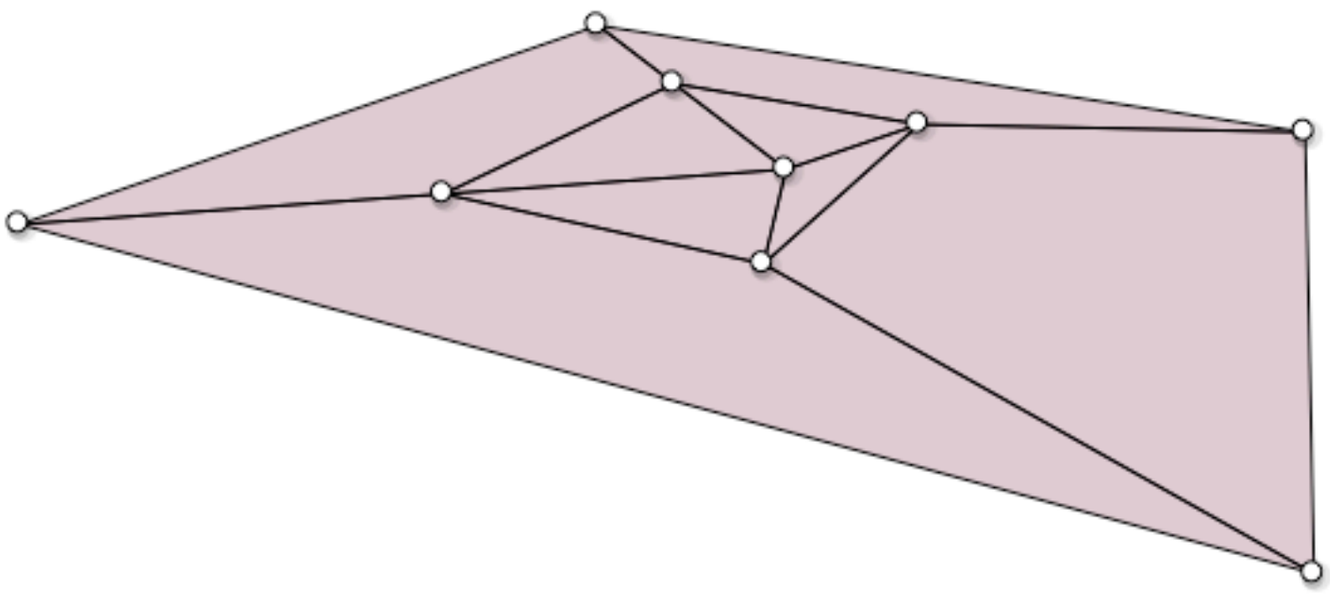
Planar Graphs in $2^1/_2$ Dimensions

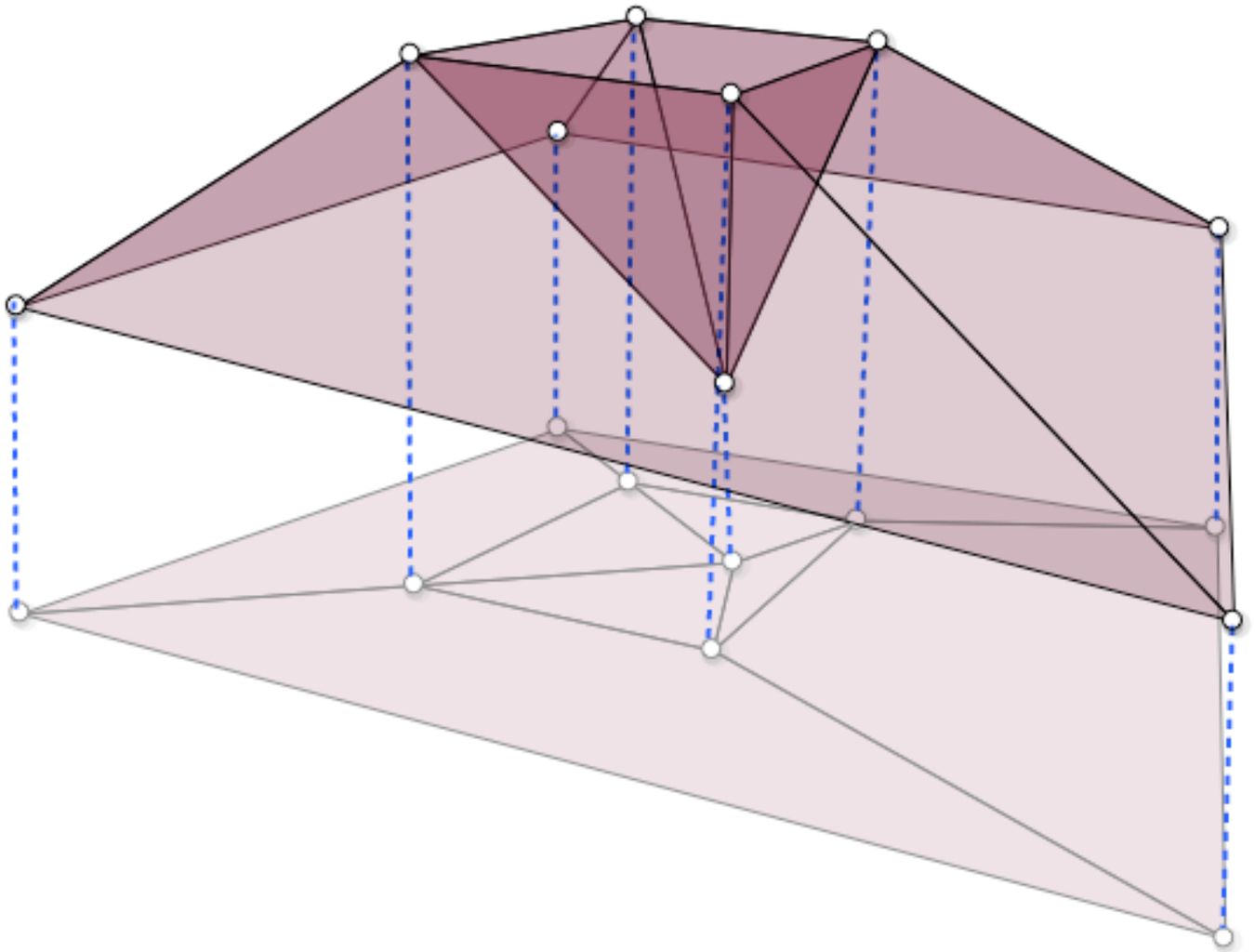
Don Sheehy

$2^1/_2$ Dimensions

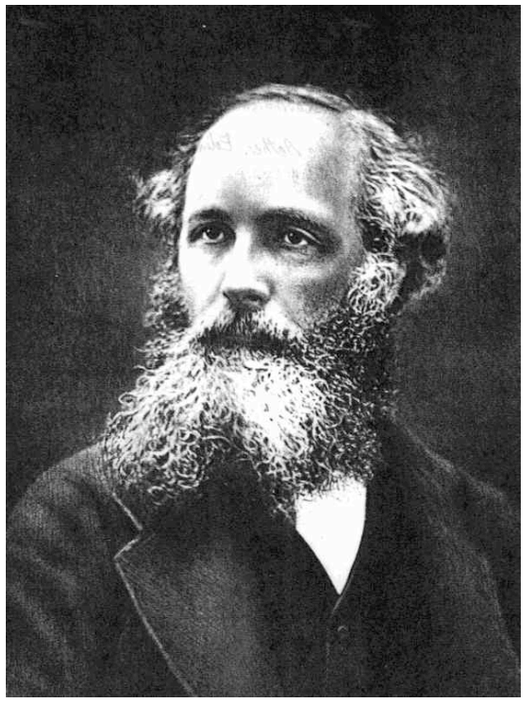
$2^1/_2$ Dimensions

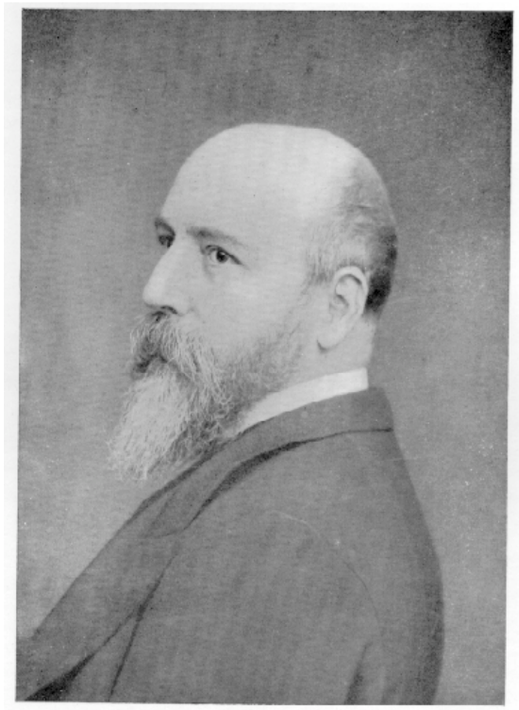


$2^{1}/_{2}$ Dimensions

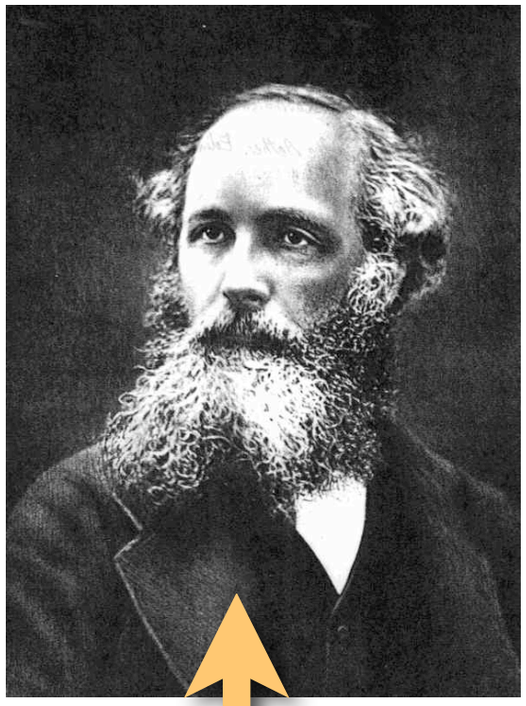


Cast of Characters

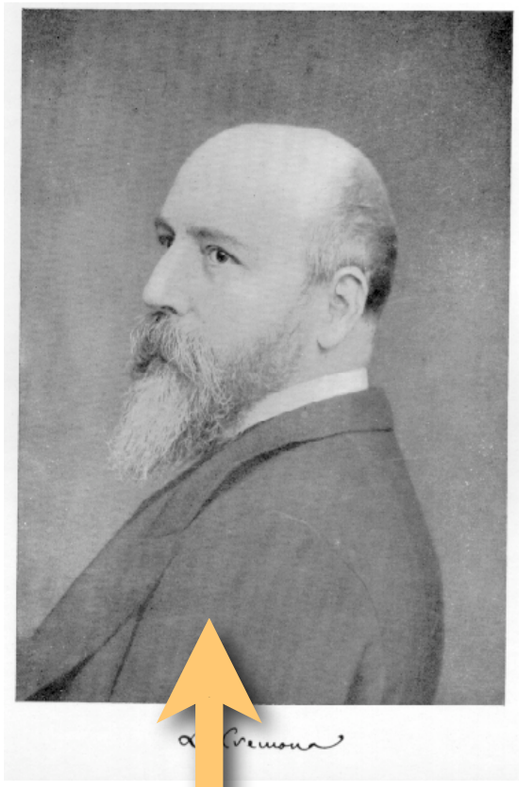




Cast of Characters

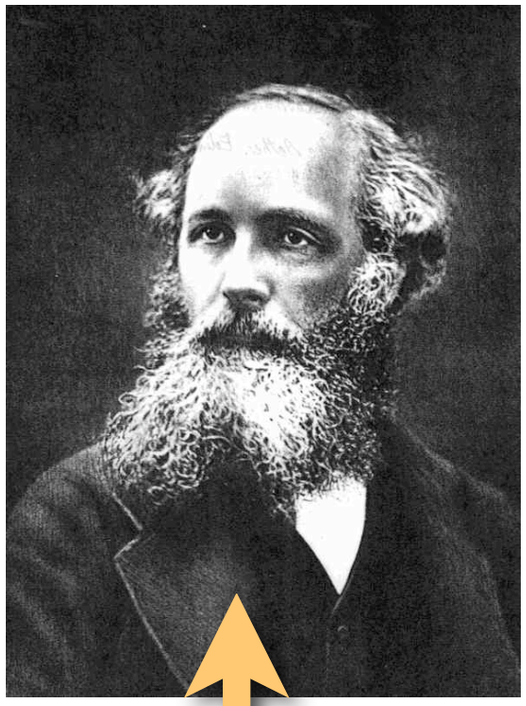


James Clerk Maxwell

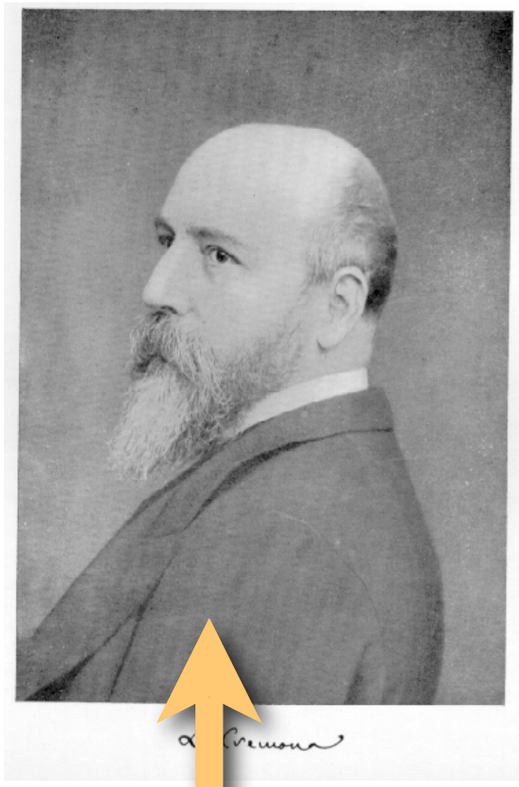


Luigi Cremona

Cast of Characters



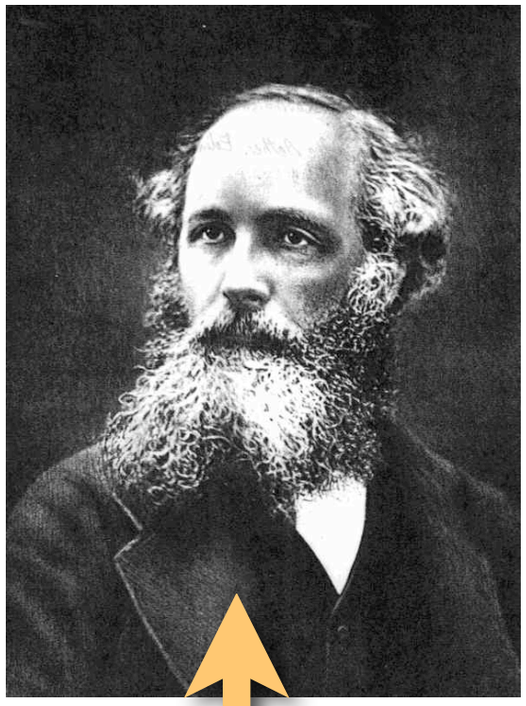
James Clerk Maxwell



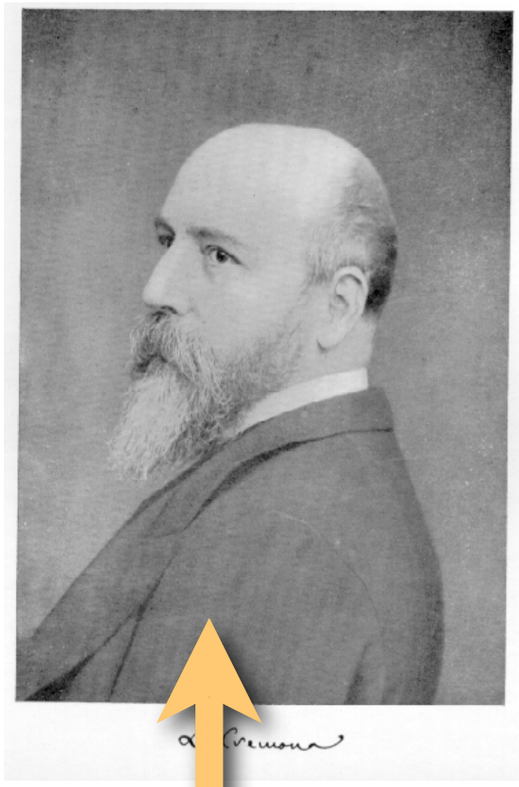
Luigi Cremona

Ernst Steinitz

Cast of Characters



James Clerk Maxwell



Luigi Cremona

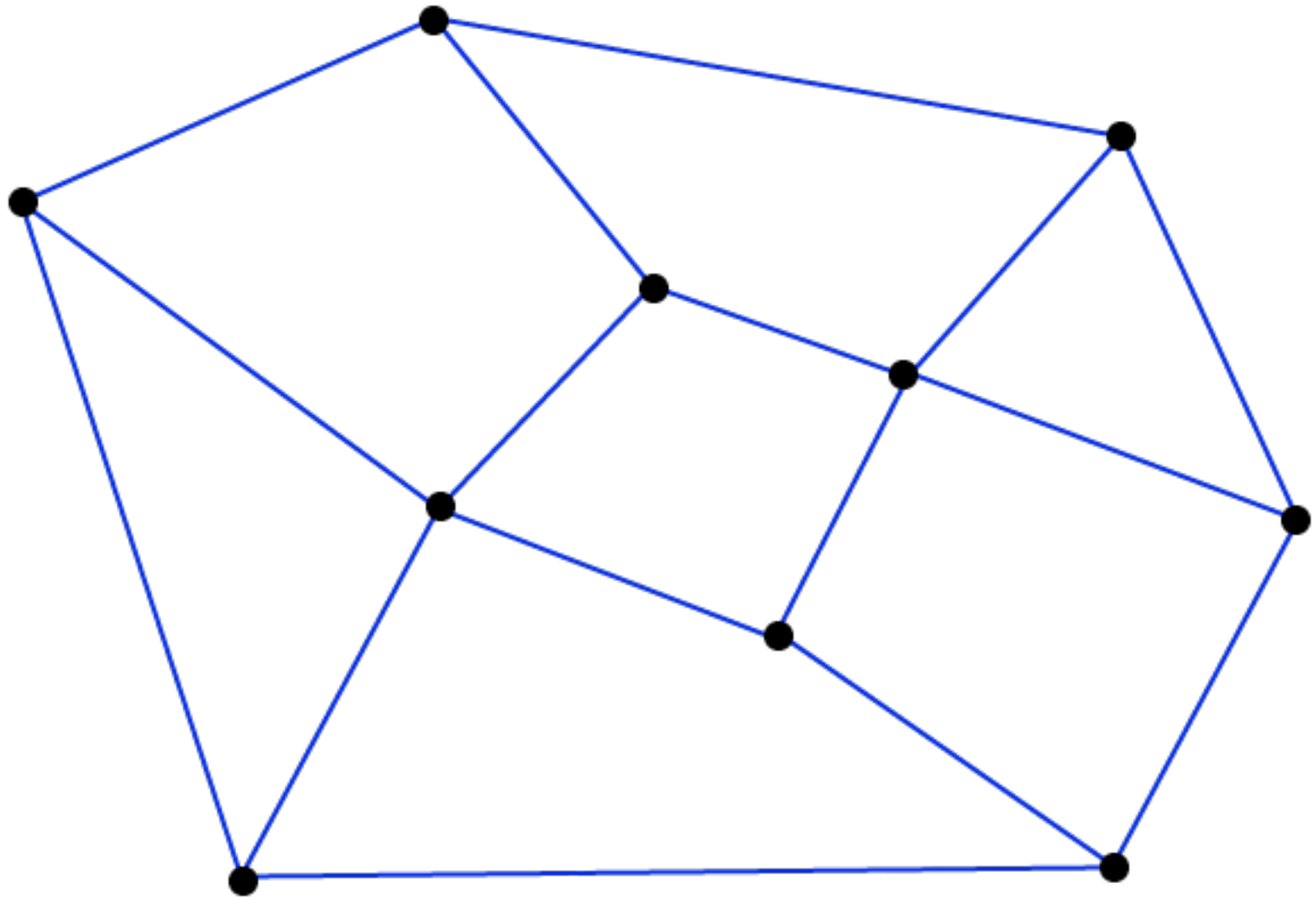
Ernst Steinitz

W. T. Tutte

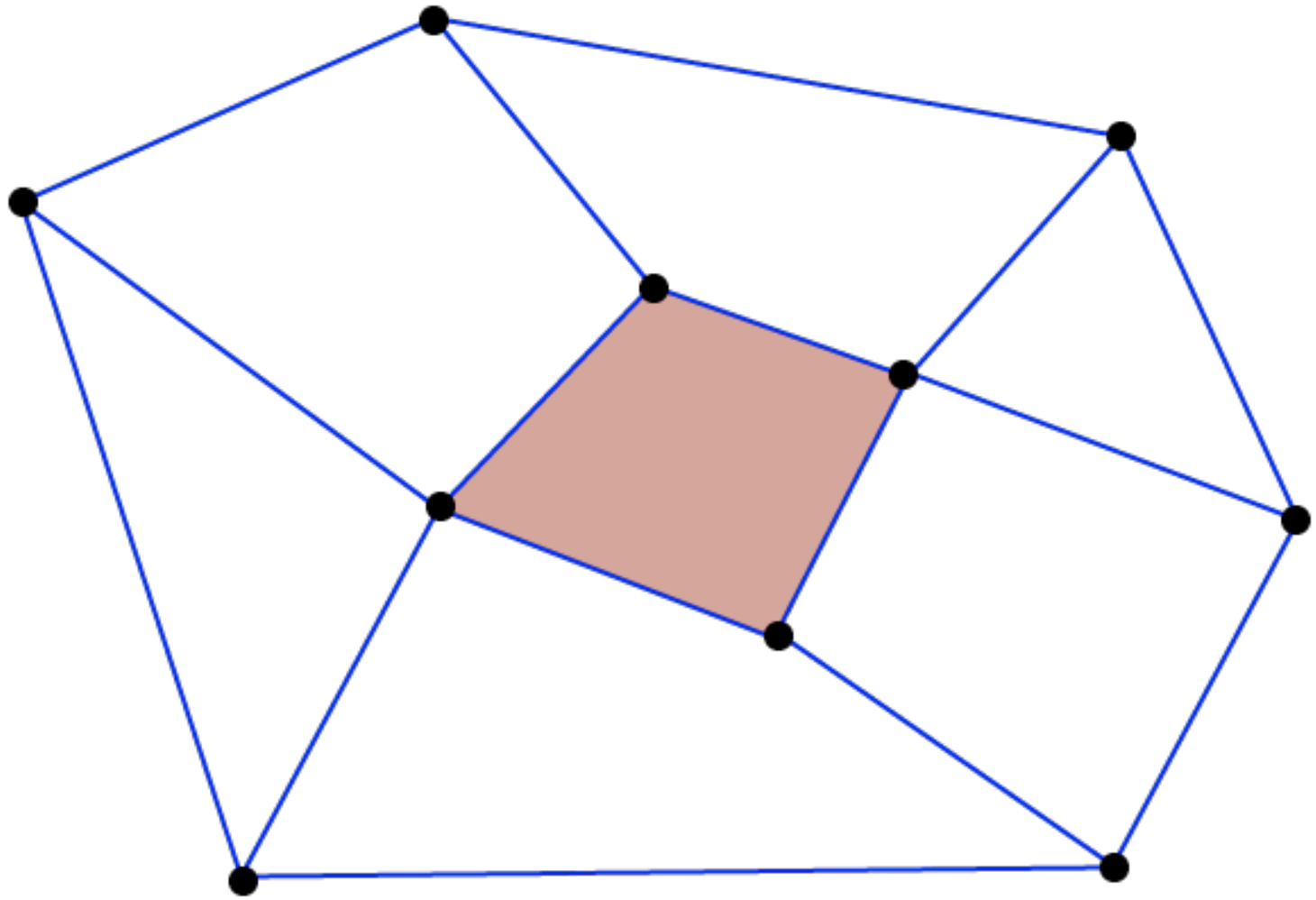
Planar Graphs

Planar Graphs

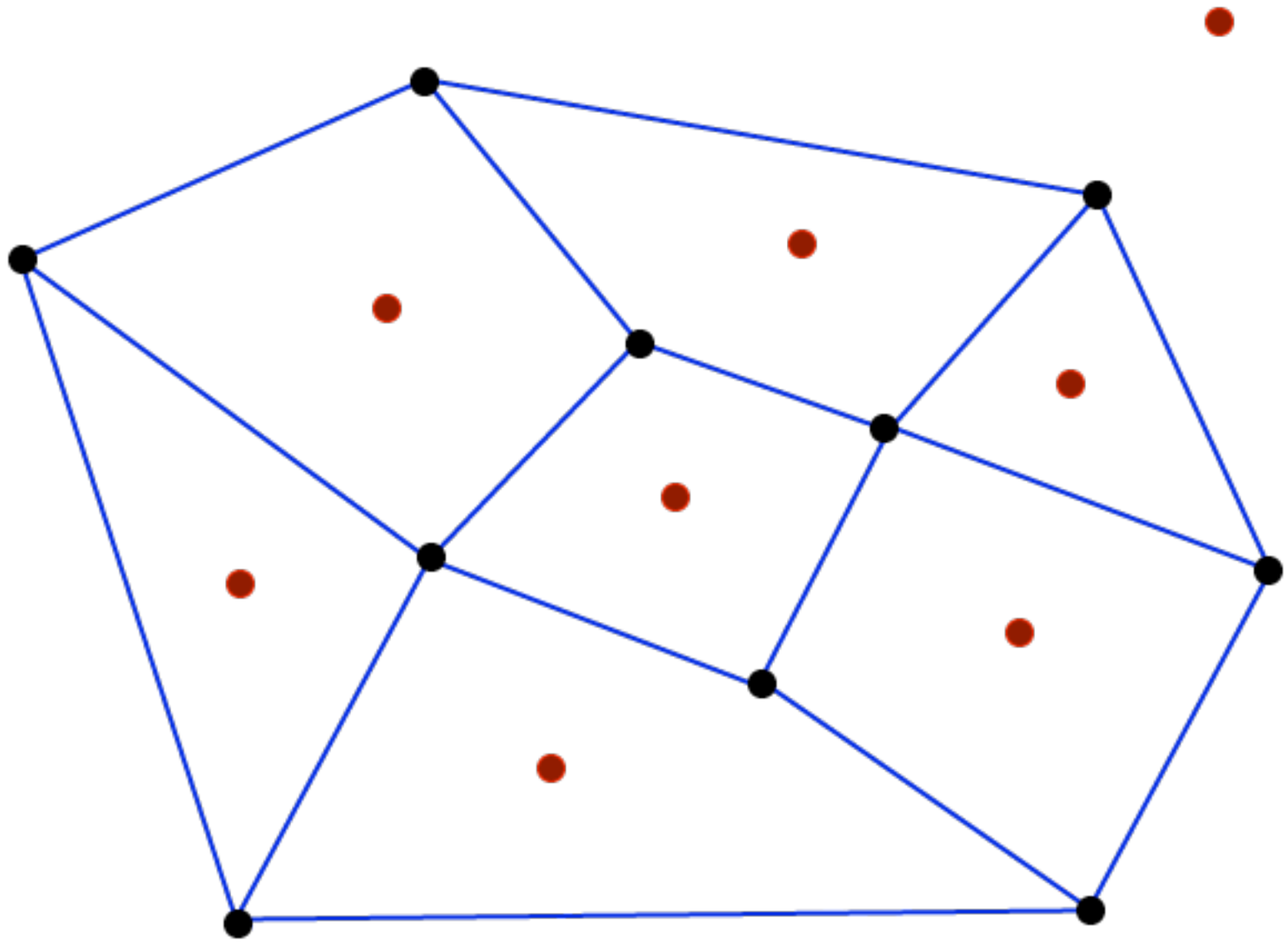
Planar Graphs



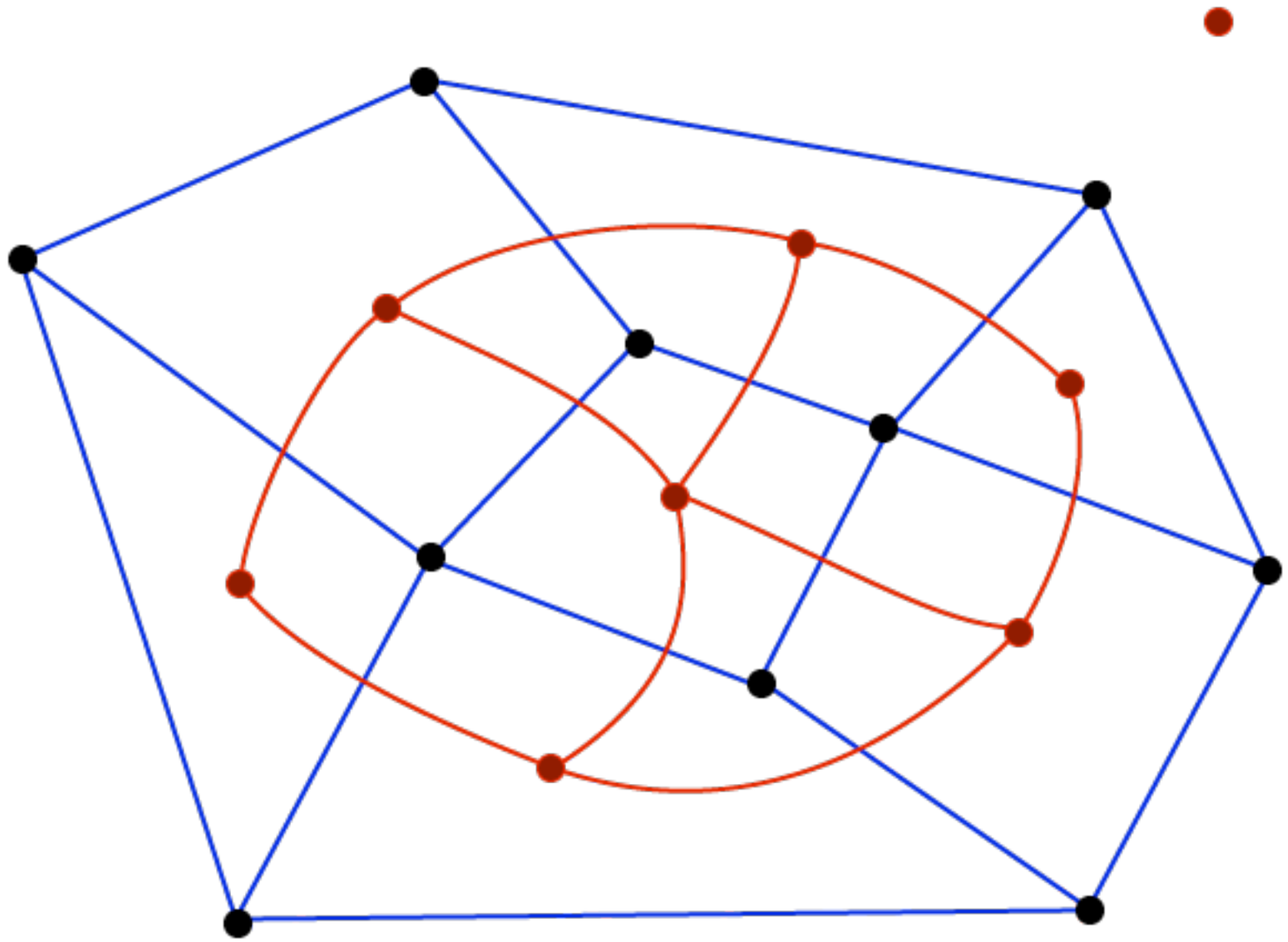
Planar Graphs



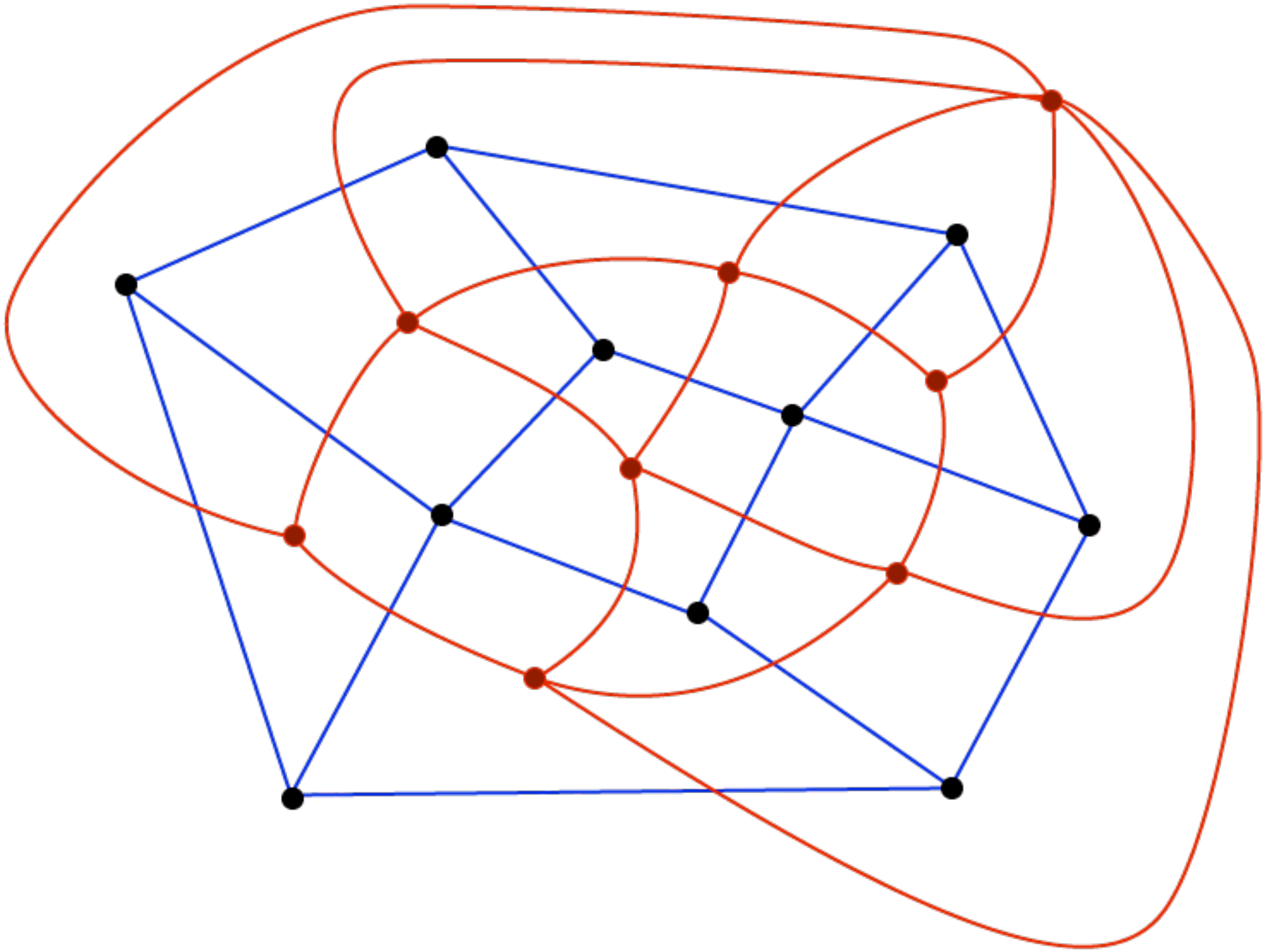
Planar Graphs



