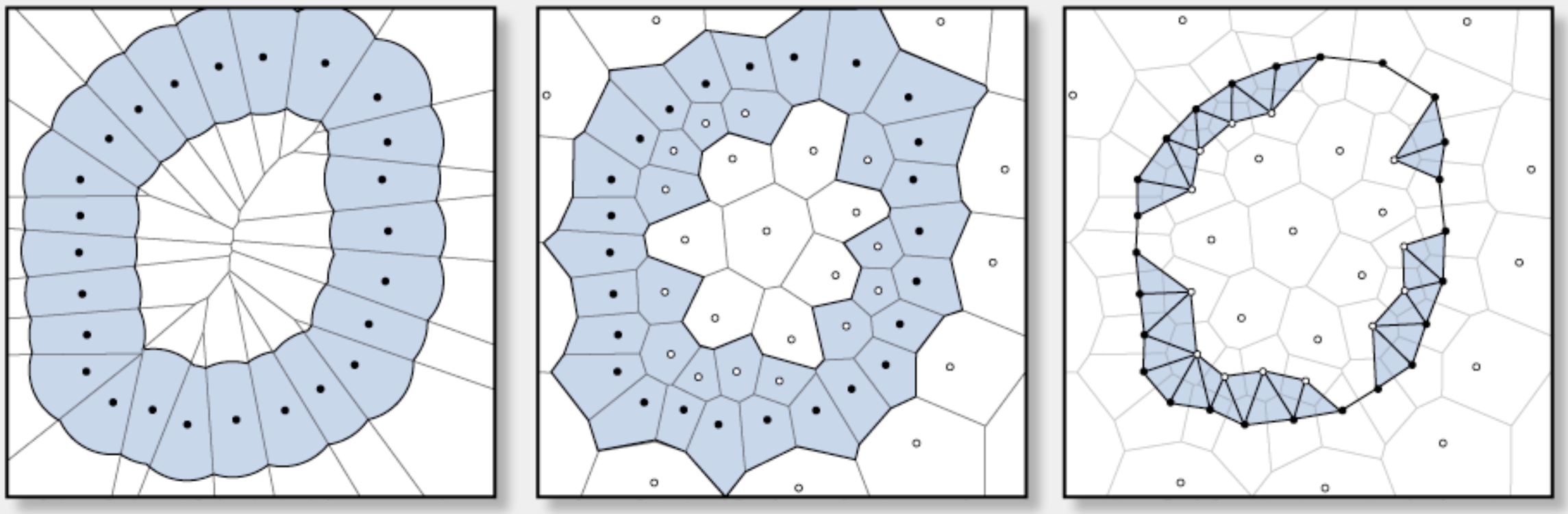
Finer refinement yields a tighter interleaving.

Special case for small scales.