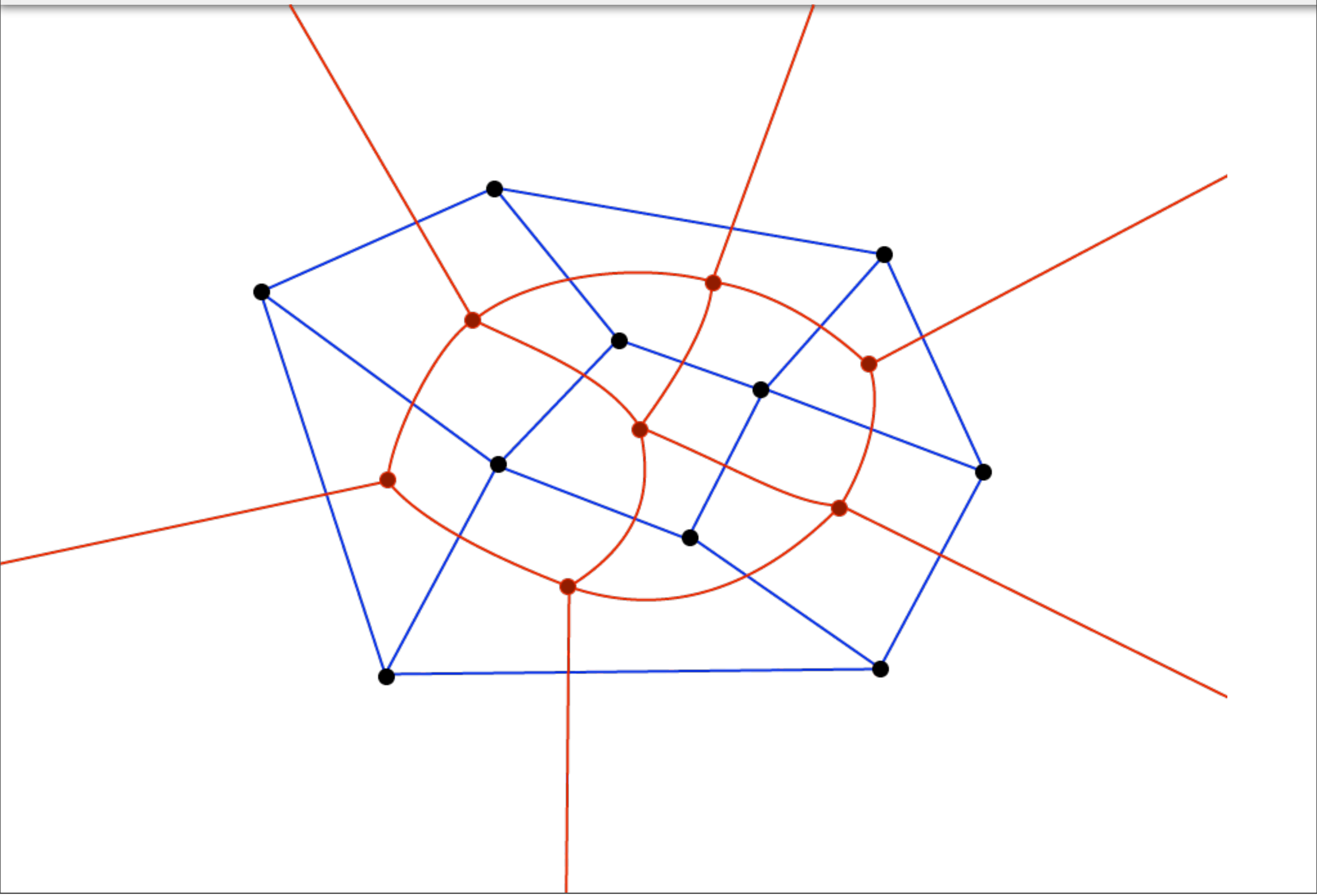
Planar Graphs



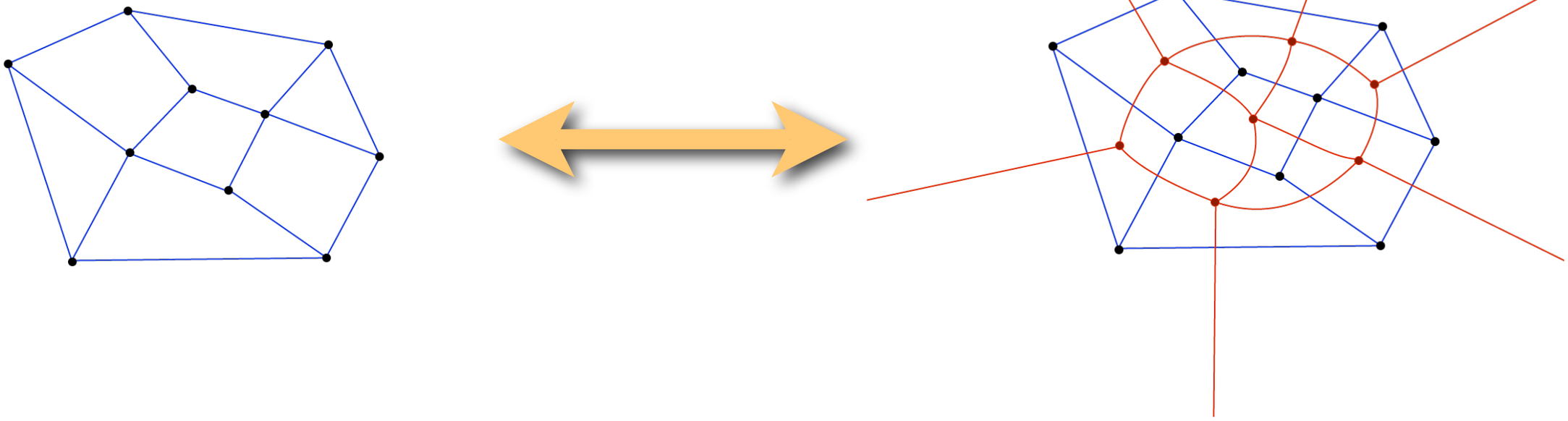
Planar Graphs



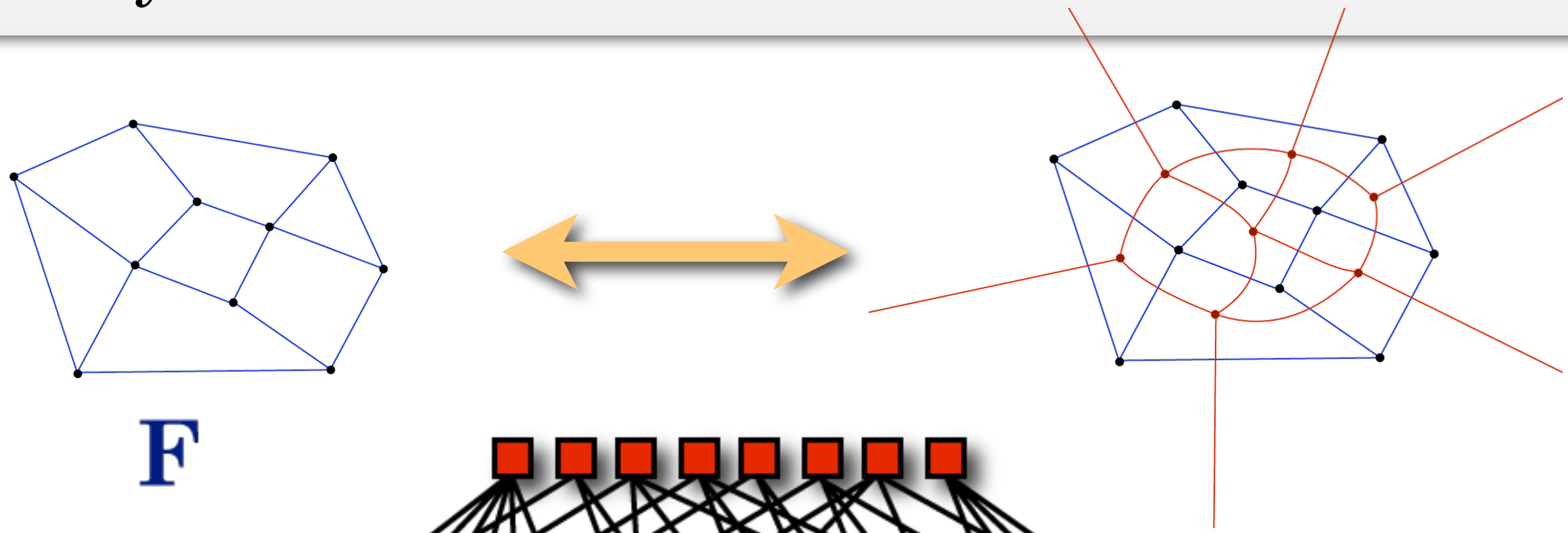
Planar Graphs



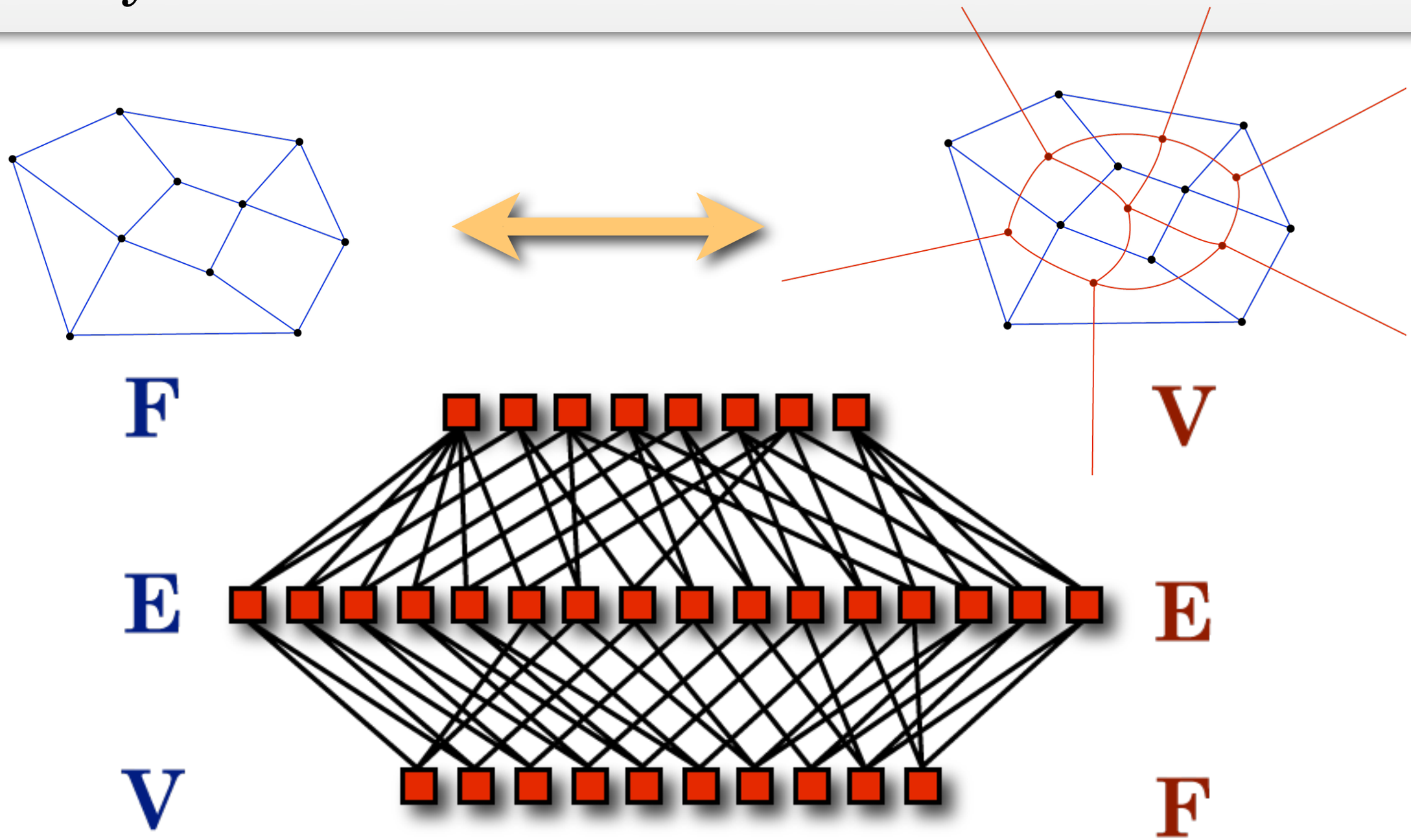
Duality



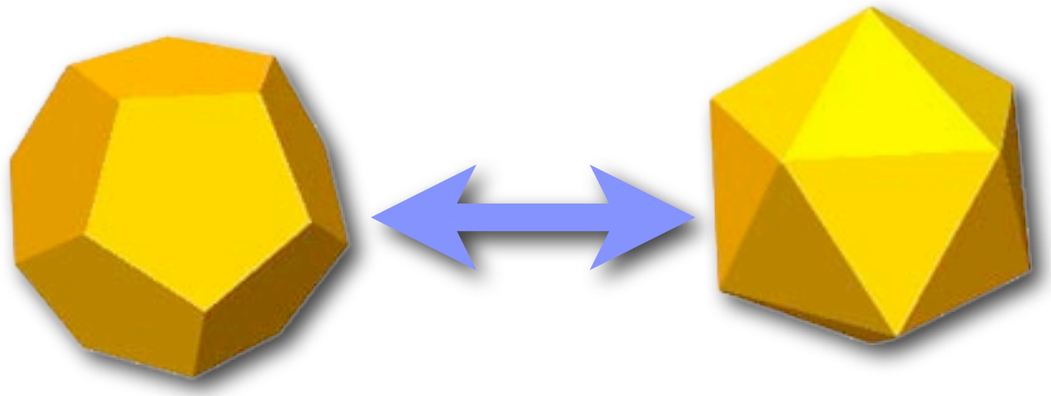
Duality

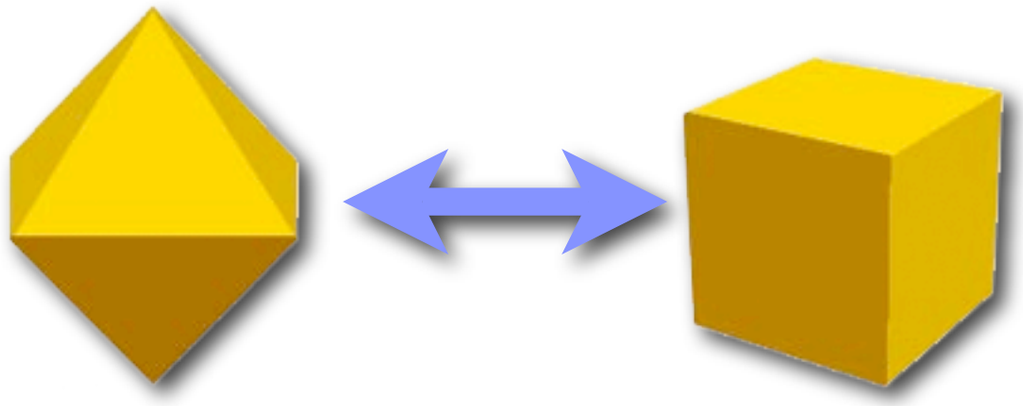


Duality



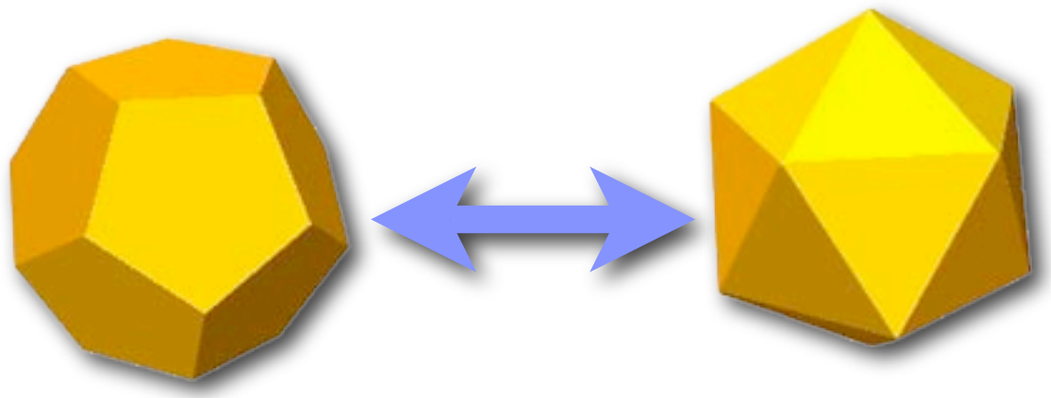
Polar Polytopes

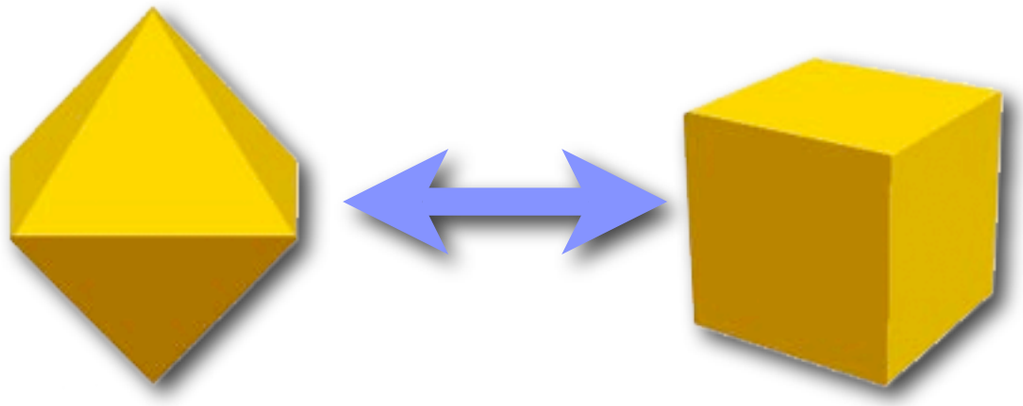




Polar Polytopes

$$A^\circ = \{x \in \mathbb{R}^d \mid a \cdot x \leq 1, \forall a \in A\}$$

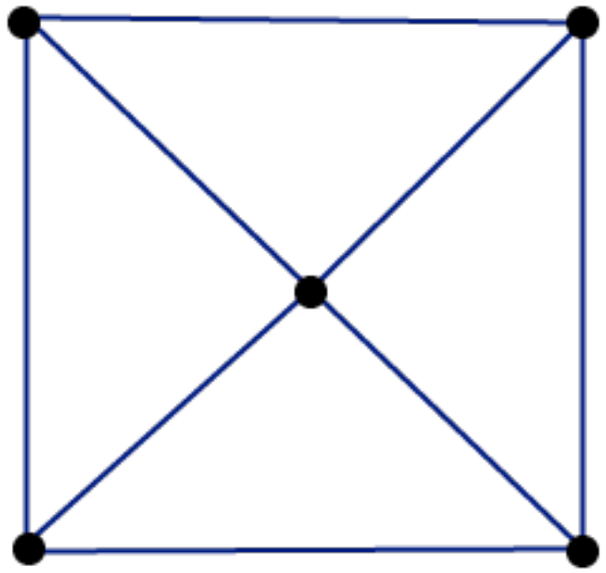




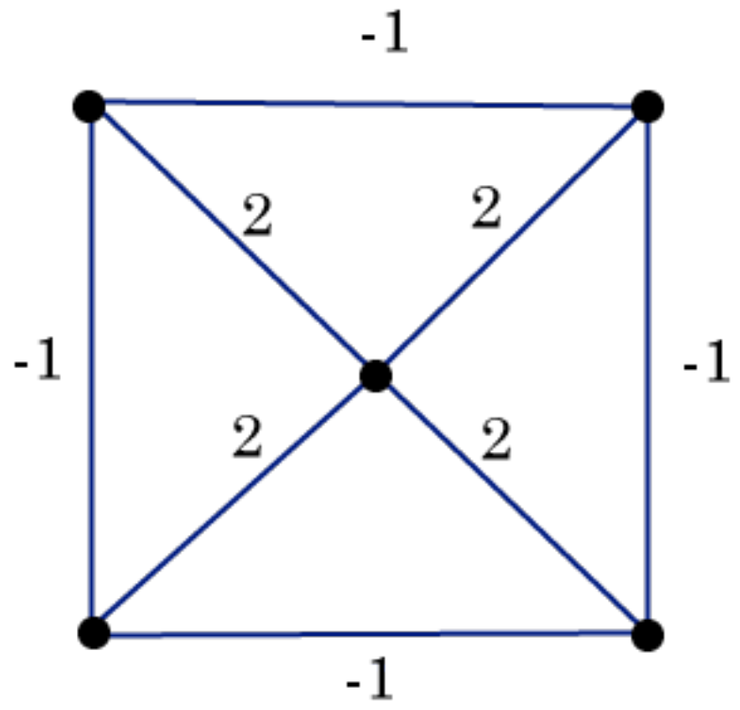
The Maxwell-Cremona Correspondence

Equilibrium Stresses

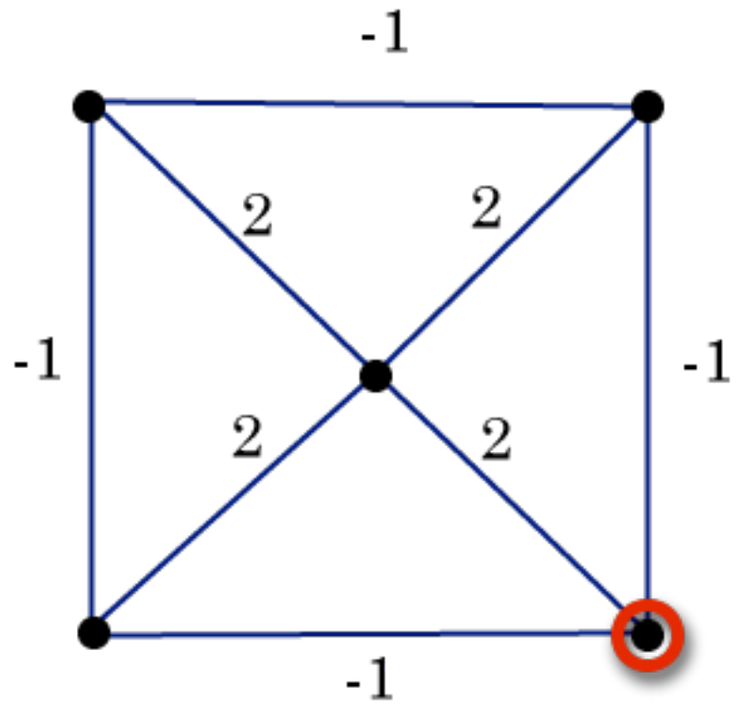
Equilibrium Stresses



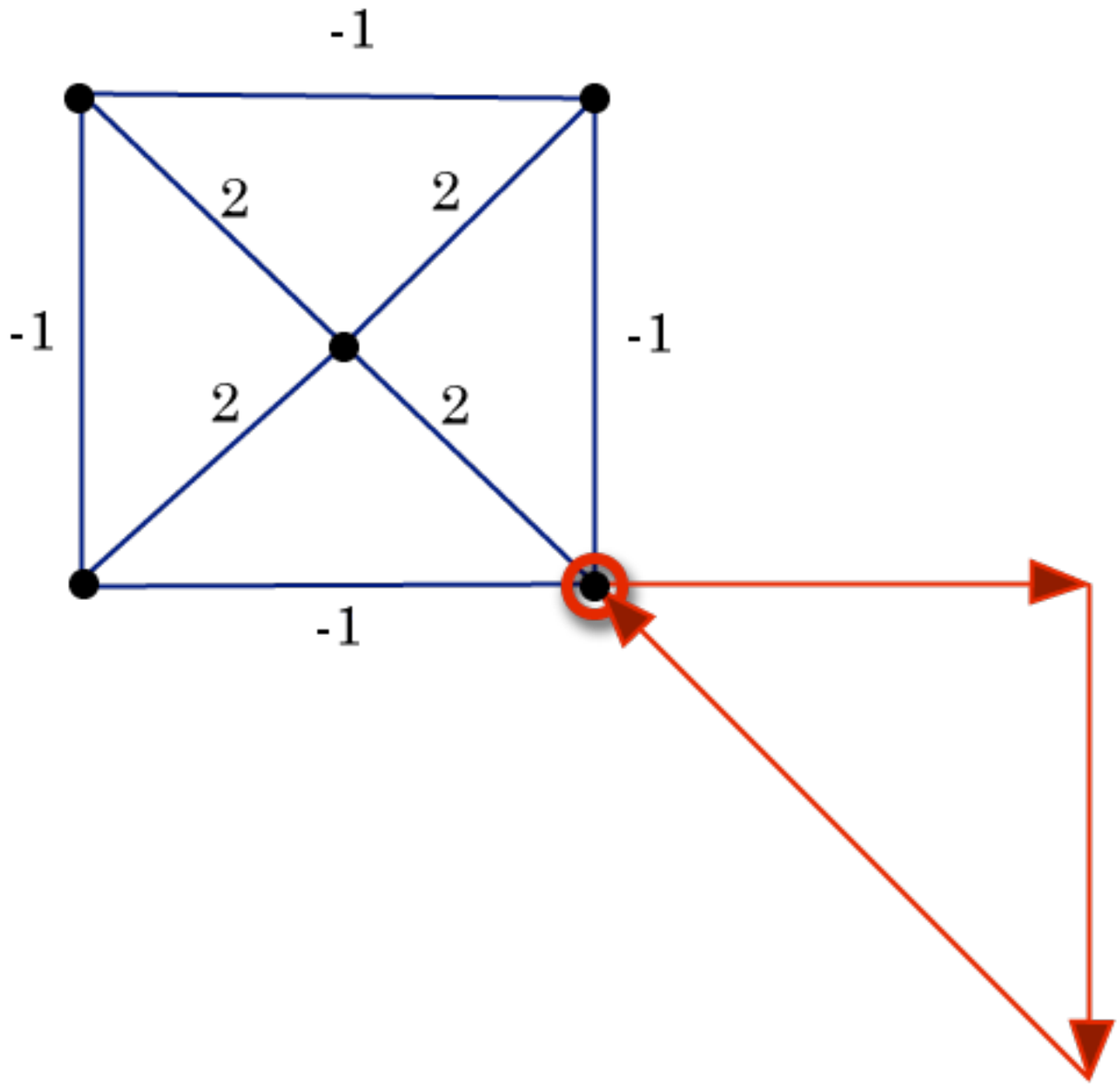
Equilibrium Stresses



Equilibrium Stresses

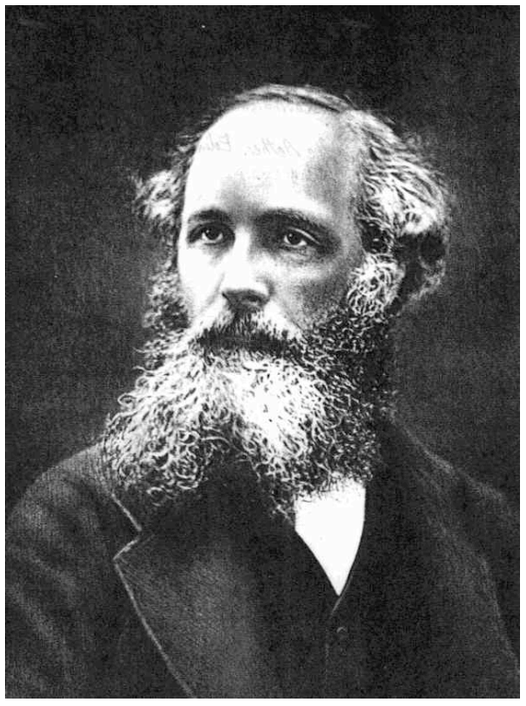


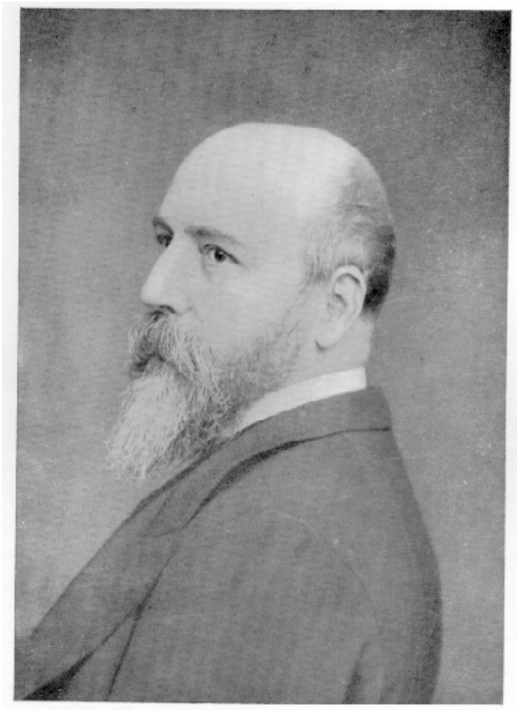
Equilibrium Stresses



The Maxwell-Cremona Correspondence

There is a 1-1 correspondence between "proper" liftings and equilibrium stresses of a planar straight line graph.

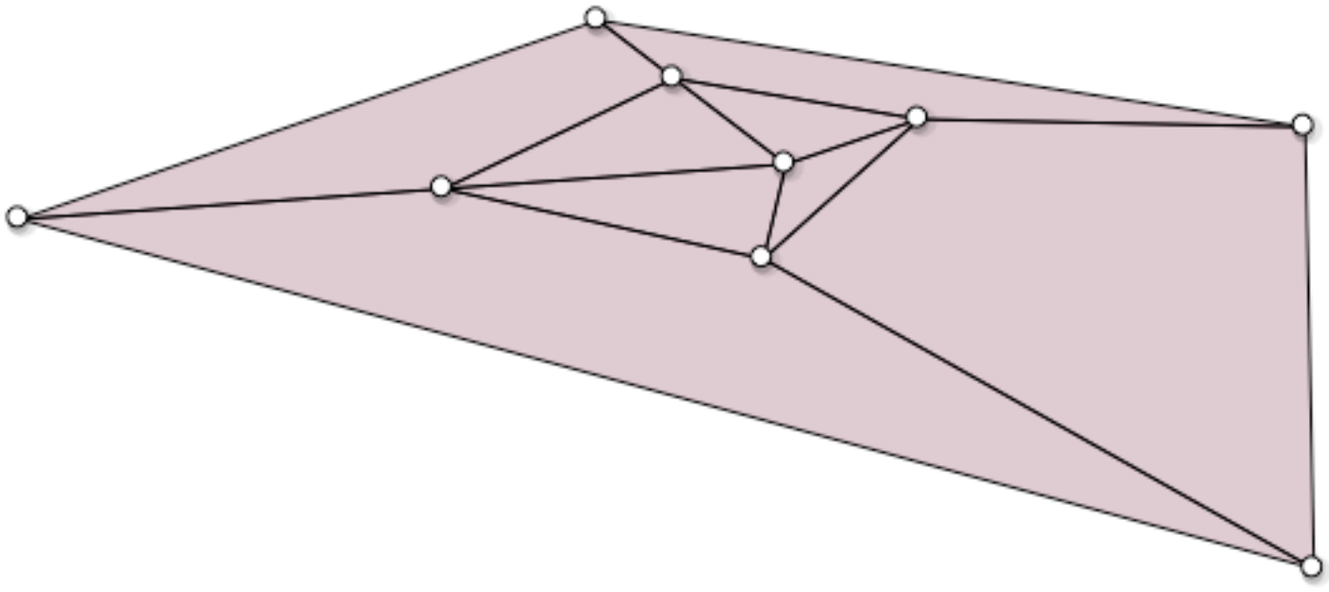




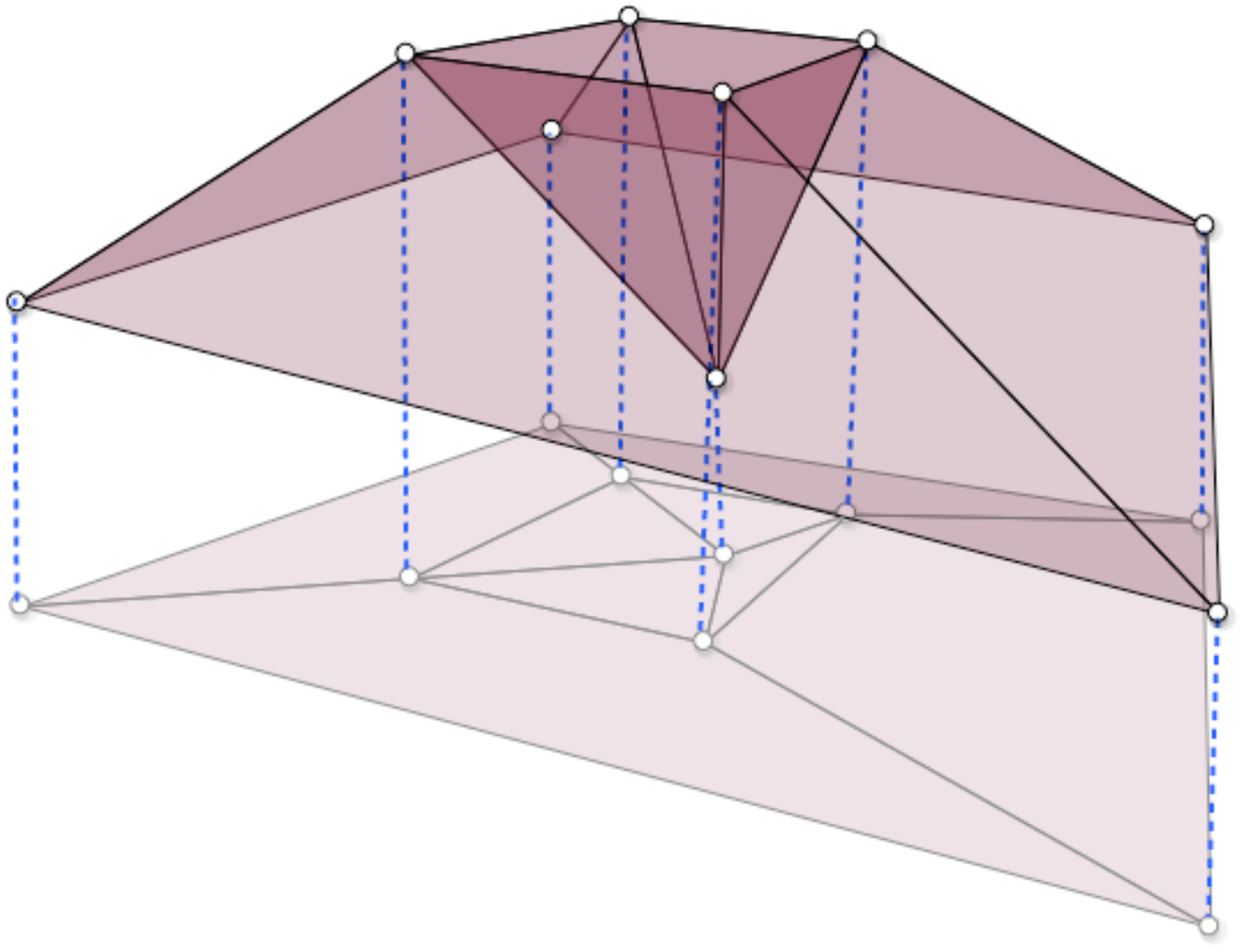
L. Cremona

The Maxwell-Cremona Correspondence

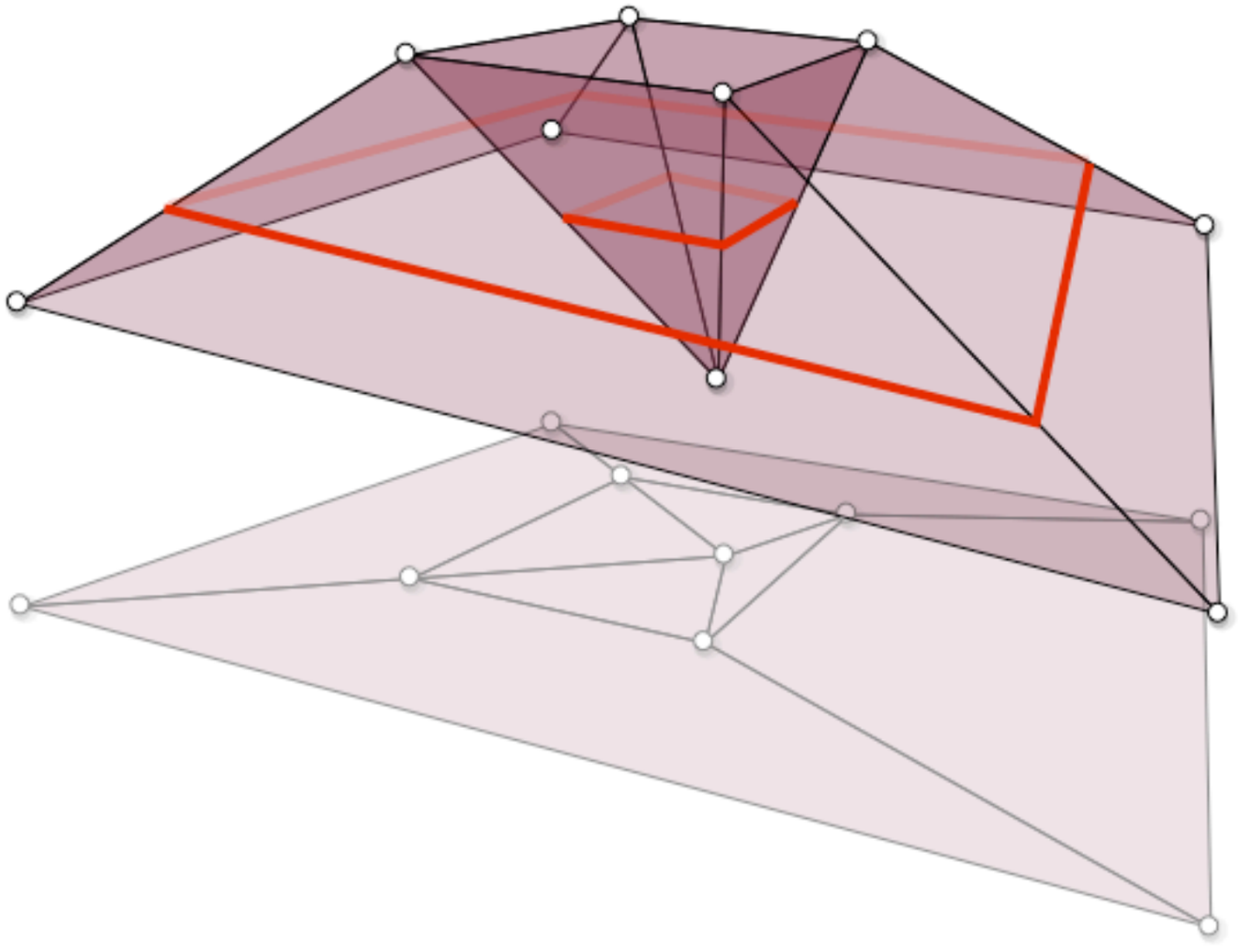
The Maxwell-Cremona Correspondence



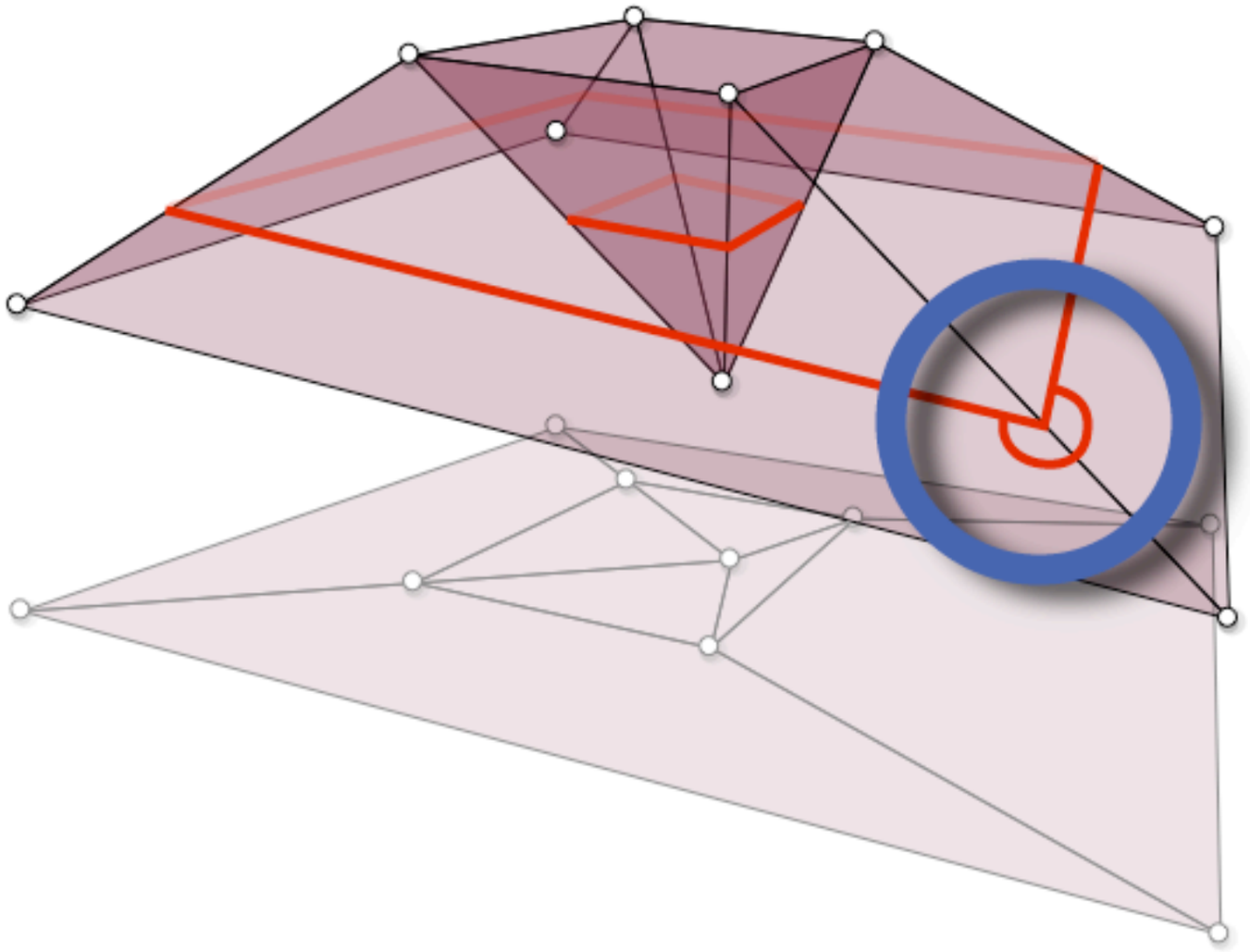
The Maxwell-Cremona Correspondence



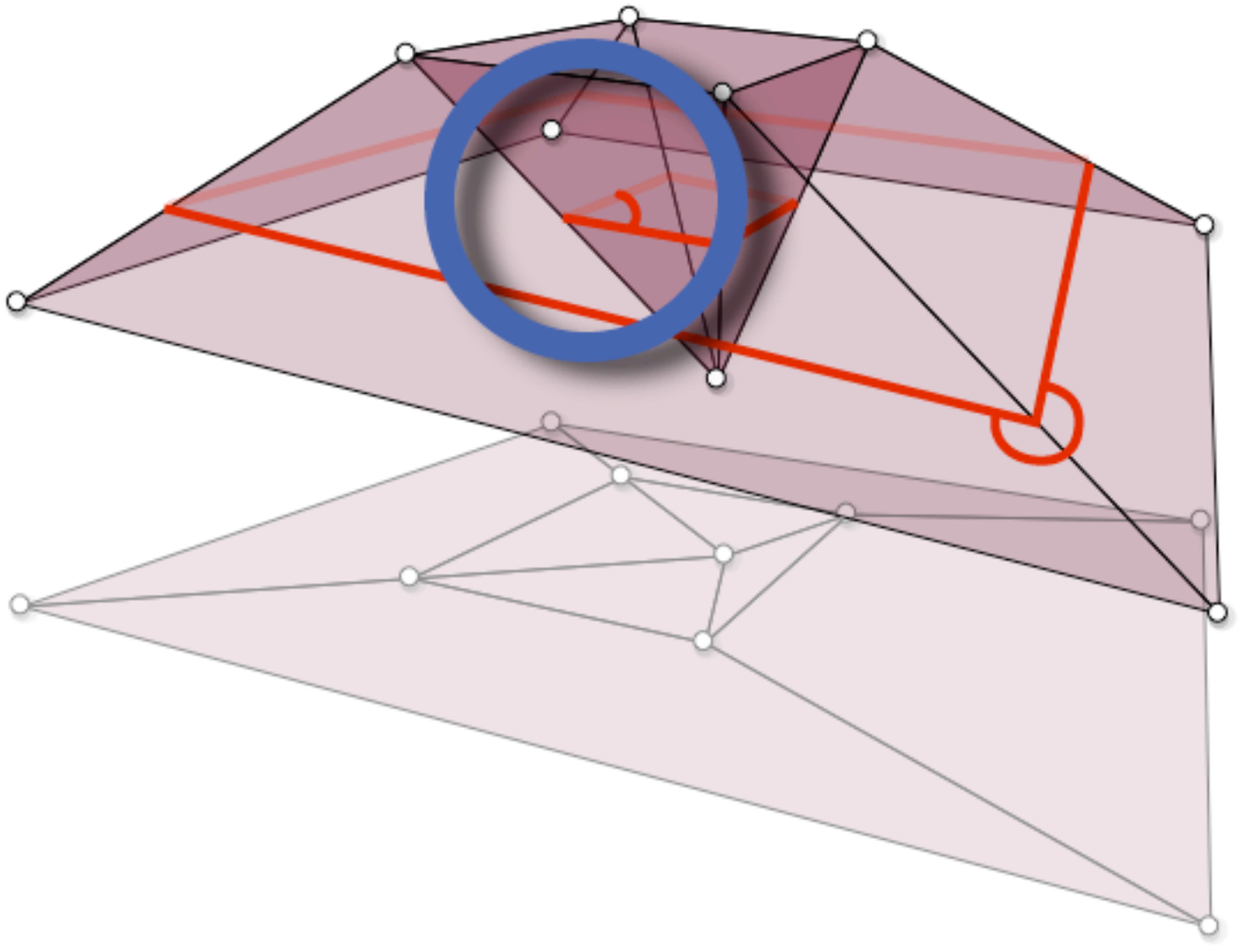
The Maxwell-Cremona Correspondence



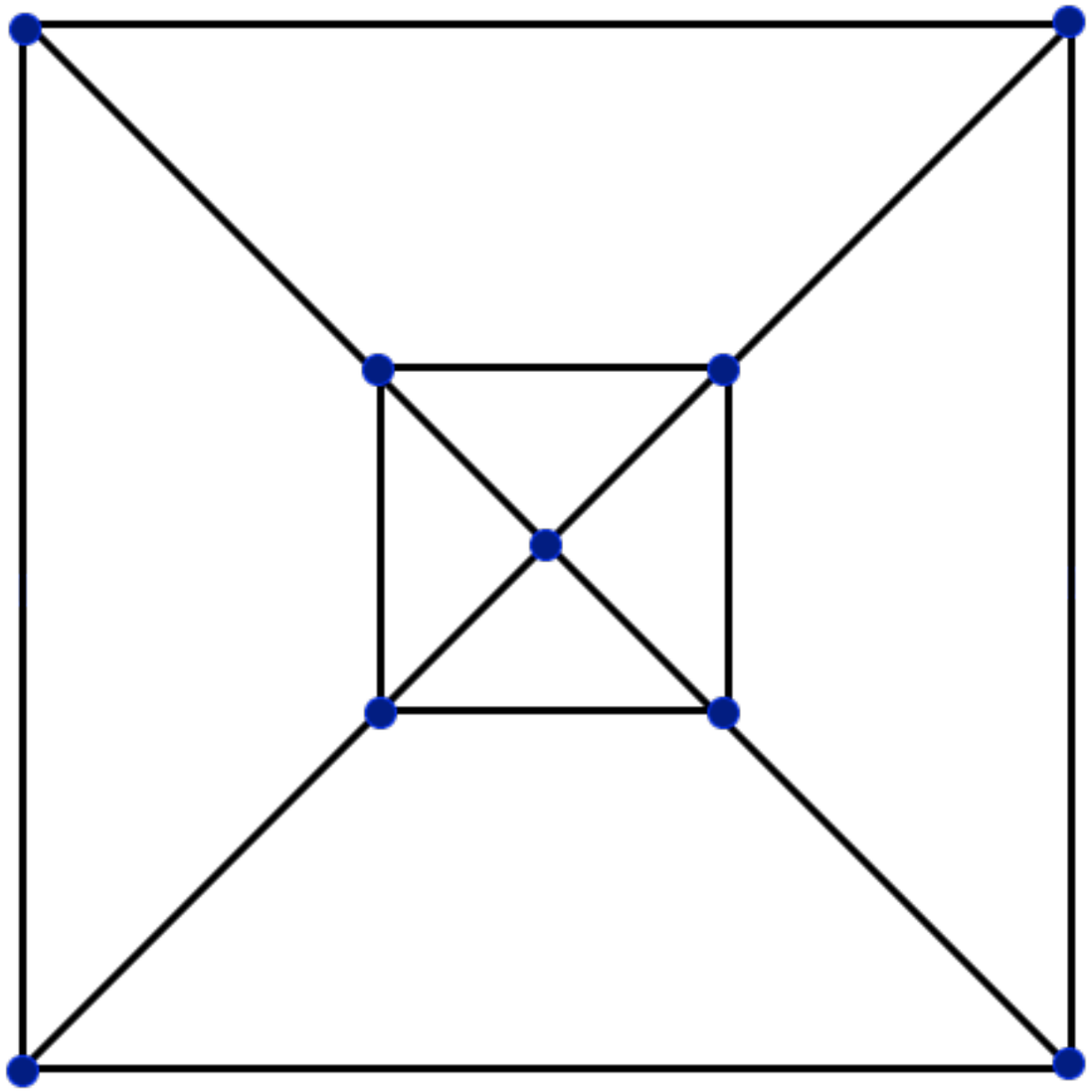
The Maxwell-Cremona Correspondence



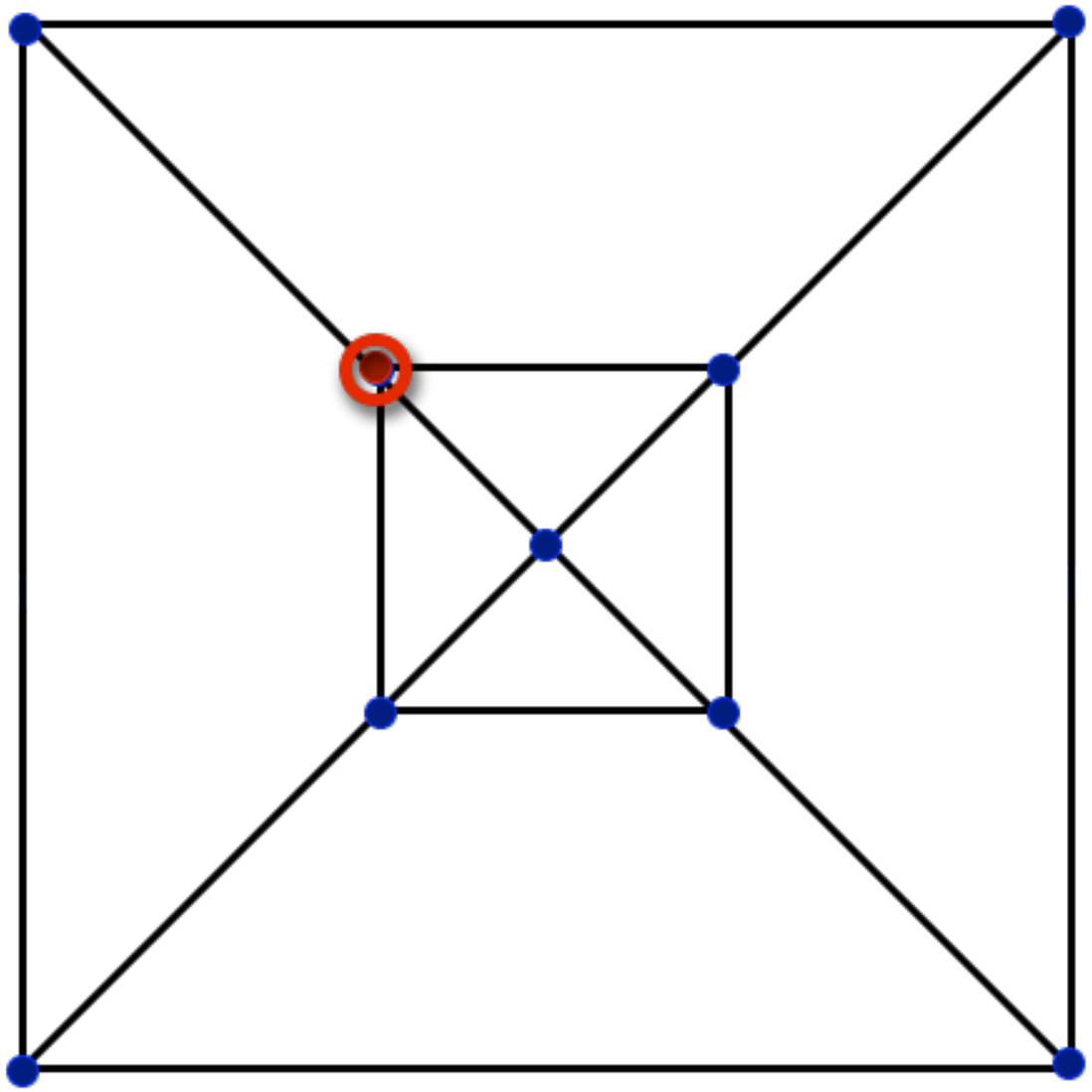
The Maxwell-Cremona Correspondence



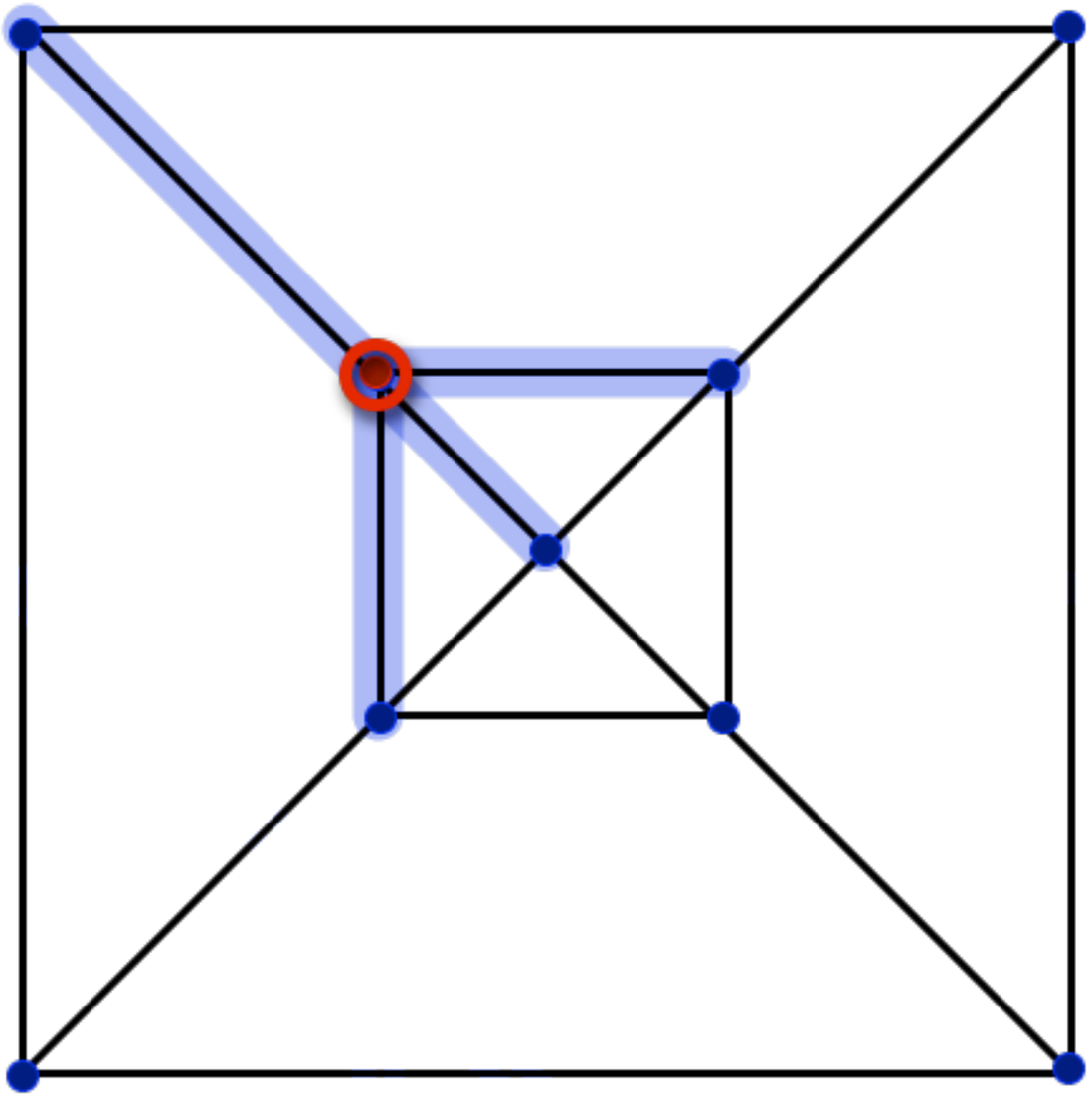
Reciprocal Diagrams from Equilibrium Stresses