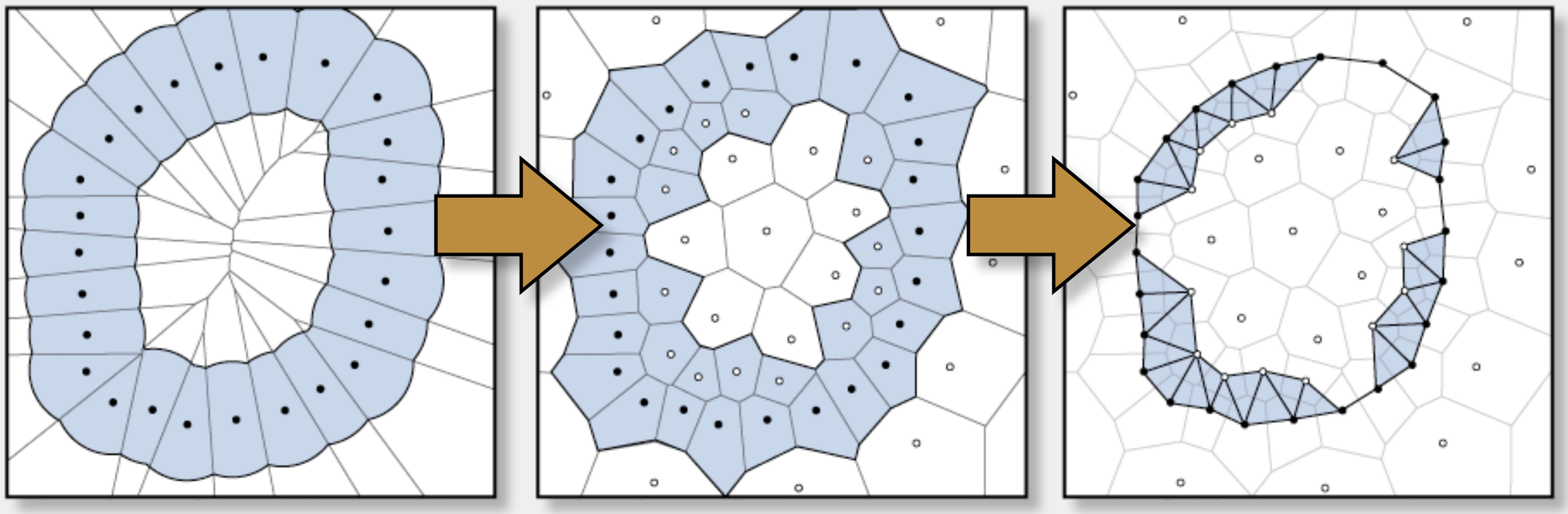
Geometric Approximation

Topologically Equivalent



Geometric Approximation

Topologically Equivalent



The Results

1. Build a mesh M.

2. Assign birth times to vertices based on distance to P (special case points very close to P).**

3. For each simplex s of Del(M), let birth(s) be the min birth time of its vertices.

4. Feed this filtered complex to the persistence algorithm.

	Approximation ratio	Complex Size
Previous Work	1	$n^{O(d)}$
Simple mesh filtration		
Over-refine the mesh		
Linear-Size Meshing		

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	Approximation ratio	Complex Size
Previous Work	1	$n^{O(d)}$
Simple mesh filtration	ρ	$2^{O(d^2)} n \log \Delta$
Over-refine the mesh		
Linear-Size Meshing		

The Results

1. Build a mesh M.

2. Assign birth times to vertices based on distance to P (special case points very close to P).**

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4. Feed this filtered complex to the persistence algorithm.

	Approximation ratio	Complex Size
Previous Work	1	$n^{O(d)}$
Simple mesh filtration	$\rho = \sim 3$	$2^{O(d^2)} n \log \Delta$
Over-refine the mesh		
Linear-Size Meshing		

The Results

1. Build a mesh M.
   Over-refine it.

2. Assign birth times to vertices based on distance to P (special case points very close to P).**

3. For each simplex s of Del(M), let birth(s) be the min birth time of its vertices.

4. Feed this filtered complex to the persistence algorithm.

	Approximation ratio	Complex Size
Previous Work	1	$n^{O(d)}$
Simple mesh filtration	$\rho = \sim 3$	$2^{O(d^2)} n \log \Delta$
Over-refine the mesh	$1+\varepsilon$	$\varepsilon^{-O(d^2)} n \log \Delta$
Linear-Size Meshing		

The Results

1. Build a mesh M.
   Over-refine it.
   Use linear-size meshing.

2. Assign birth times to vertices based on distance to P (special case points very close to P).**

3. For each simplex s of Del(M), let birth(s) be the min birth time of its vertices.

4. Feed this filtered complex to the persistence algorithm.

	Approximation ratio	Complex Size
Previous Work	1	$n^{O(d)}$
Simple mesh filtration	$\rho = \sim 3$	$2^{O(d^2)} n \log \Delta$
Over-refine the mesh	$1+\varepsilon$	$\varepsilon^{-O(d^2)} n \log \Delta$
Linear-Size Meshing	$1+\varepsilon+3\theta$	$(\varepsilon\theta)^{-O(d^2)} n$

Thank you.