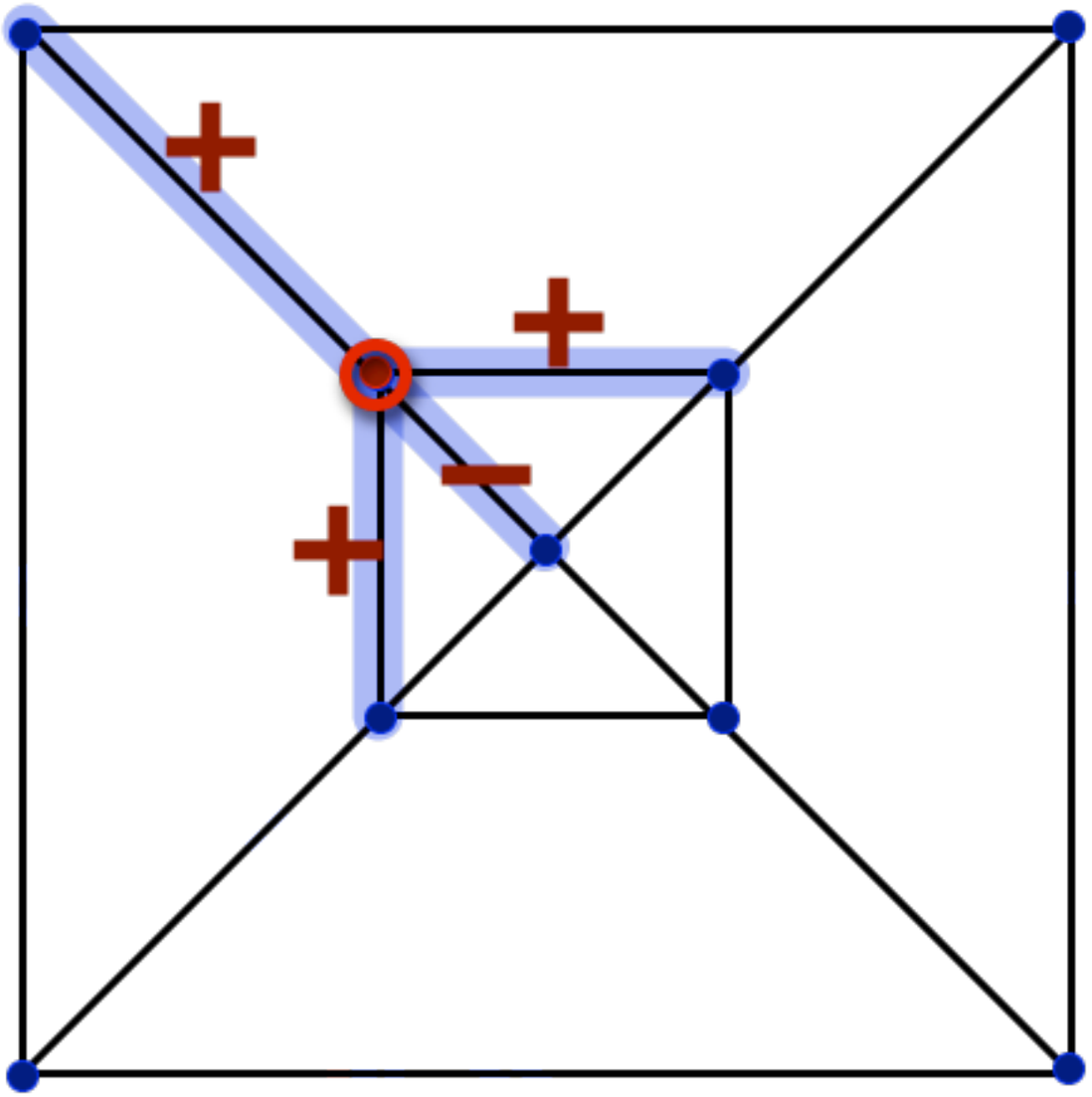
Reciprocal Diagrams from Equilibrium Stresses



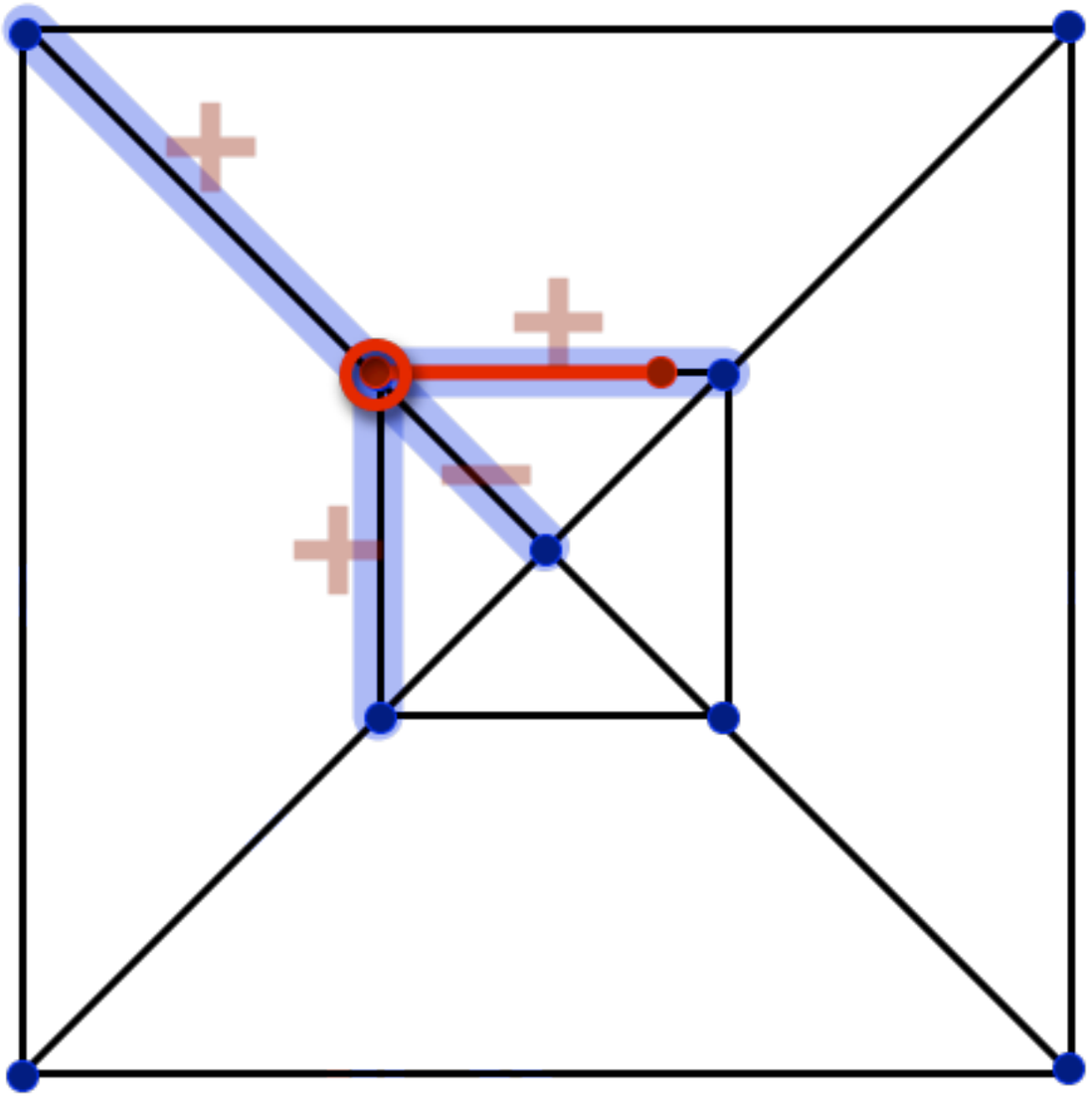
Reciprocal Diagrams from Equilibrium Stresses



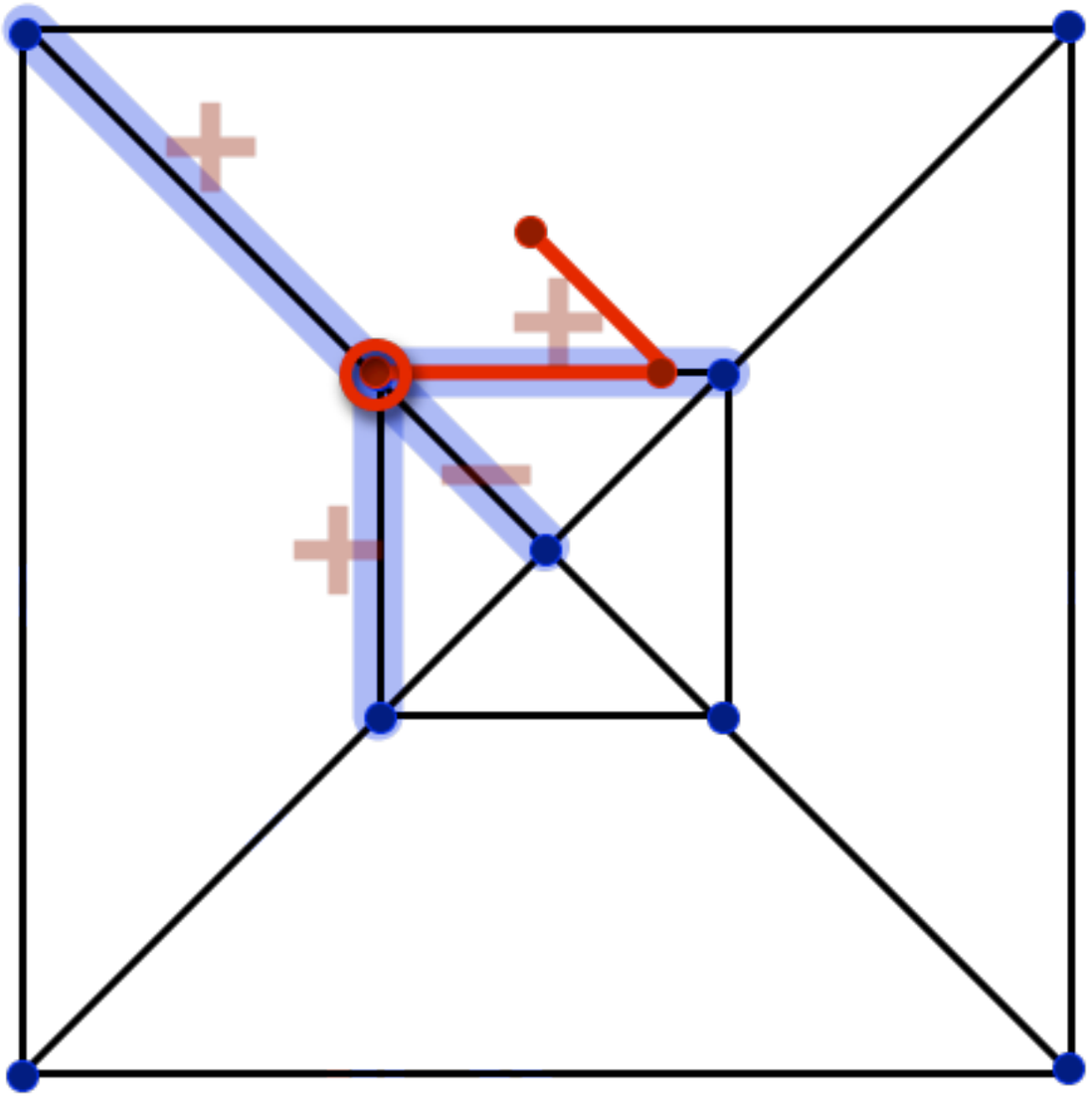
Reciprocal Diagrams from Equilibrium Stresses



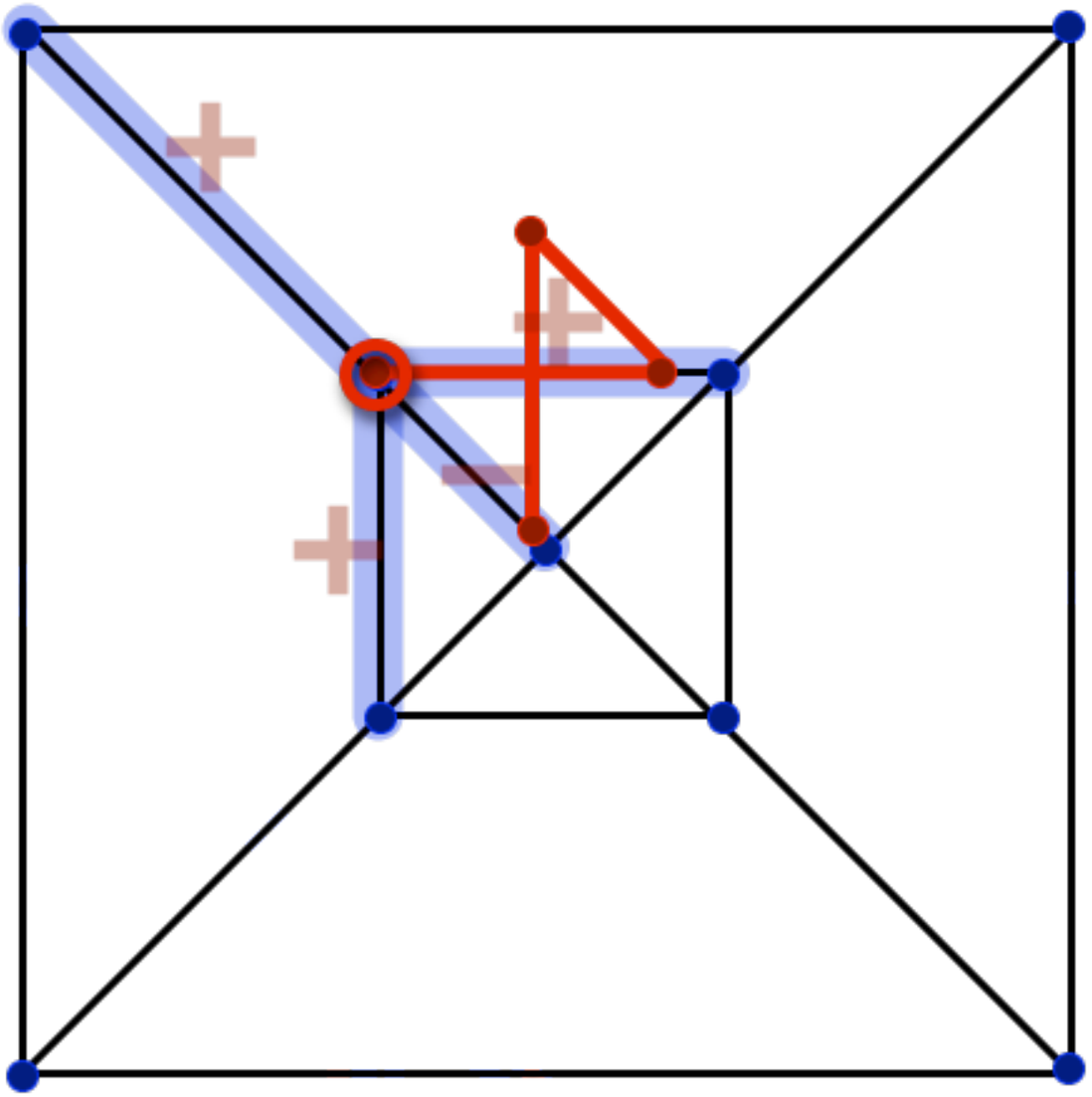
Reciprocal Diagrams from Equilibrium Stresses



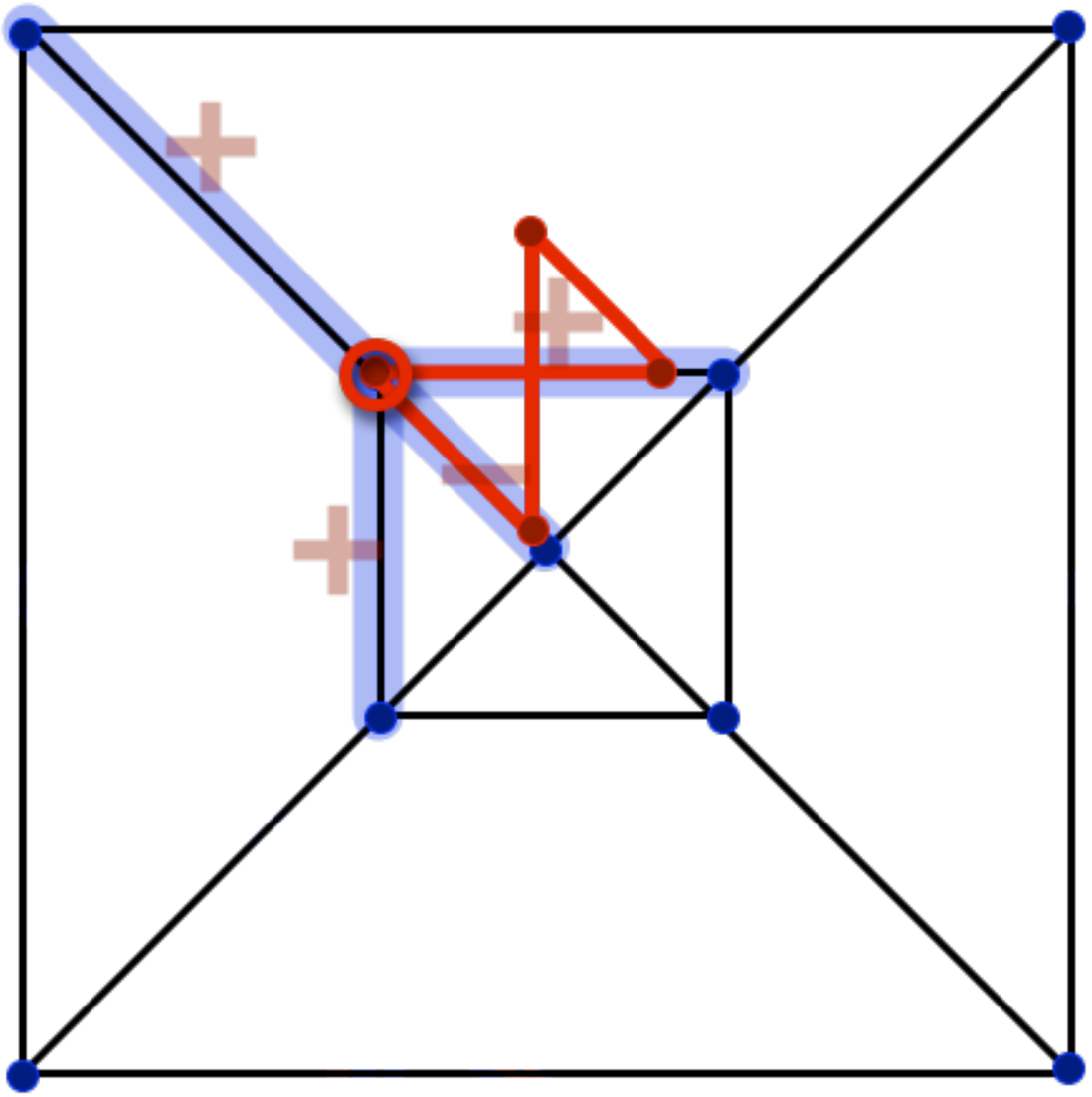
Reciprocal Diagrams from Equilibrium Stresses



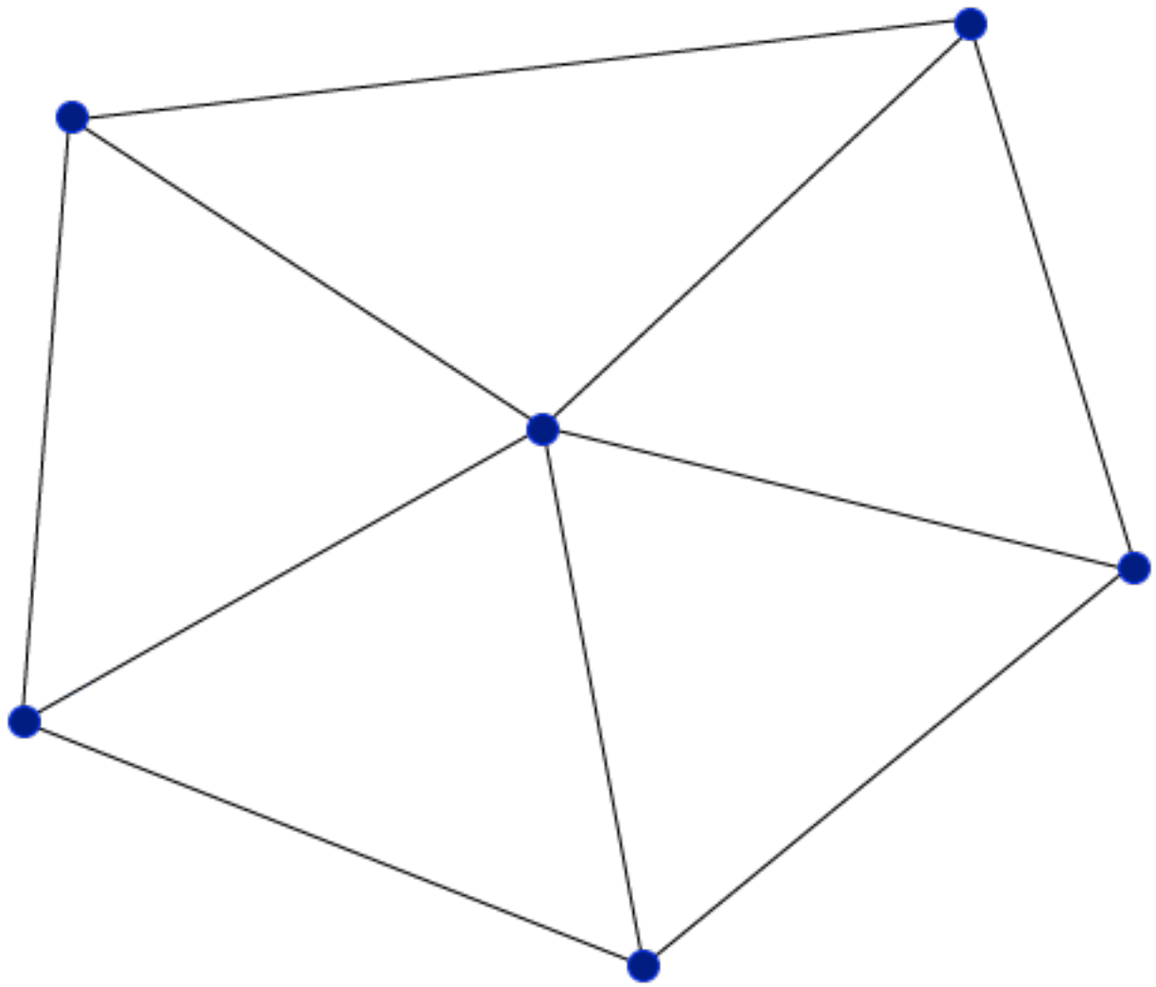
Reciprocal Diagrams from Equilibrium Stresses



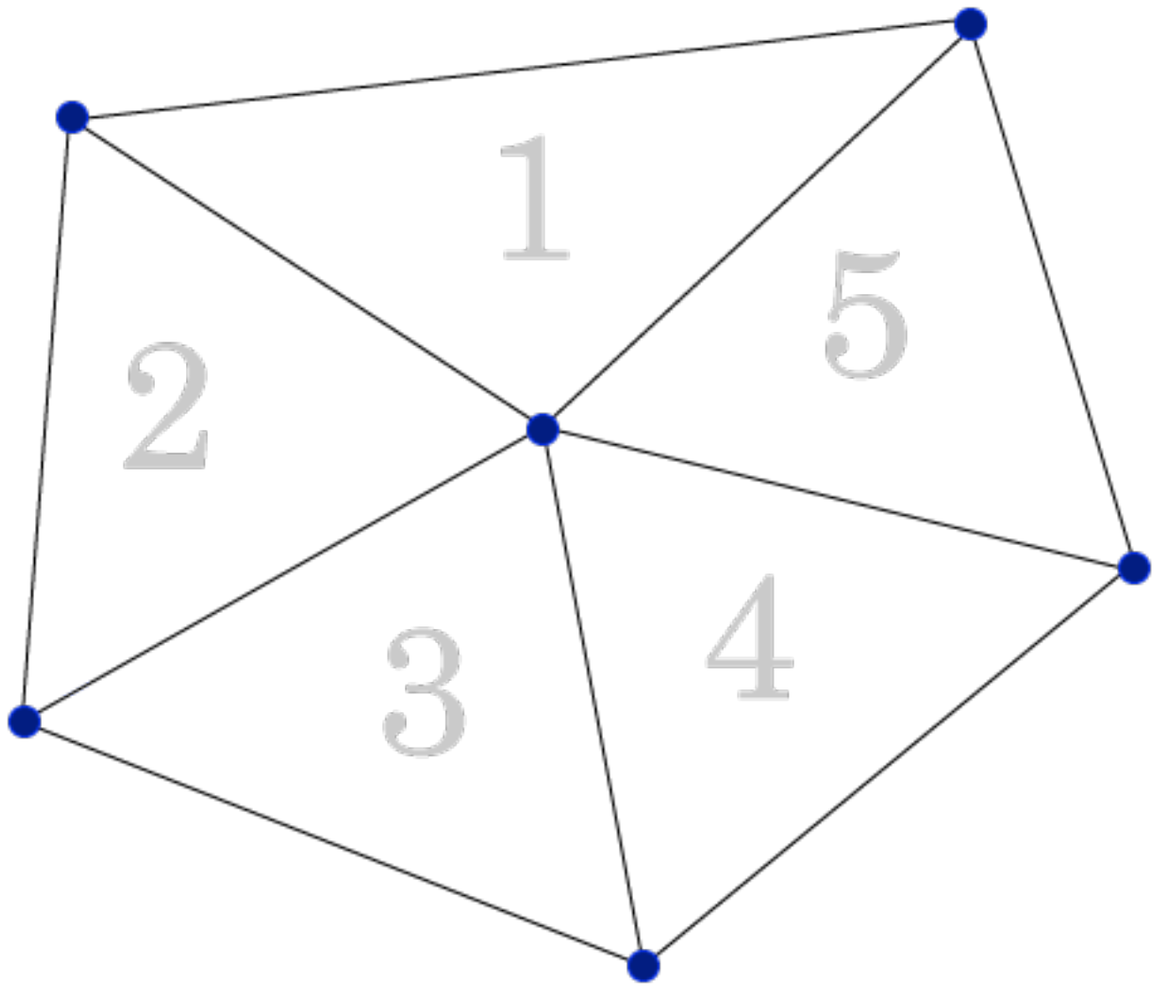
Reciprocal Diagrams from Equilibrium Stresses



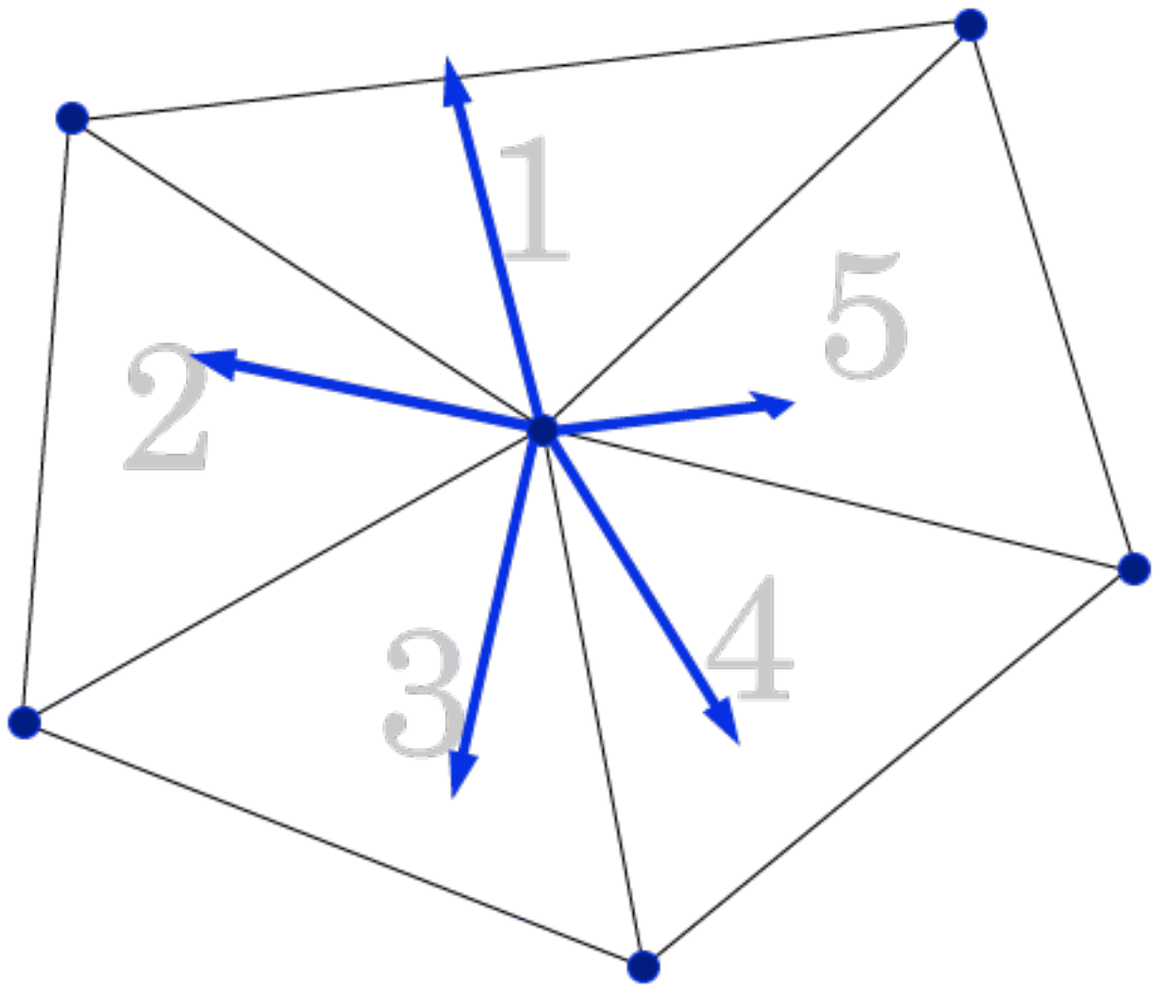
Reciprocal Diagrams from Liftings



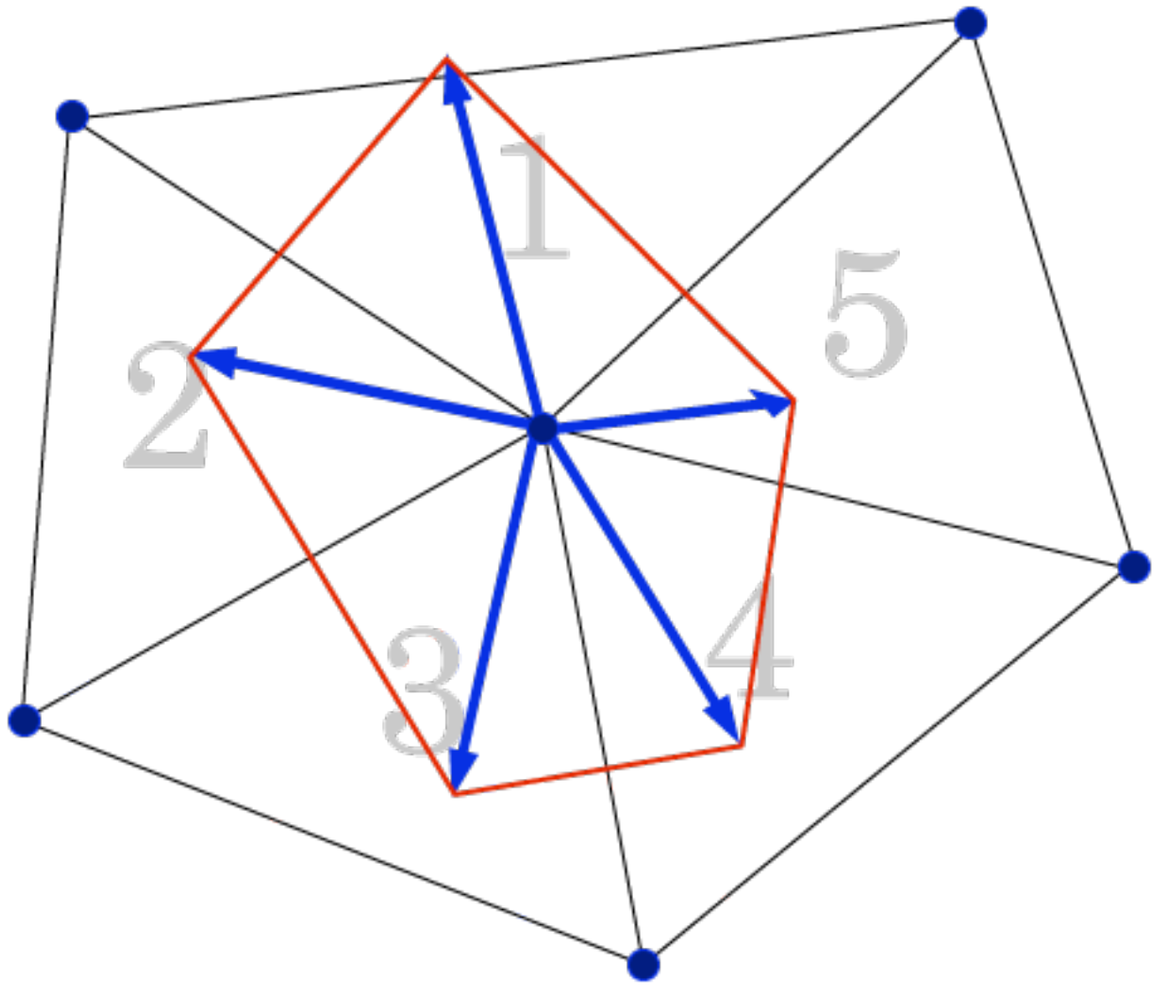
Reciprocal Diagrams from Liftings



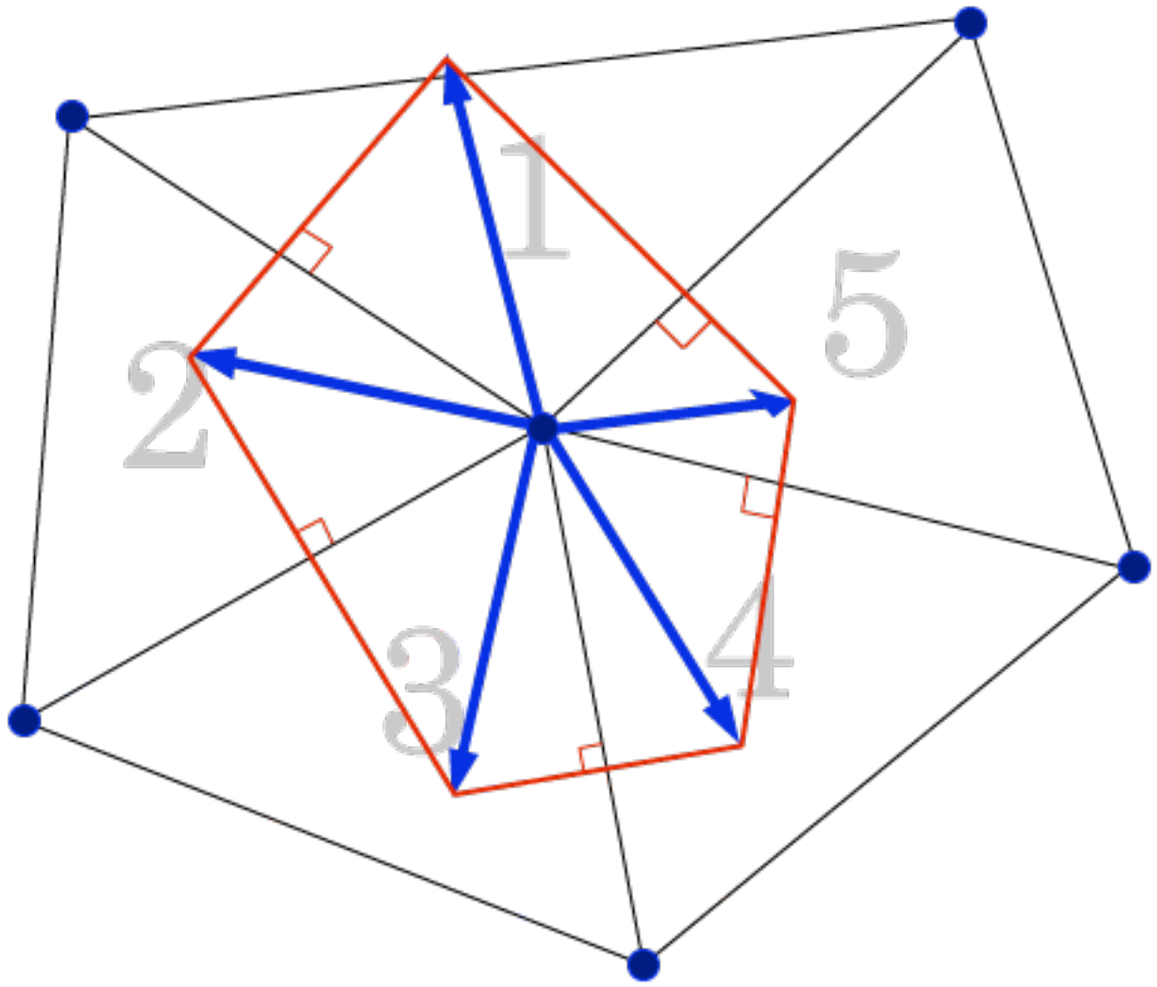
Reciprocal Diagrams from Liftings



Reciprocal Diagrams from Liftings



Reciprocal Diagrams from Liftings



The Maxwell-Cremona Corresondence

Equilibrium Stresses

Reciprocal Diagrams

Liftings

Other Famous Reciprocal Diagrams

Delaunay Triangulation

Voronoi Diagram

Other Famous Reciprocal Diagrams

Weighted Delaunay Triangulation

Weighted Voronoi Diagram

Other Famous Reciprocal Diagrams

Weighted Delaunay Triangulation

2½ dimensional polarity

Weighted Voronoi Diagram

How to Draw a Graph

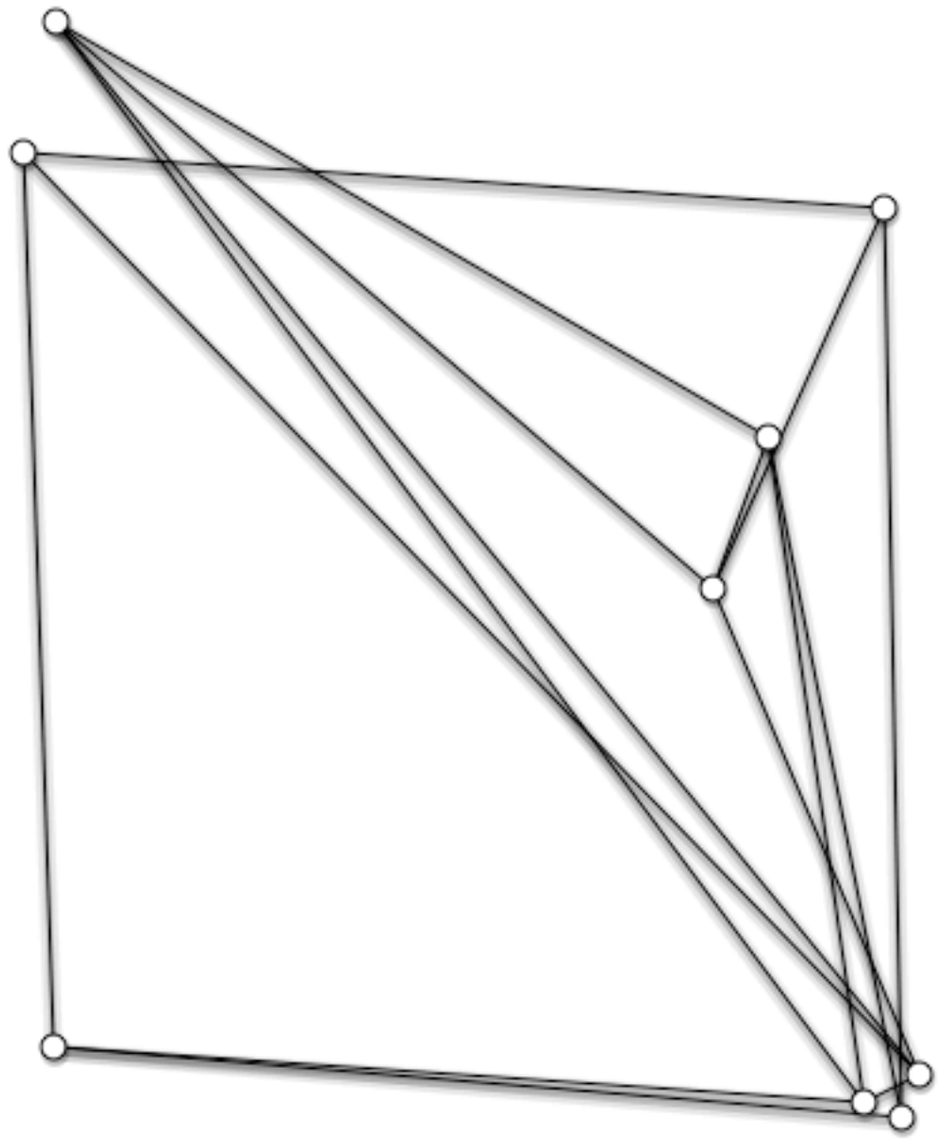
Tutte's Algorithm

1. Fix one face of a simple, planar, 3-connected graph in convex position.
2. Place each other vertex at the barycenter (centroid) of its neighbors.

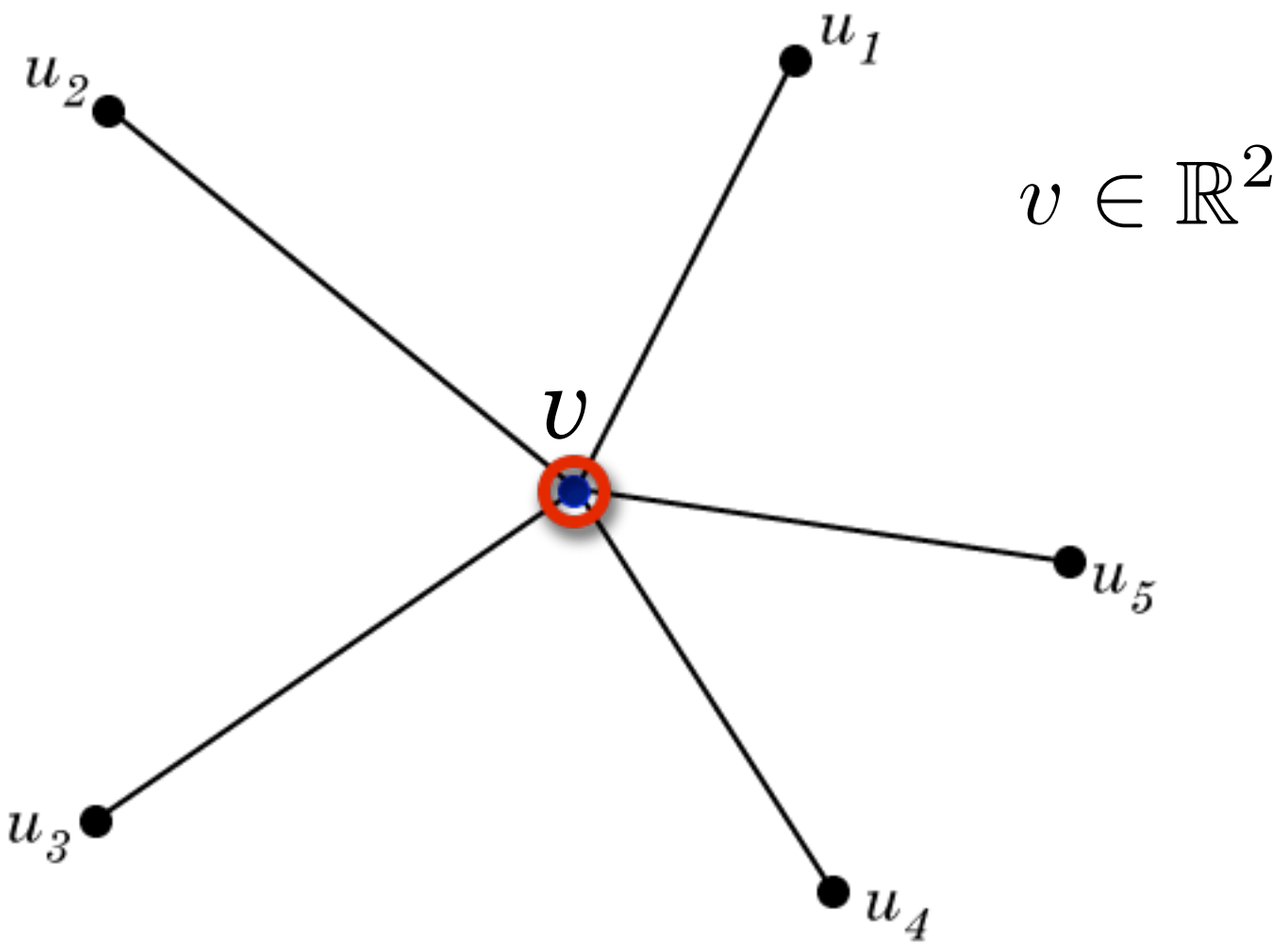
The result is a non-crossing, convex drawing.

Spring Interpretation

Spring Interpretation

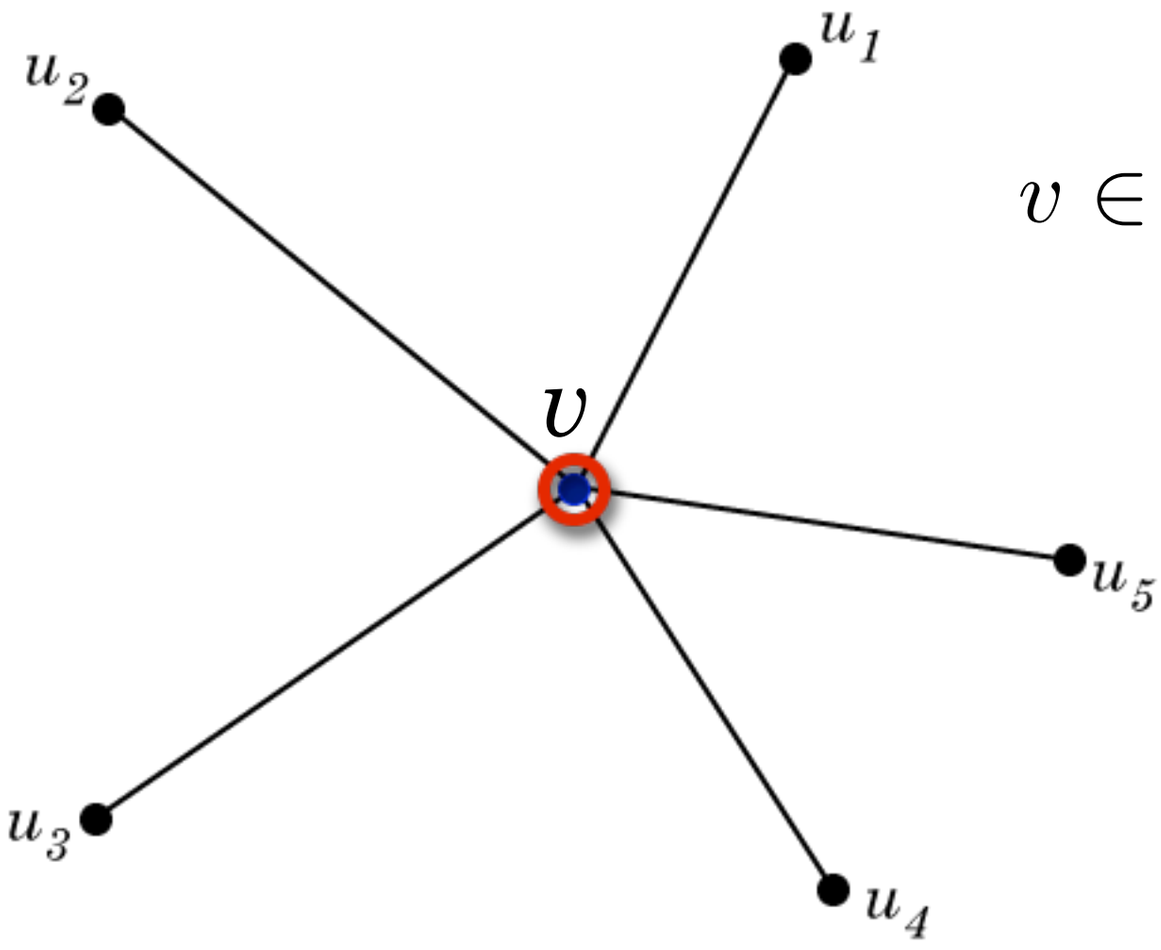


Computing Forces



$$v \in \mathbb{R}^2$$

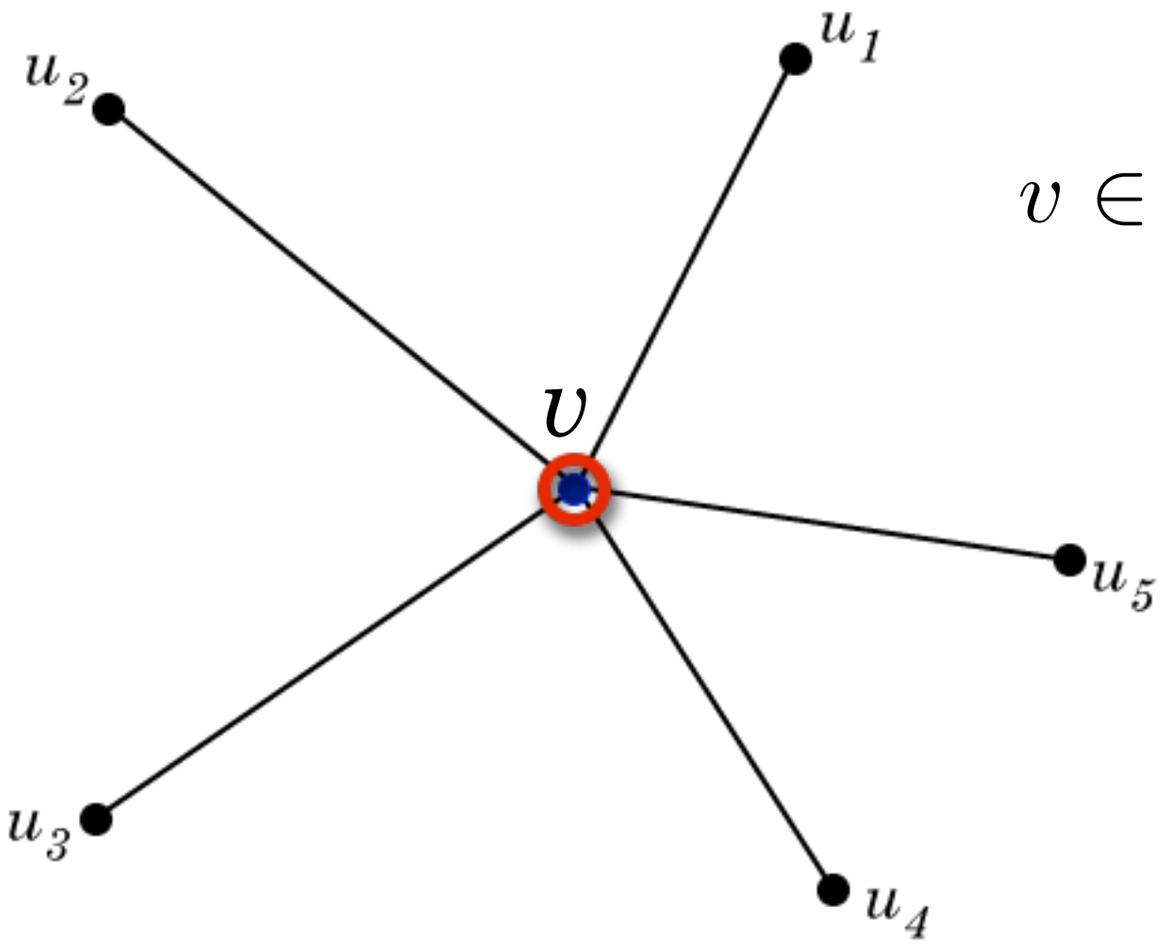
Computing Forces



$$v \in \mathbb{R}^2$$

$$F_v = \sum_{u \sim v} (v - u)$$

Computing Forces

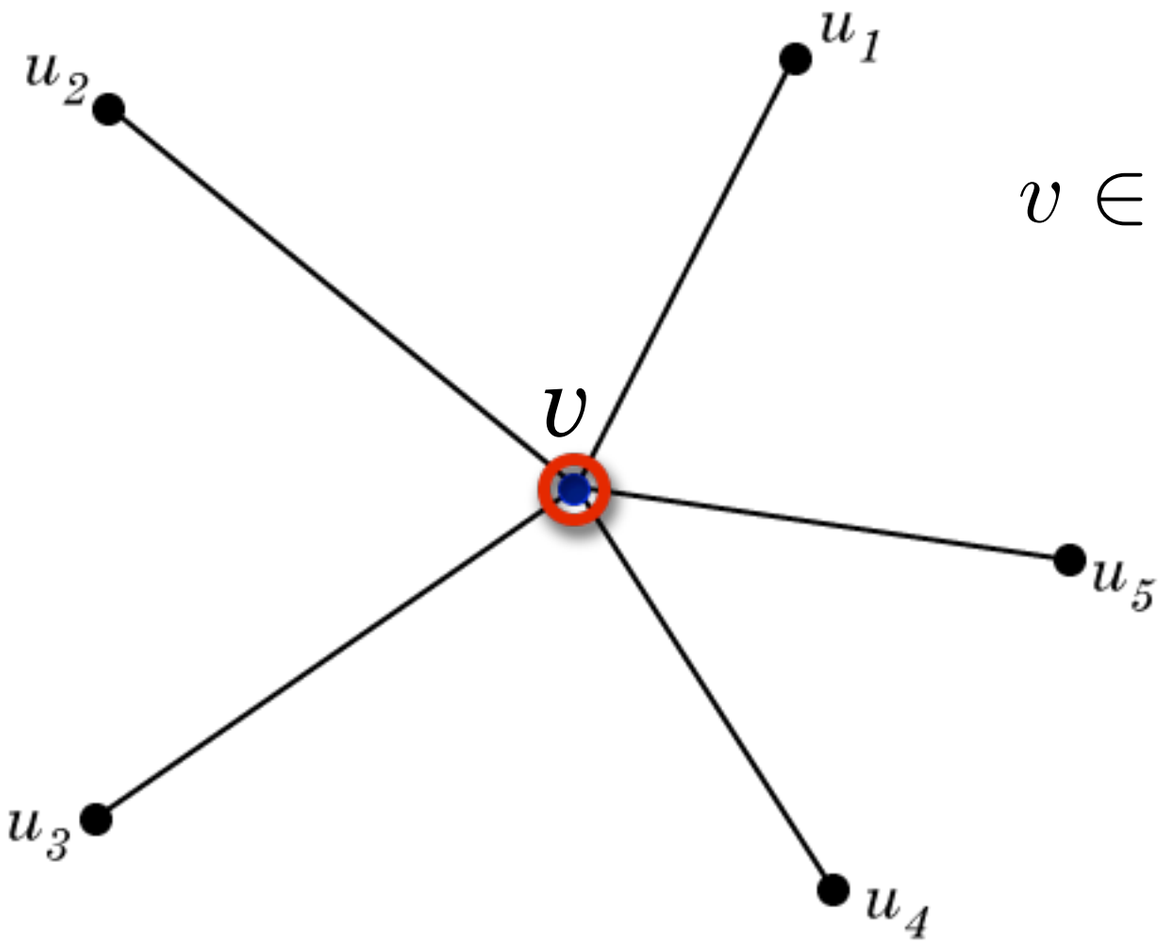


$$v \in \mathbb{R}^2$$

$$F_v = \sum_{u \sim v} (v - u)$$

$$= d_v v - \sum_{u \sim v} u$$

Computing Forces



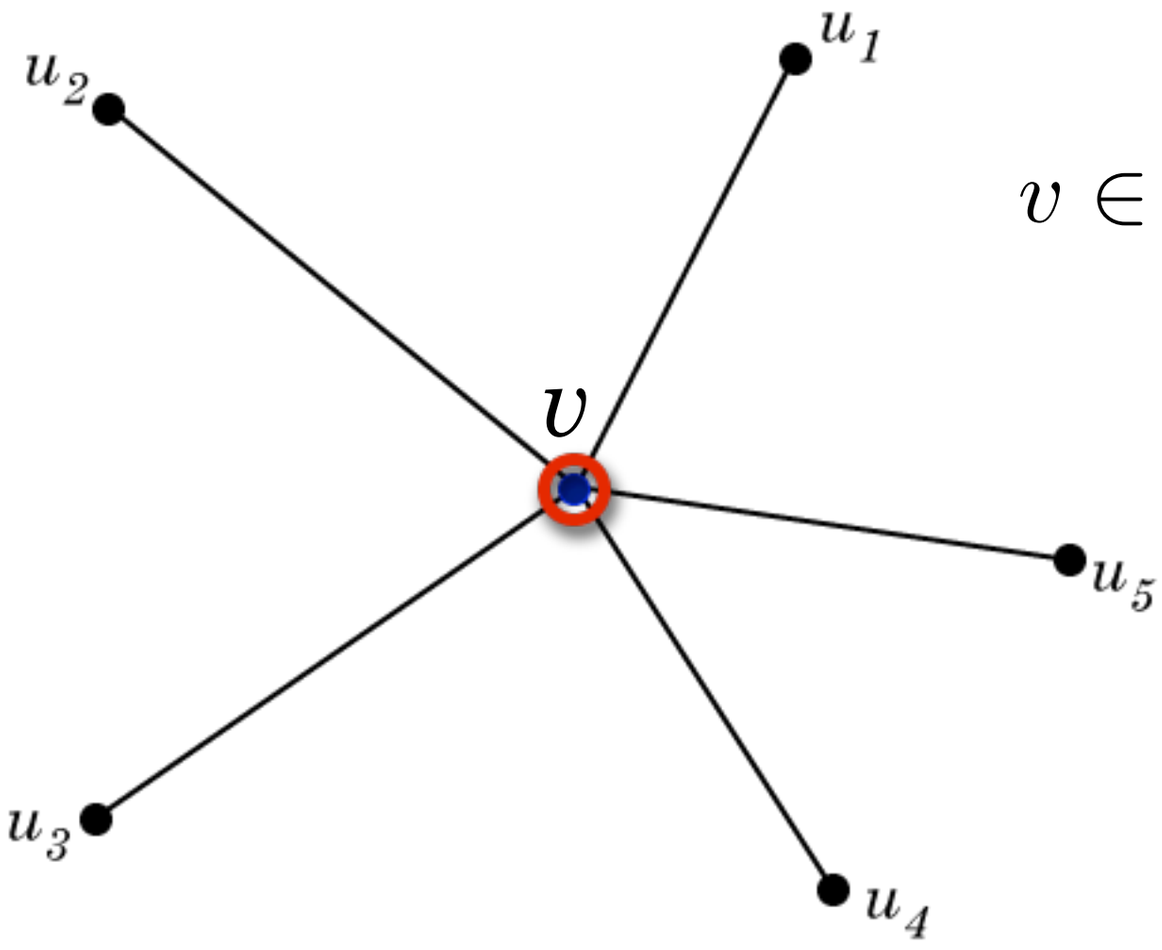
$$v \in \mathbb{R}^2$$

$$F_v = \sum_{u \sim v} (v - u)$$

$$= d_v v - \sum_{u \sim v} u$$

$$F = LV$$

Computing Forces



$$v \in \mathbb{R}^2$$

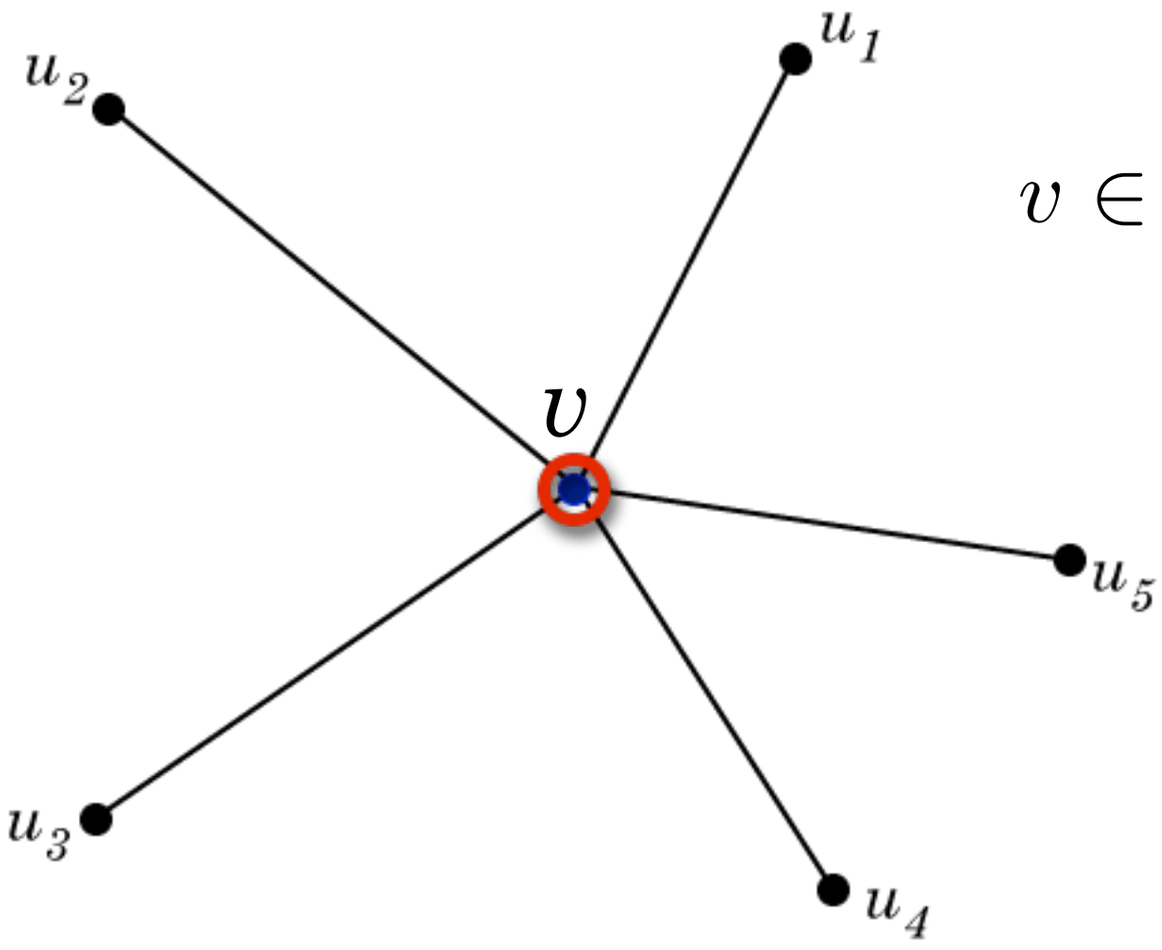
$$F_v = \sum_{u \sim v} (v - u)$$

$$= d_v v - \sum_{u \sim v} u$$

$$F = LV$$

$$L = D - A$$

Computing Forces



$$v \in \mathbb{R}^2$$

$$F_v = \sum_{u \sim v} (v - u)$$

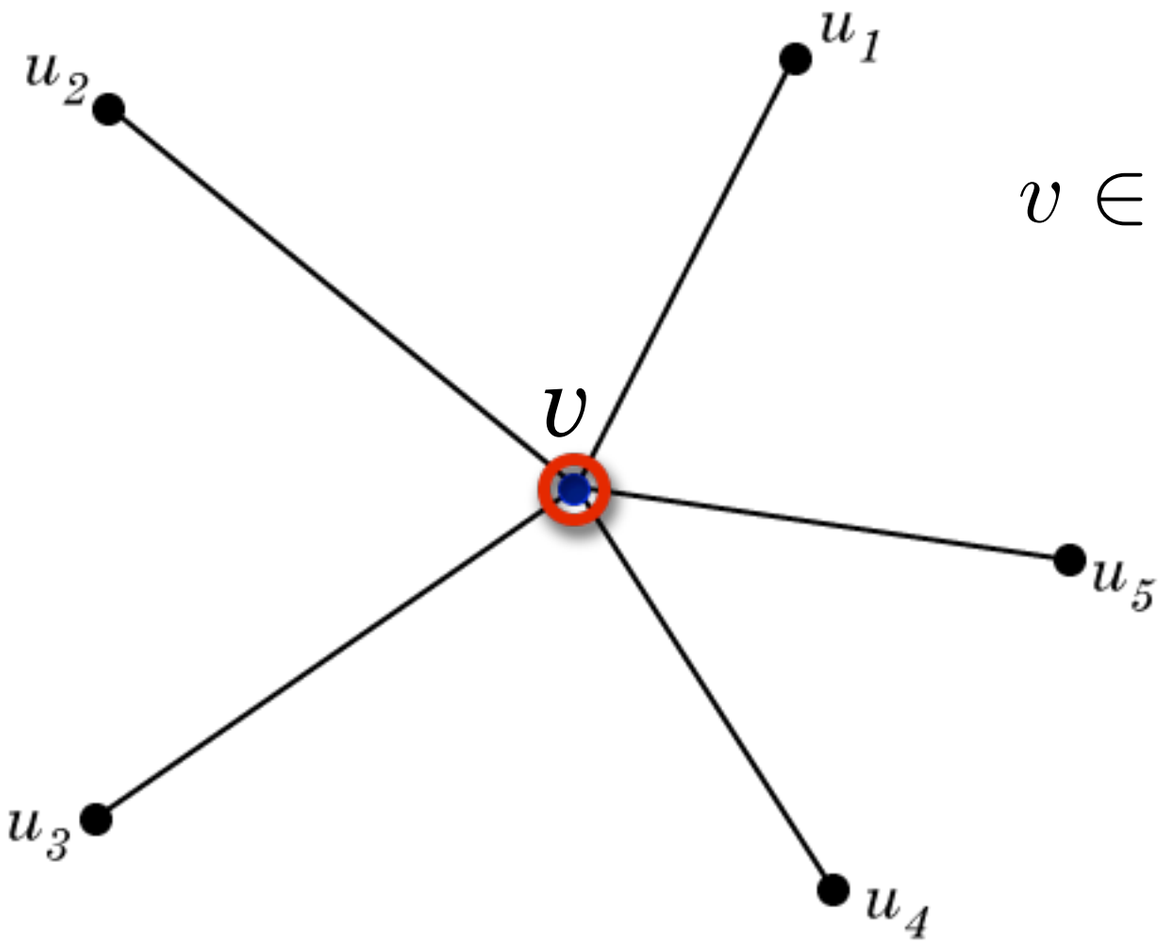
$$= d_v v - \sum_{u \sim v} u$$

$$F = LV$$

$$L = D - A$$

degrees

Computing Forces



$$v \in \mathbb{R}^2$$

$$F_v = \sum_{u \sim v} (v - u)$$

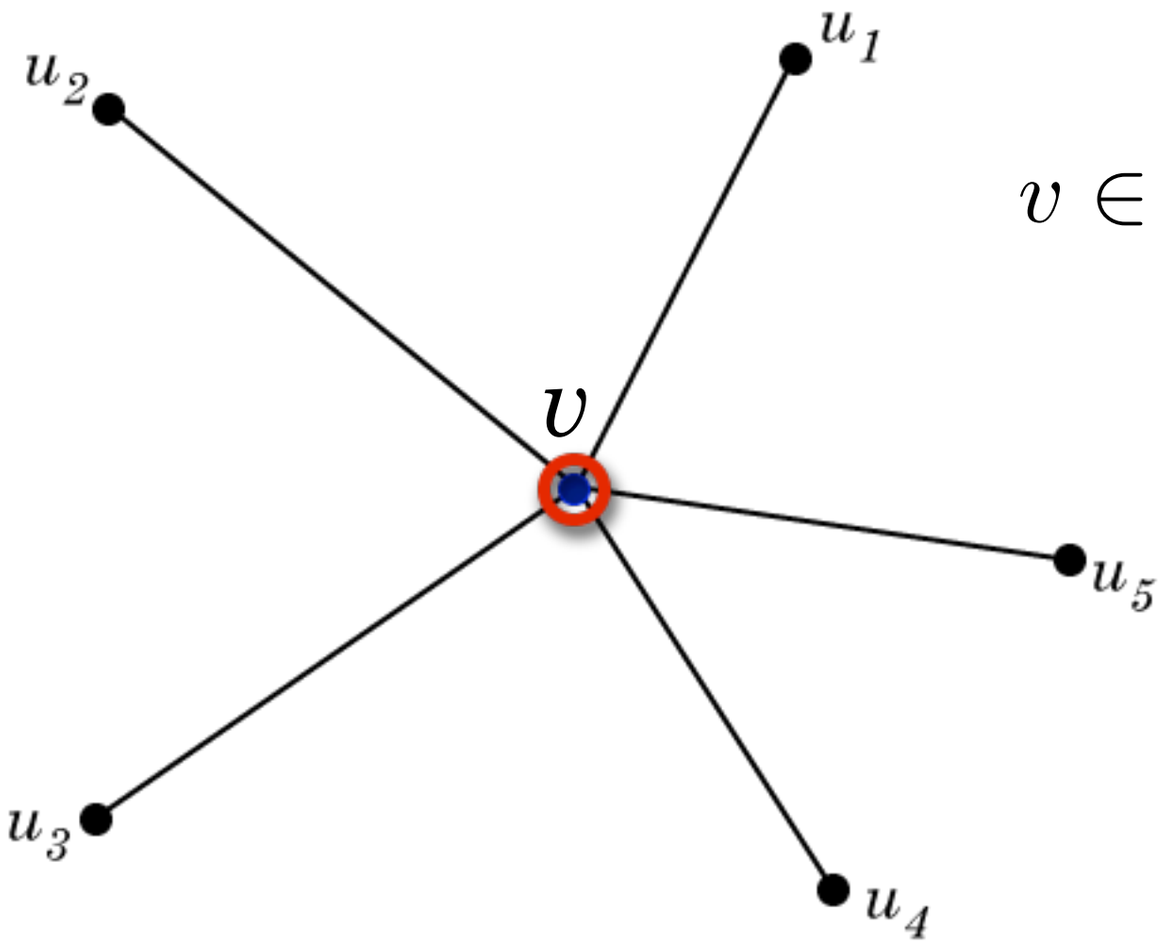
$$= d_v v - \sum_{u \sim v} u$$

$$F = LV$$

$$L = D - A$$

degrees *adjacency*

Computing Forces



$$v \in \mathbb{R}^2$$

$$F_v = \sum_{u \sim v} (v - u)$$

$$= d_v v - \sum_{u \sim v} u$$

$$F = LV$$

$$L = D - A$$

degrees *adjacency*

The Laplacian!

Computing Forces

$$LV = F$$

Computing Forces

$$LV = F = 0?$$

Computing Forces

$LV = F = 0$?

V_1: boundary

V_2: interior

Computing Forces

$$LV = F = 0?$$

V_1: boundary
V_2: interior

$$\begin{bmatrix} L_1 & B^T \\ B & L_2 \end{bmatrix} \begin{bmatrix} V_1 \\ V_2 \end{bmatrix} = \begin{bmatrix} F' \\ 0 \end{bmatrix}$$

Computing Forces

$$LV = F = 0?$$

V_1: boundary
V_2: interior

$$\begin{bmatrix} L_1 & B^T \\ B & L_2 \end{bmatrix} \begin{bmatrix} V_1 \\ V_2 \end{bmatrix} = \begin{bmatrix} F' \\ 0 \end{bmatrix}$$

$$BV_1 + L_2V_2 = 0$$

Computing Forces

$$LV = F = 0?$$

V_1: boundary
V_2: interior

$$\begin{bmatrix} L_1 & B^T \\ B & L_2 \end{bmatrix} \begin{bmatrix} V_1 \\ V_2 \end{bmatrix} = \begin{bmatrix} F' \\ 0 \end{bmatrix}$$

$$BV_1 + L_2V_2 = 0$$

$$V_2 = -L_2^{-1}B\ V_1$$

Computing Forces

$$LV = F = 0?$$

V_1: boundary
V_2: interior

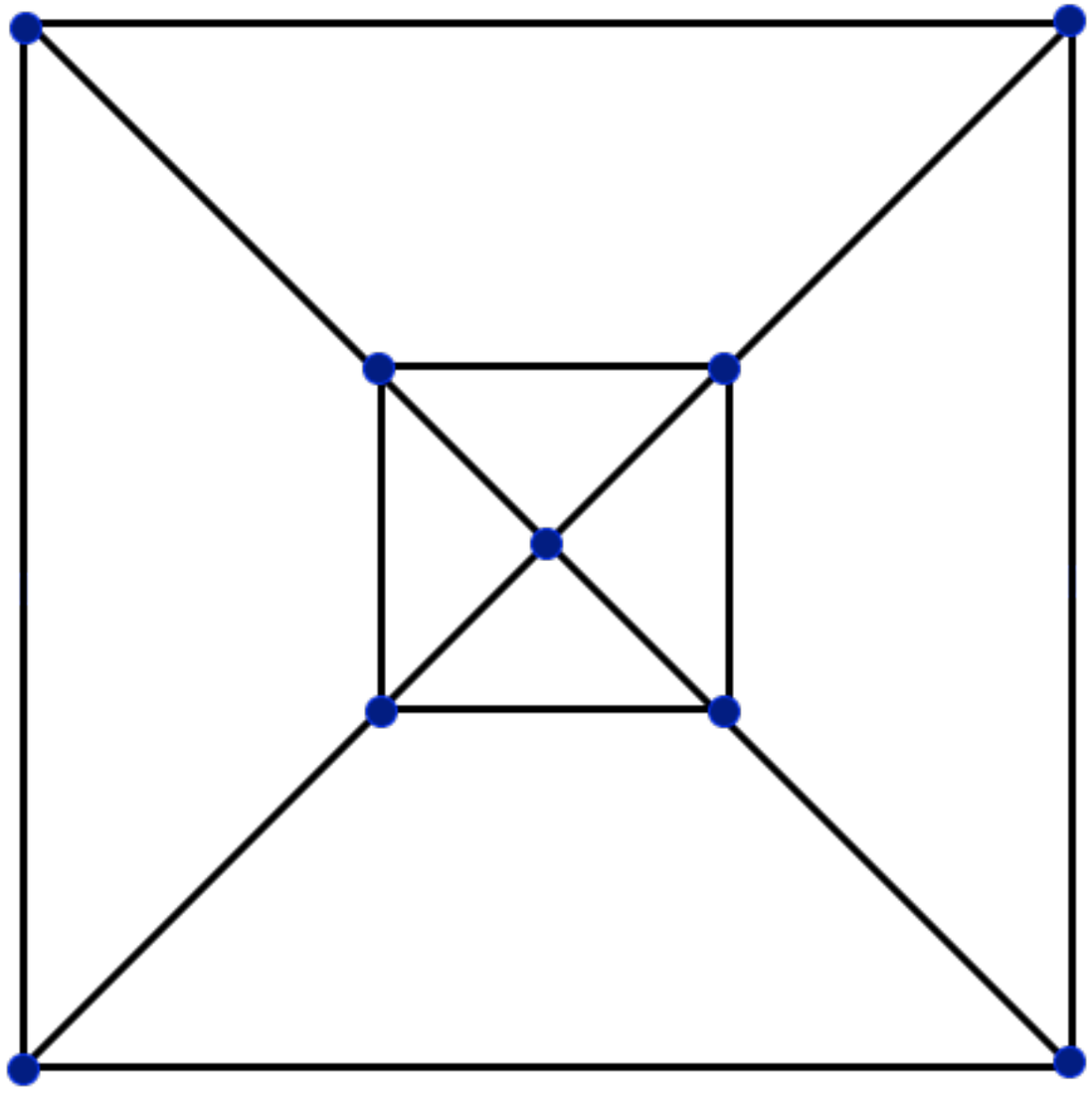
$$\begin{bmatrix} L_1 & B^T \\ B & L_2 \end{bmatrix} \begin{bmatrix} V_1 \\ V_2 \end{bmatrix} = \begin{bmatrix} F' \\ 0 \end{bmatrix}$$

$$BV_1 + L_2V_2 = 0$$

$$V_2 = (-L_2^{-1}B)V_1$$

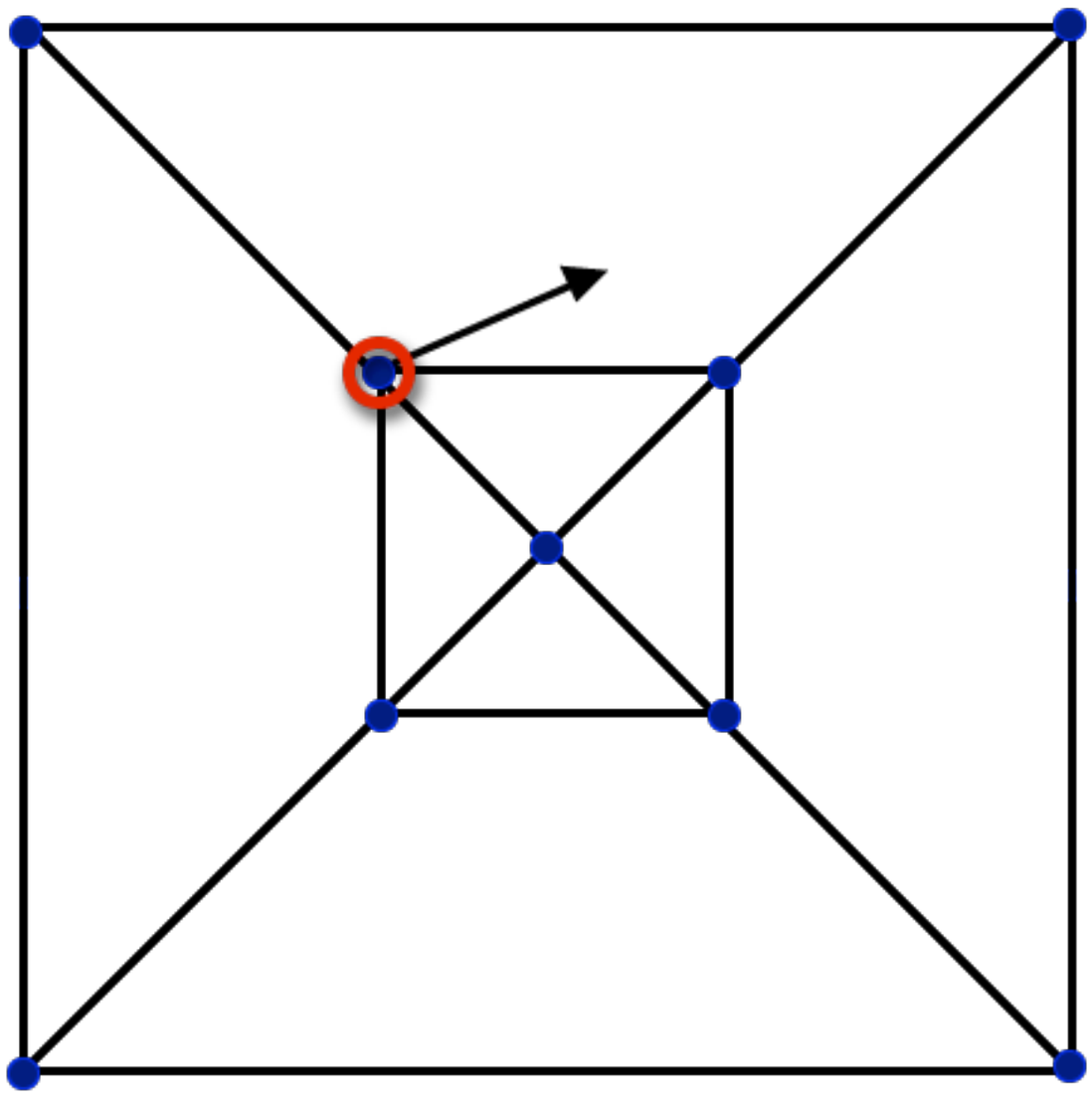
Monotone Paths

Pick a direction and a vertex. There is a monotone path in that direction from the vertex to the boundary.



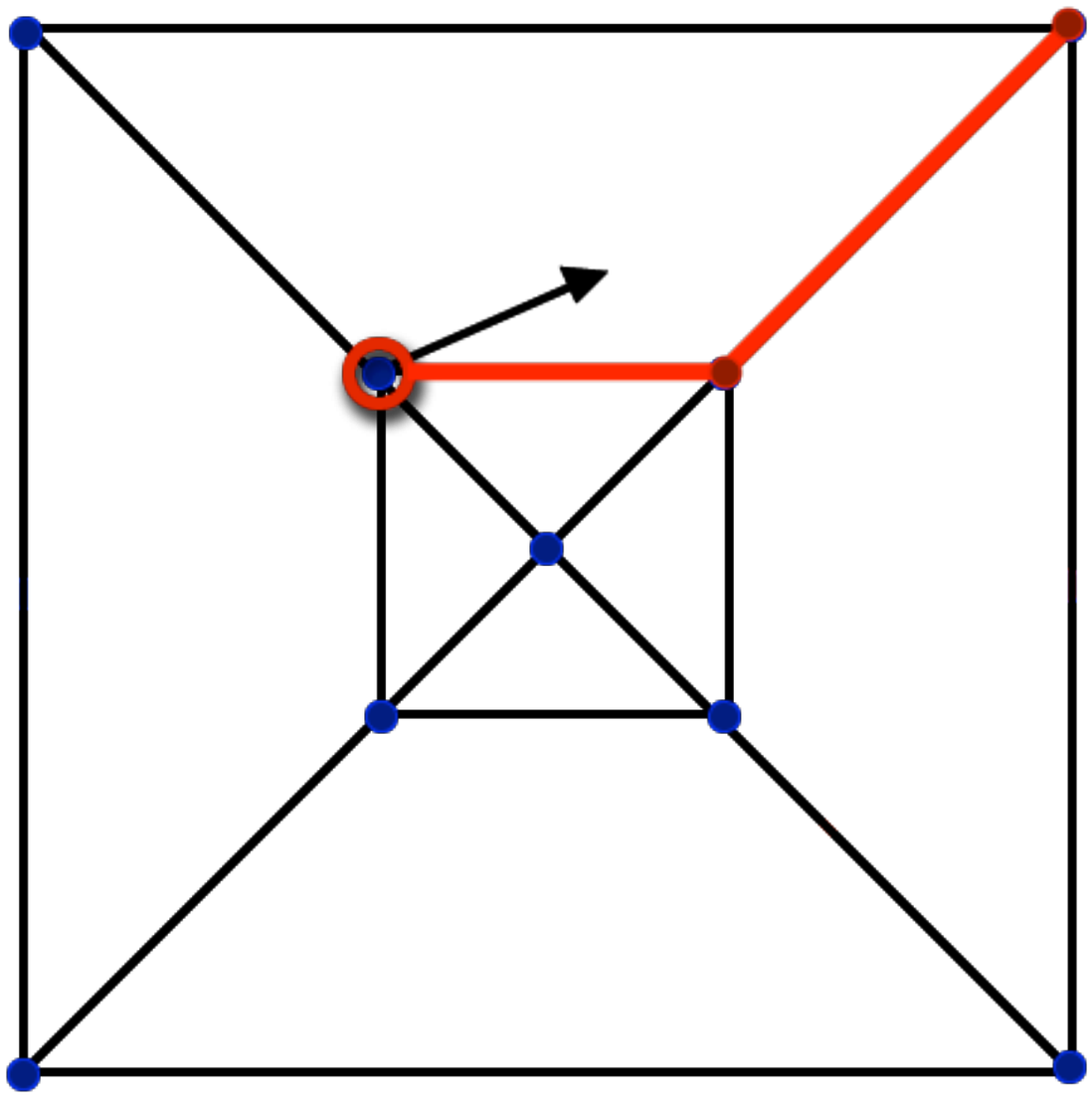
Monotone Paths

Pick a direction and a vertex. There is a monotone path in that direction from the vertex to the boundary.



Monotone Paths

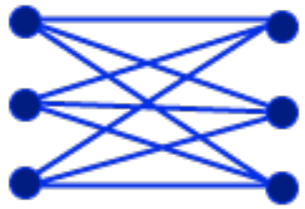
Pick a direction and a vertex. There is a monotone path in that direction from the vertex to the boundary.



Planar, 3-Connected Graphs

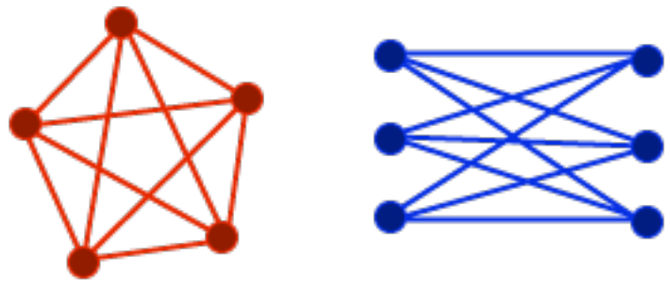
Planar, 3-Connected Graphs

➡ No K_5 or $K_{3,3}$ minors

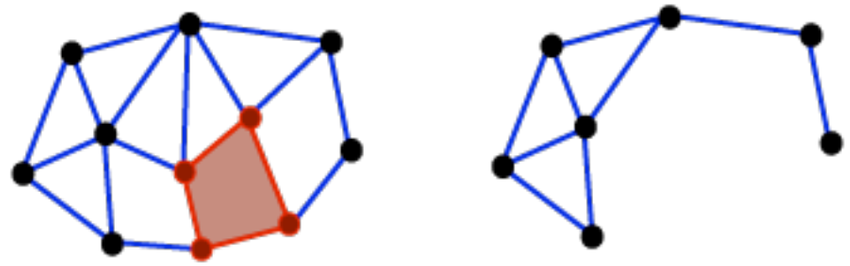


Planar, 3-Connected Graphs

- No K_5 or $K_{3,3}$ minors

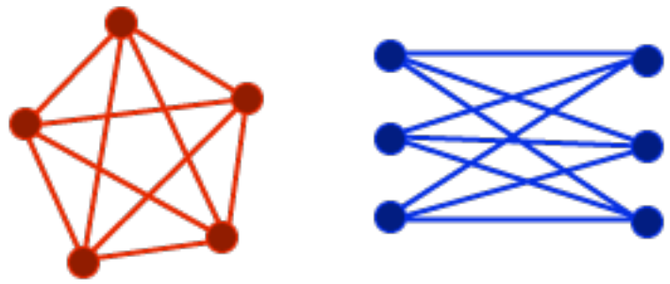


- Removing a face does not disconnect the graph.

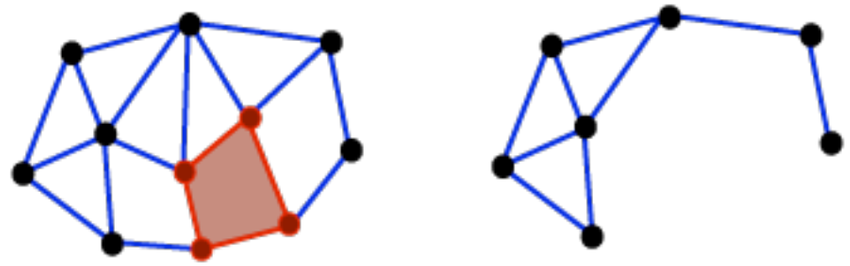


Planar, 3-Connected Graphs

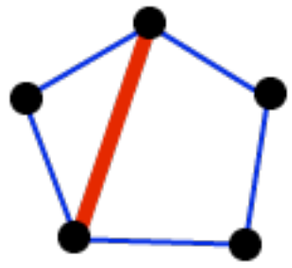
- No K_5 or $K_{3,3}$ minors



- Removing a face does not disconnect the graph.

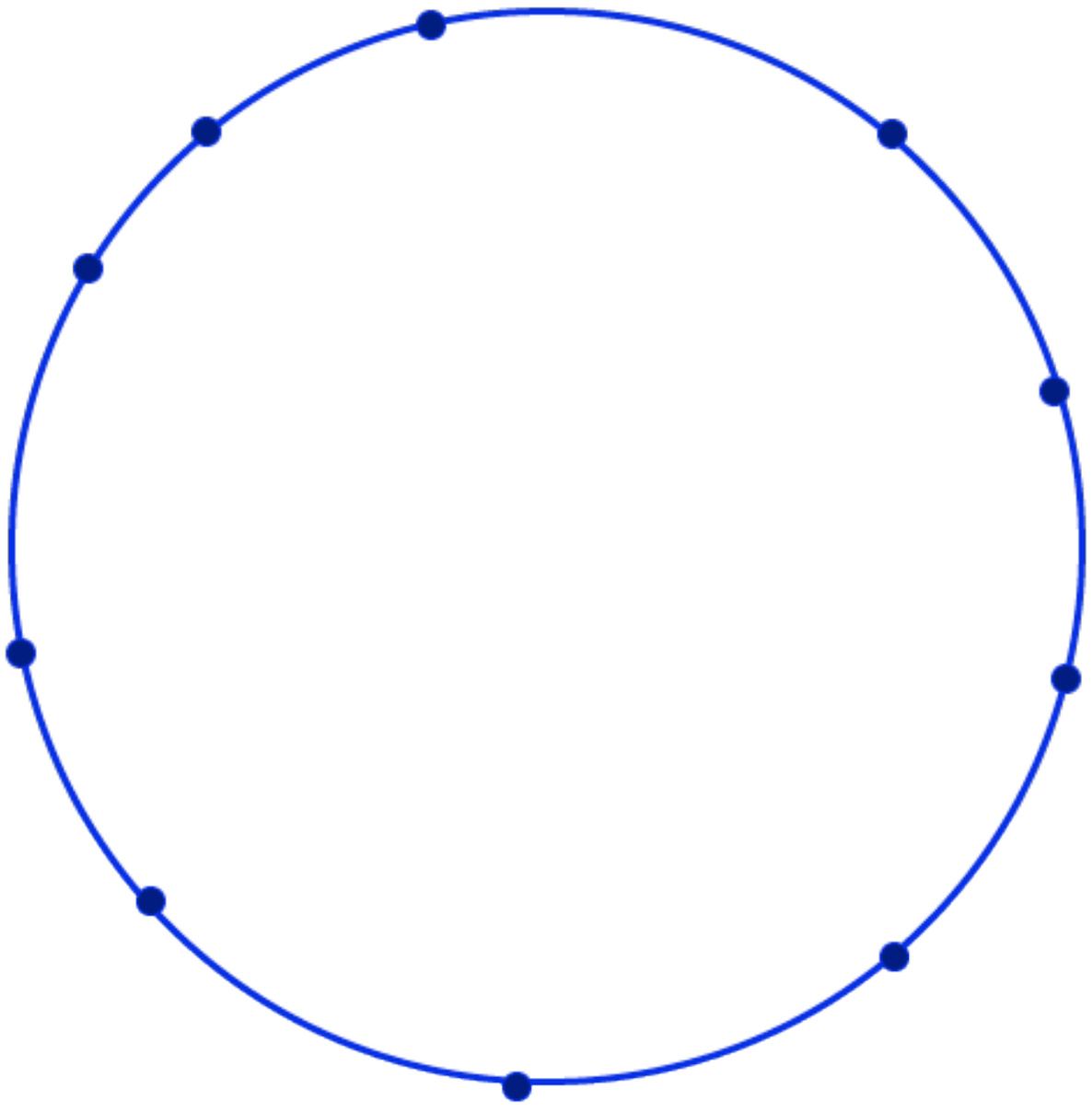


- No face has a diagonal.



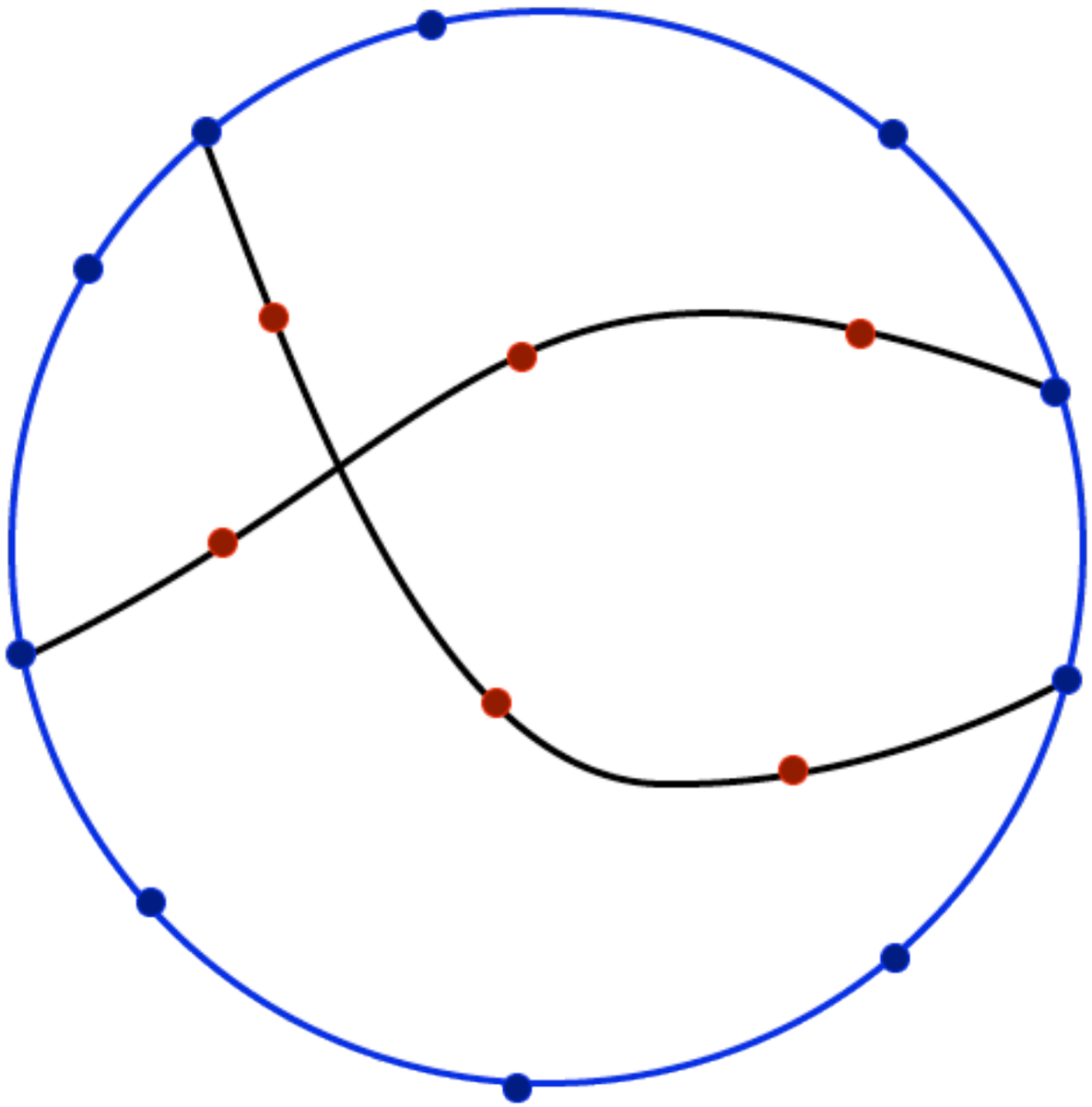
Double Crossing a Face

Lemma: No two disjoint paths have interleaved endpoints on a face.



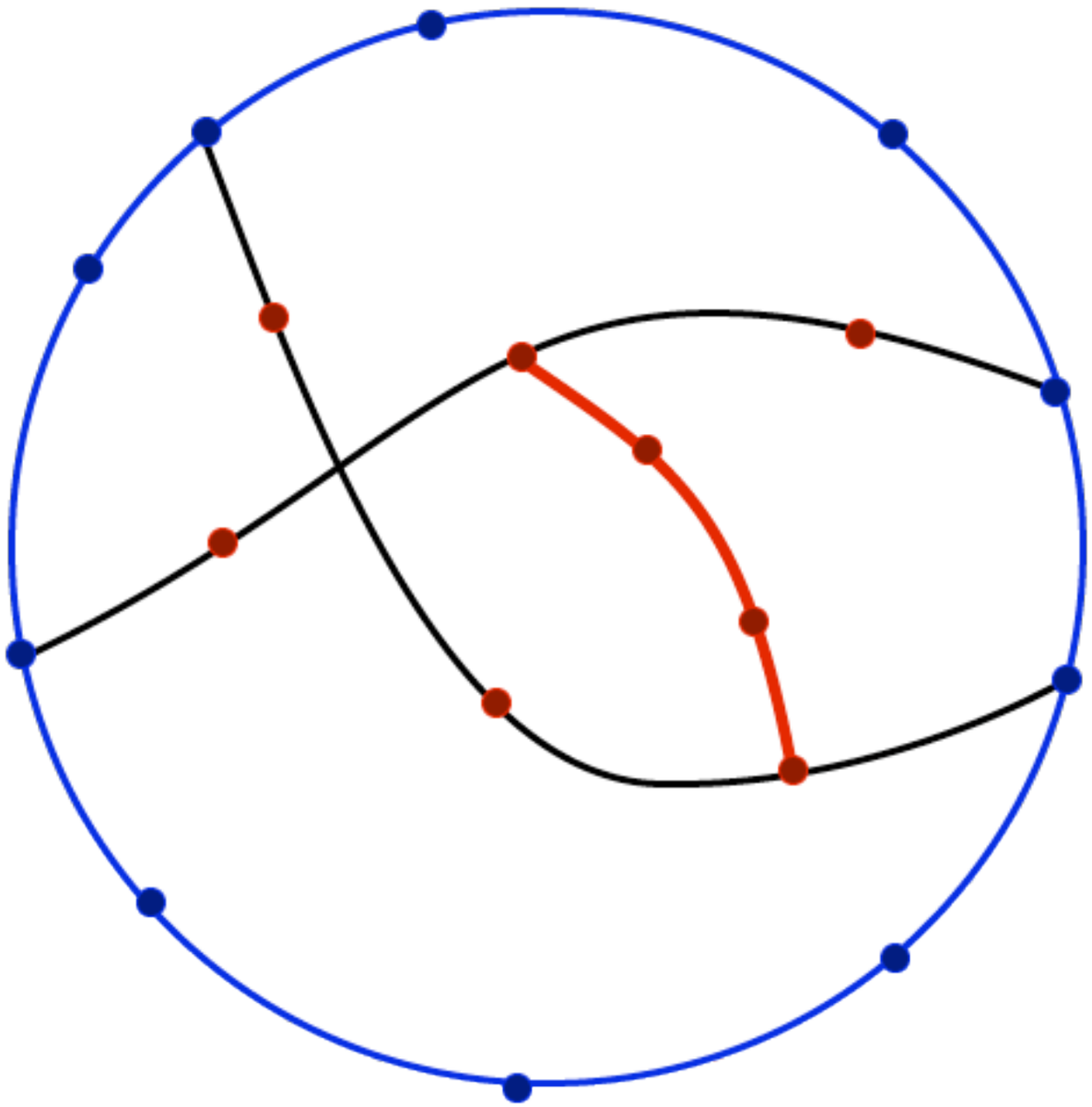
Double Crossing a Face

Lemma: No two disjoint paths have interleaved endpoints on a face.



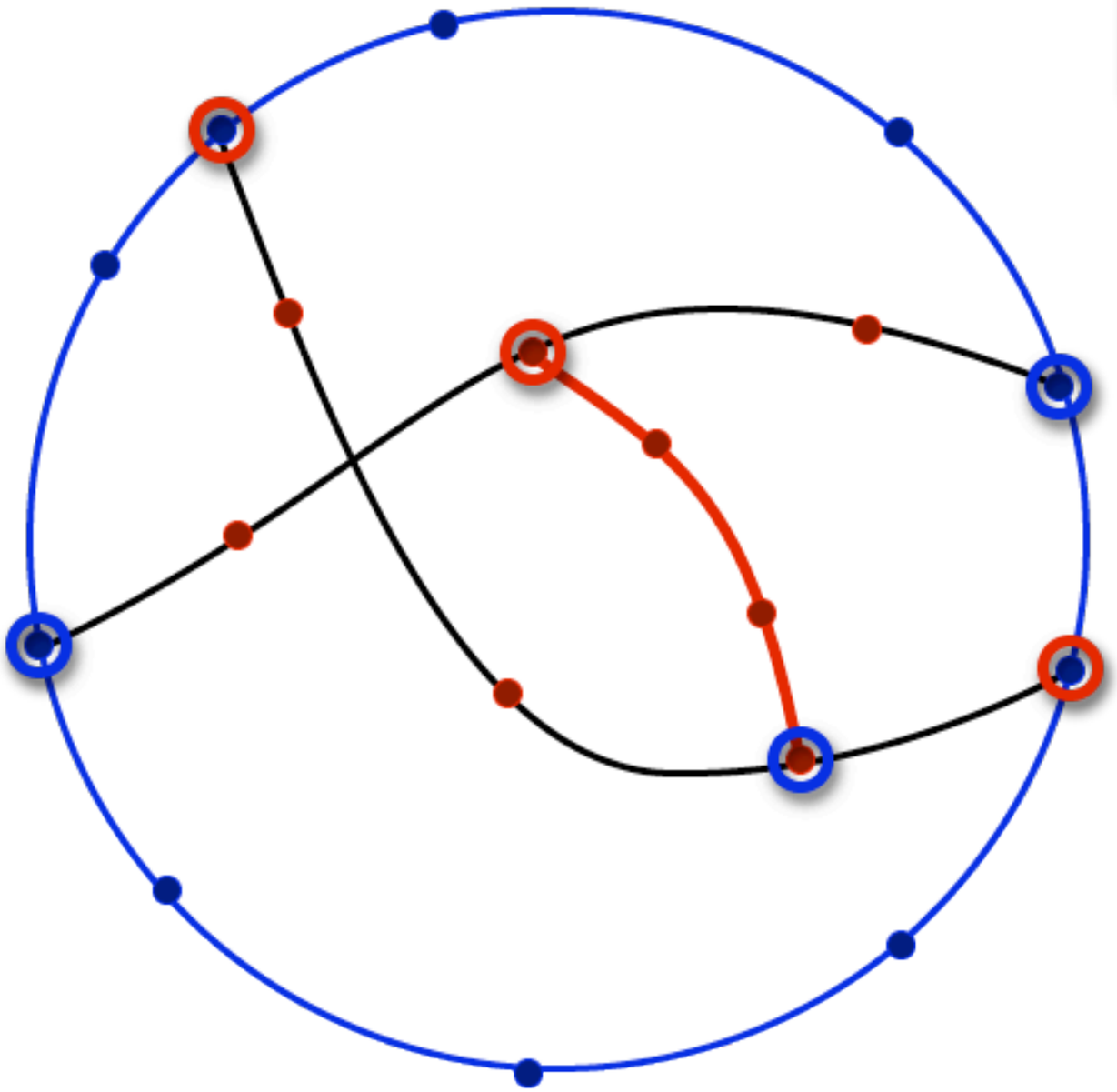
Double Crossing a Face

Lemma: No two disjoint paths have interleaved endpoints on a face.



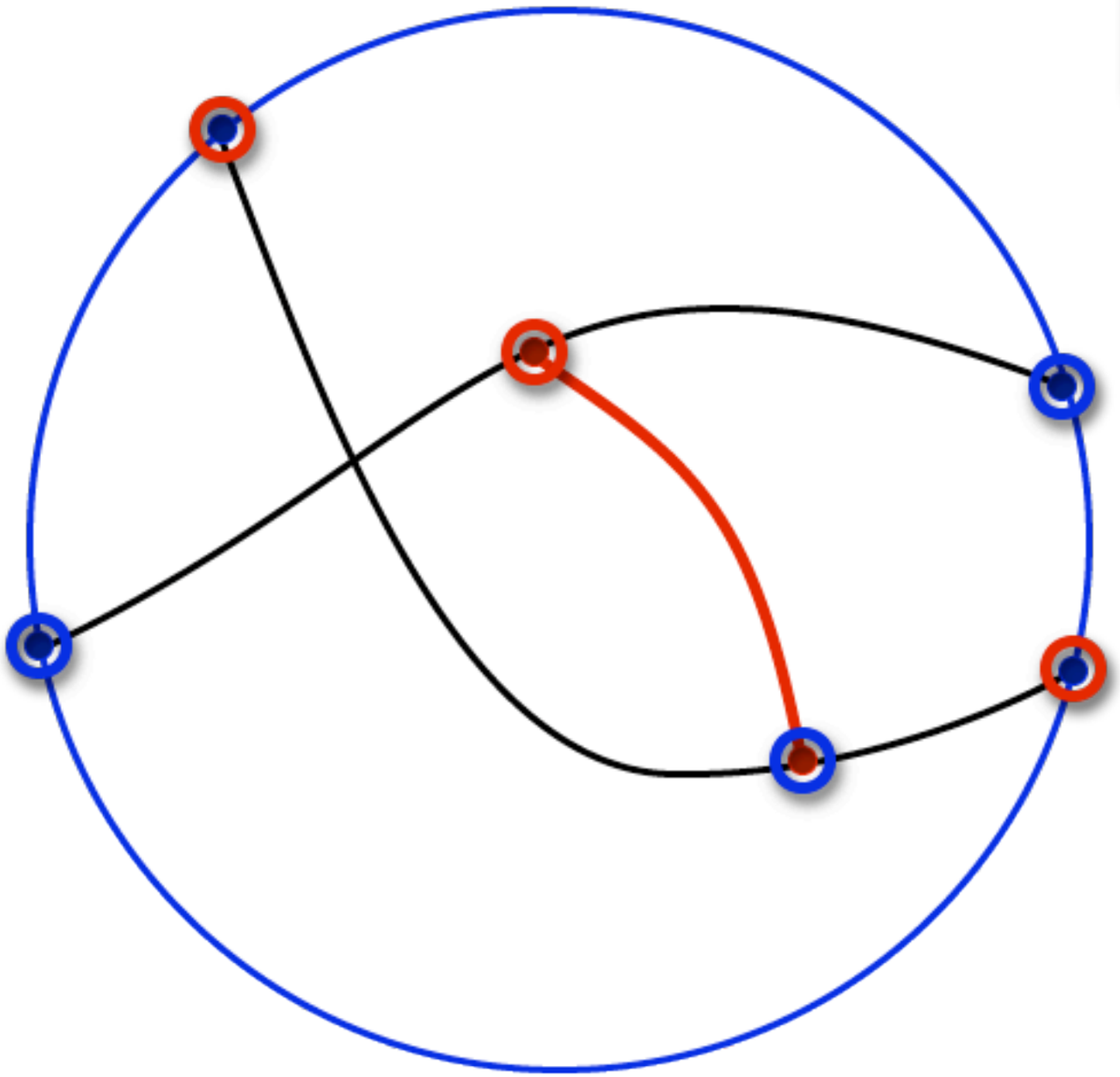
Double Crossing a Face

Lemma: No two disjoint paths have interleaved endpoints on a face.



Double Crossing a Face

Lemma: No two disjoint paths have interleaved endpoints on a face.

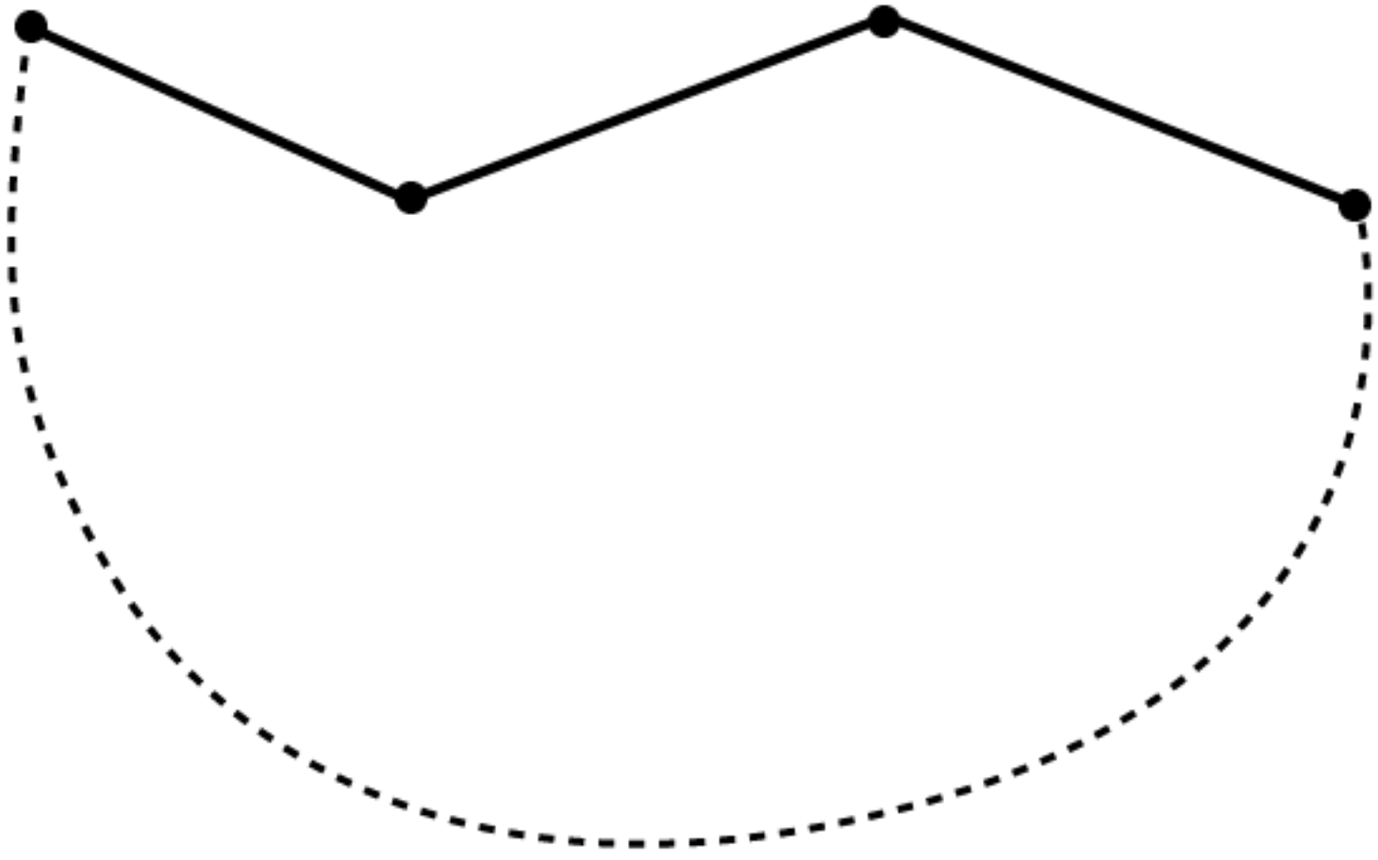


Tutte's Algorithm

No ZigZags

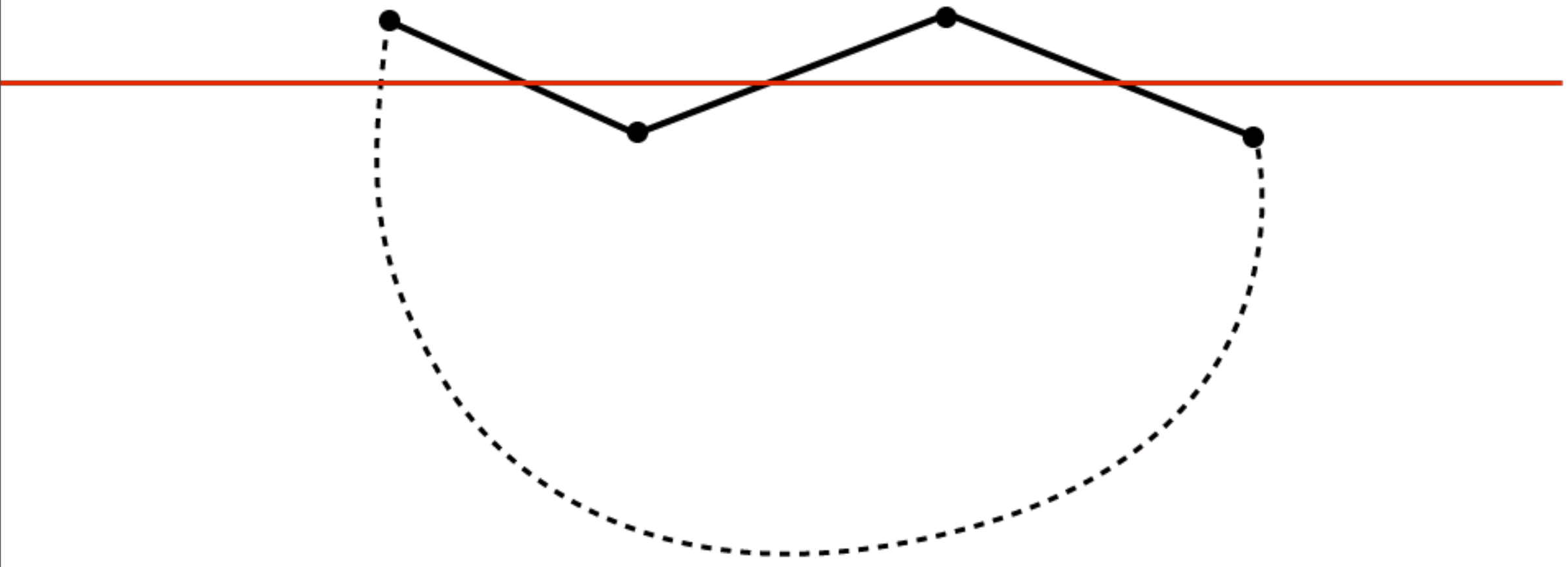
Tutte's Algorithm

No ZigZags



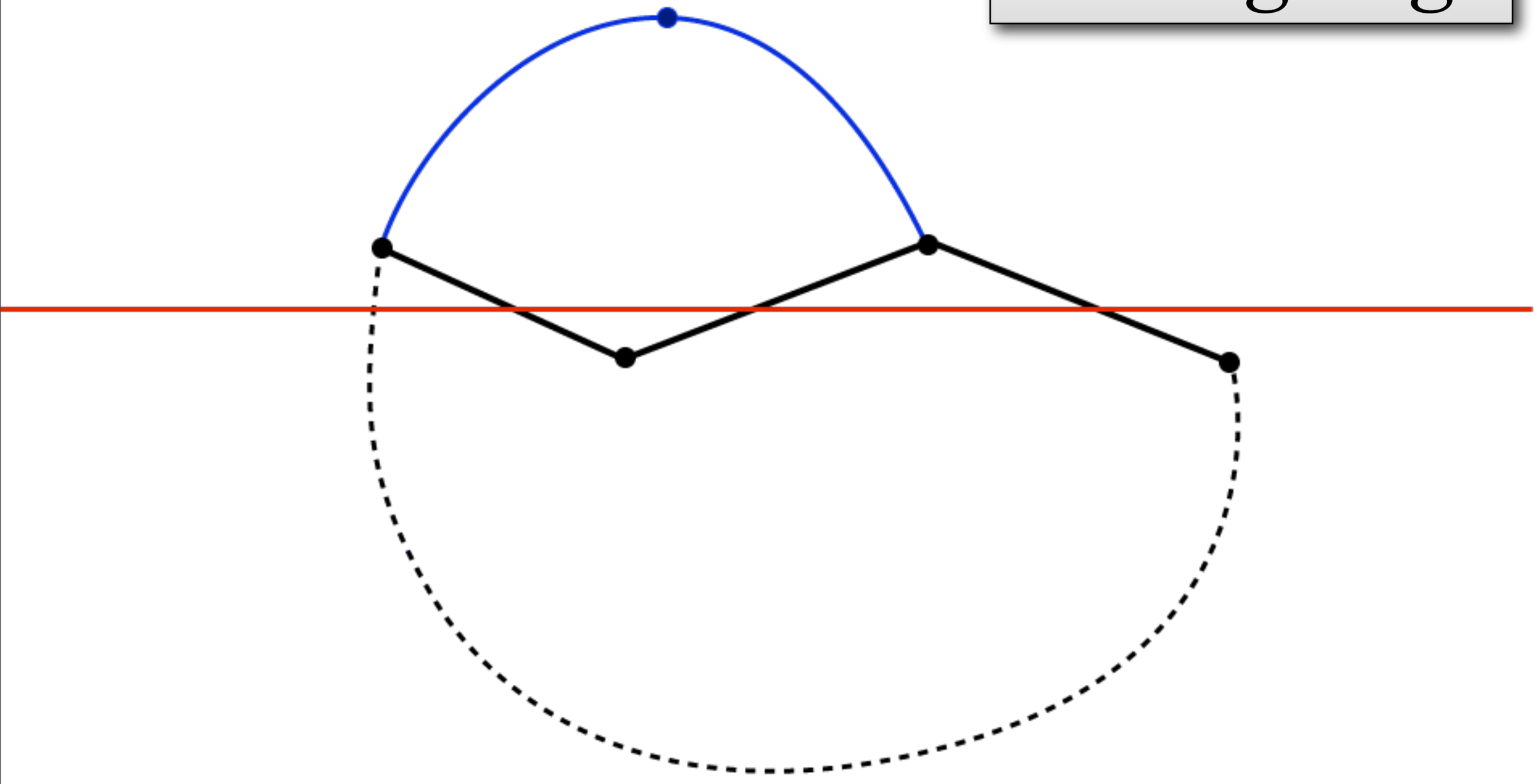
Tutte's Algorithm

No ZigZags



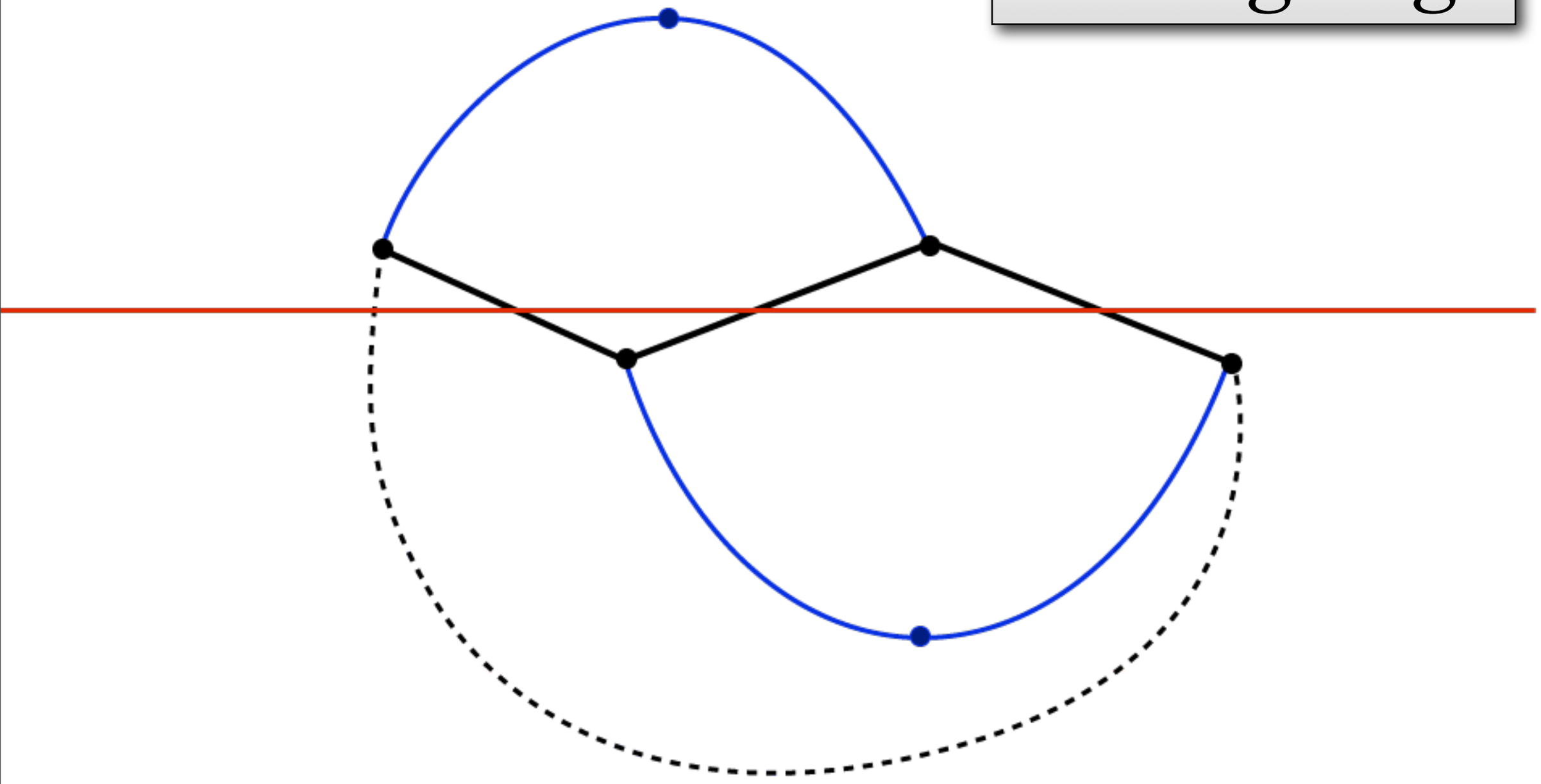
Tutte's Algorithm

No ZigZags



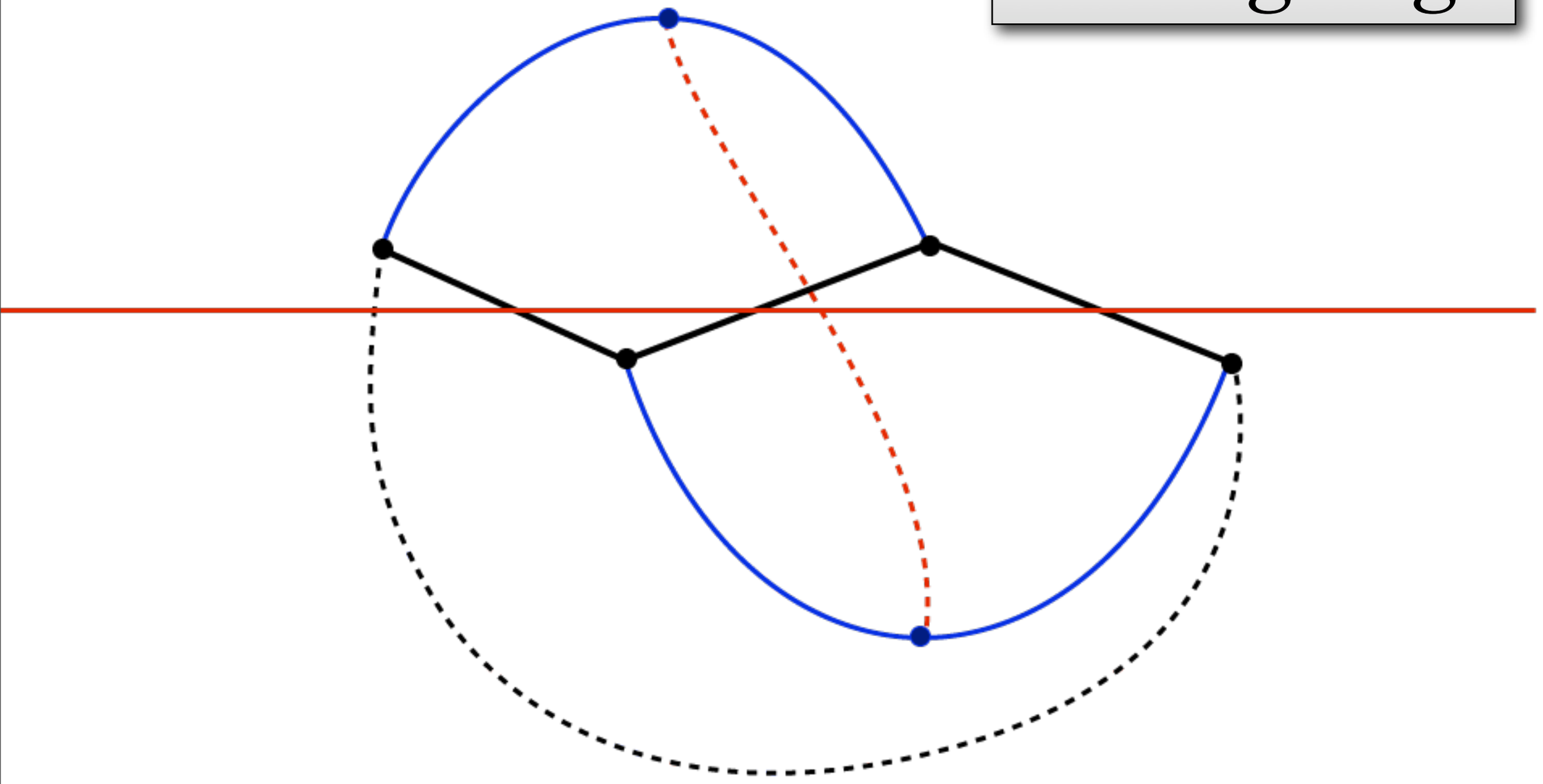
Tutte's Algorithm

No ZigZags



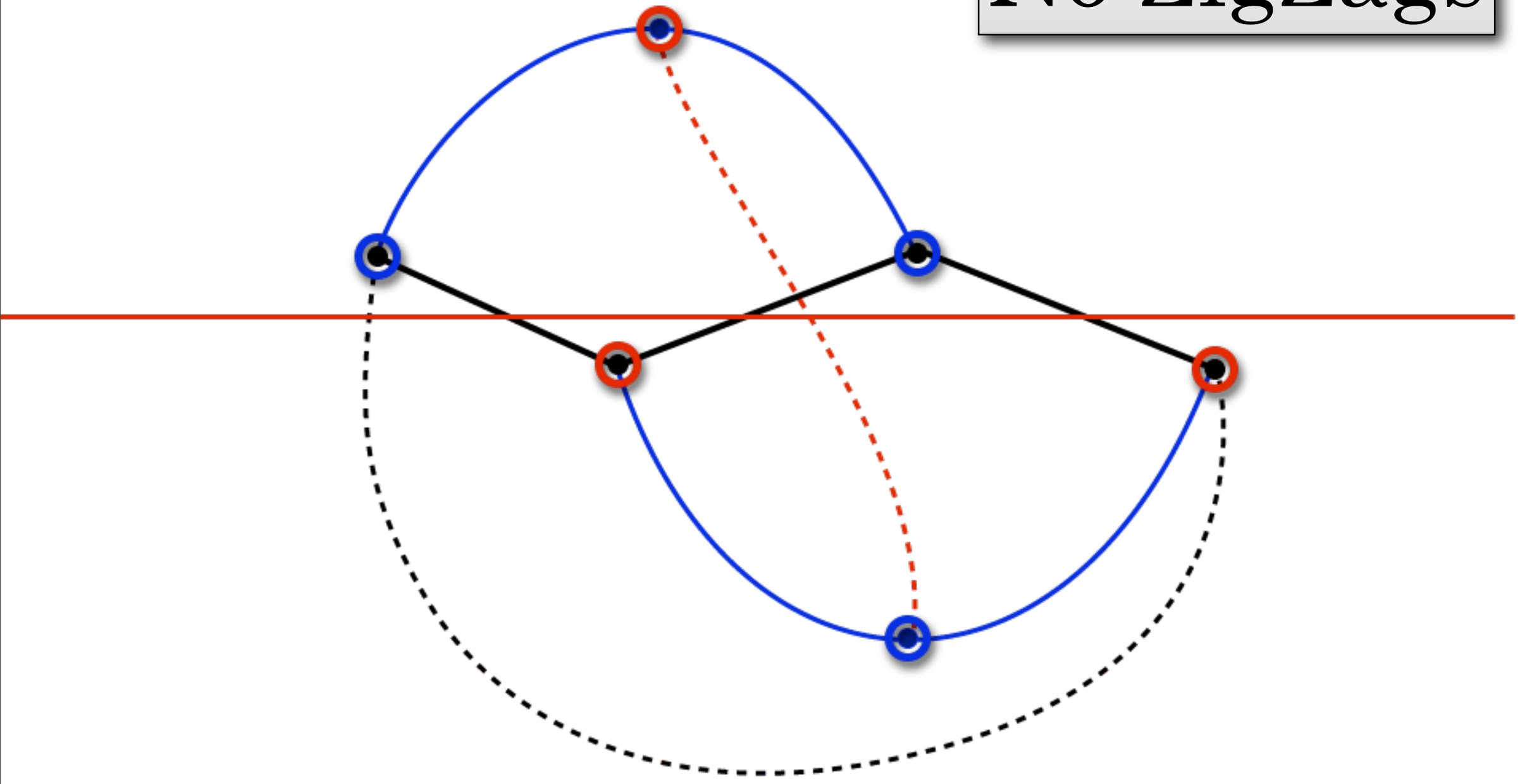
Tutte's Algorithm

No ZigZags



Tutte's Algorithm

No ZigZags

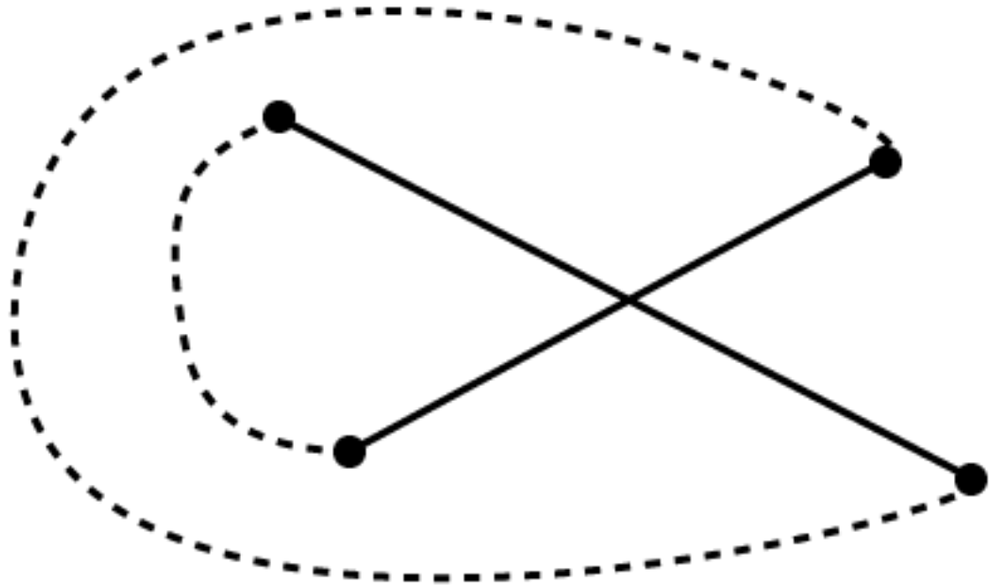


Tutte's Algorithm

No Crossings

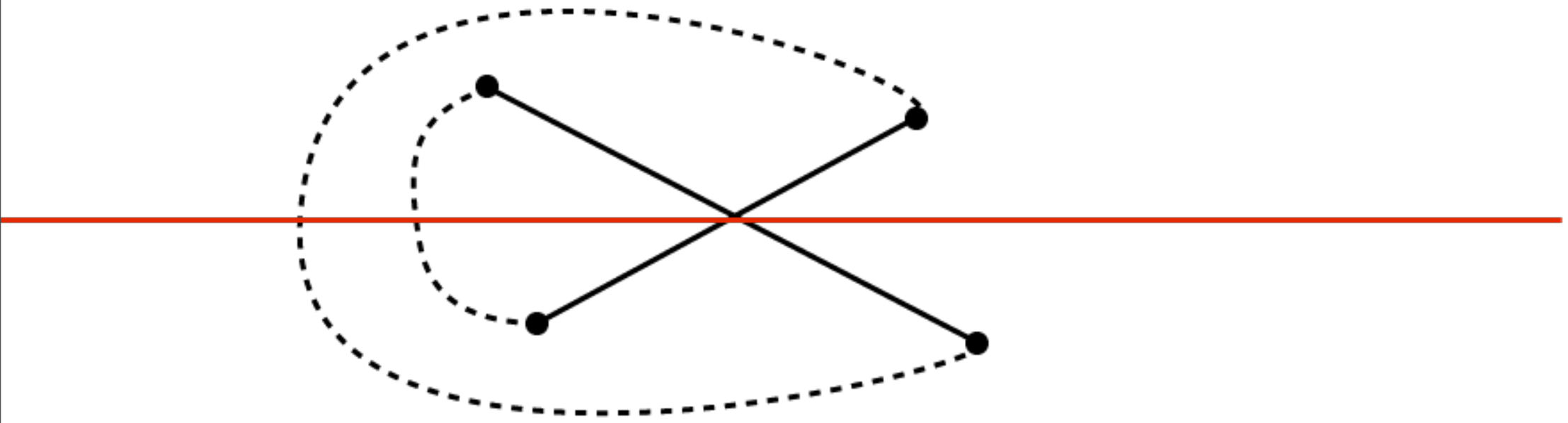
Tutte's Algorithm

No Crossings



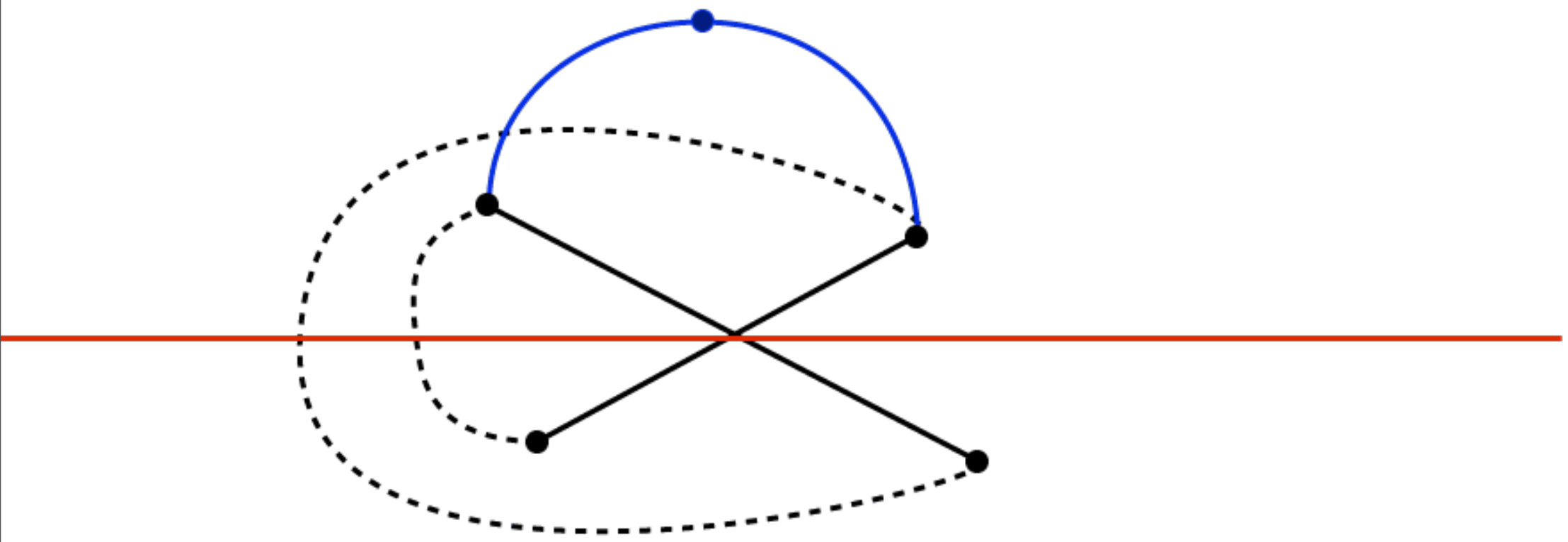
Tutte's Algorithm

No Crossings



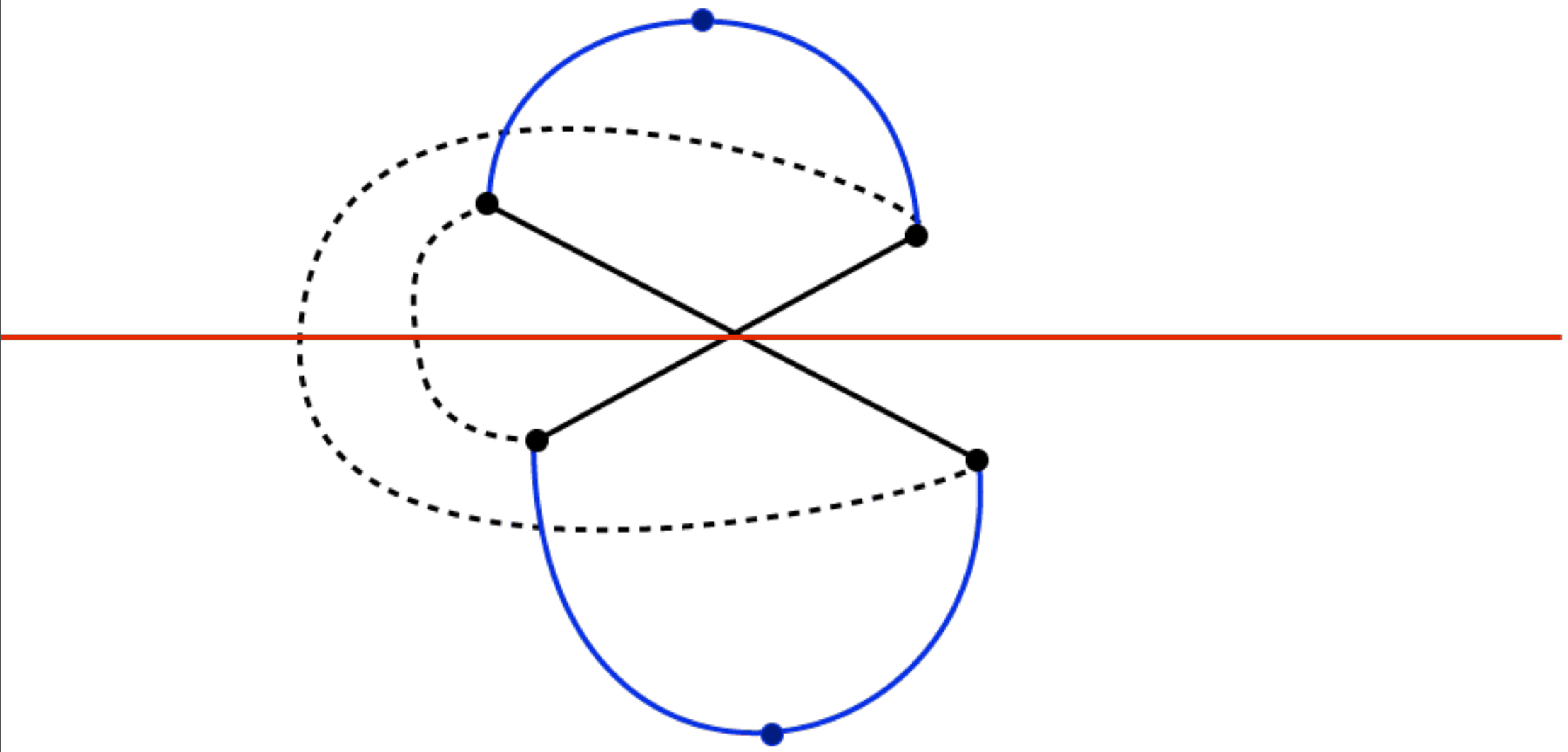
Tutte's Algorithm

No Crossings



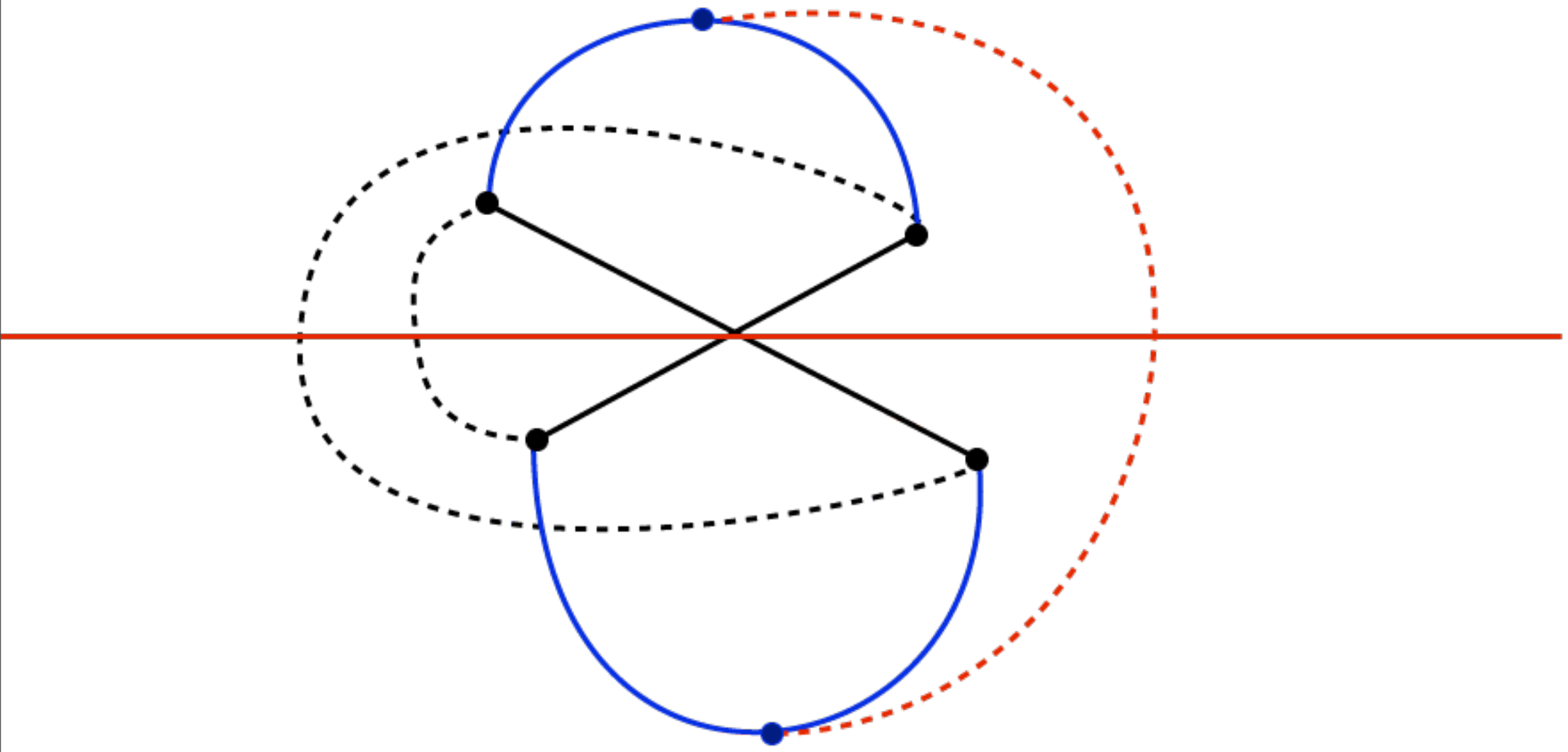
Tutte's Algorithm

No Crossings



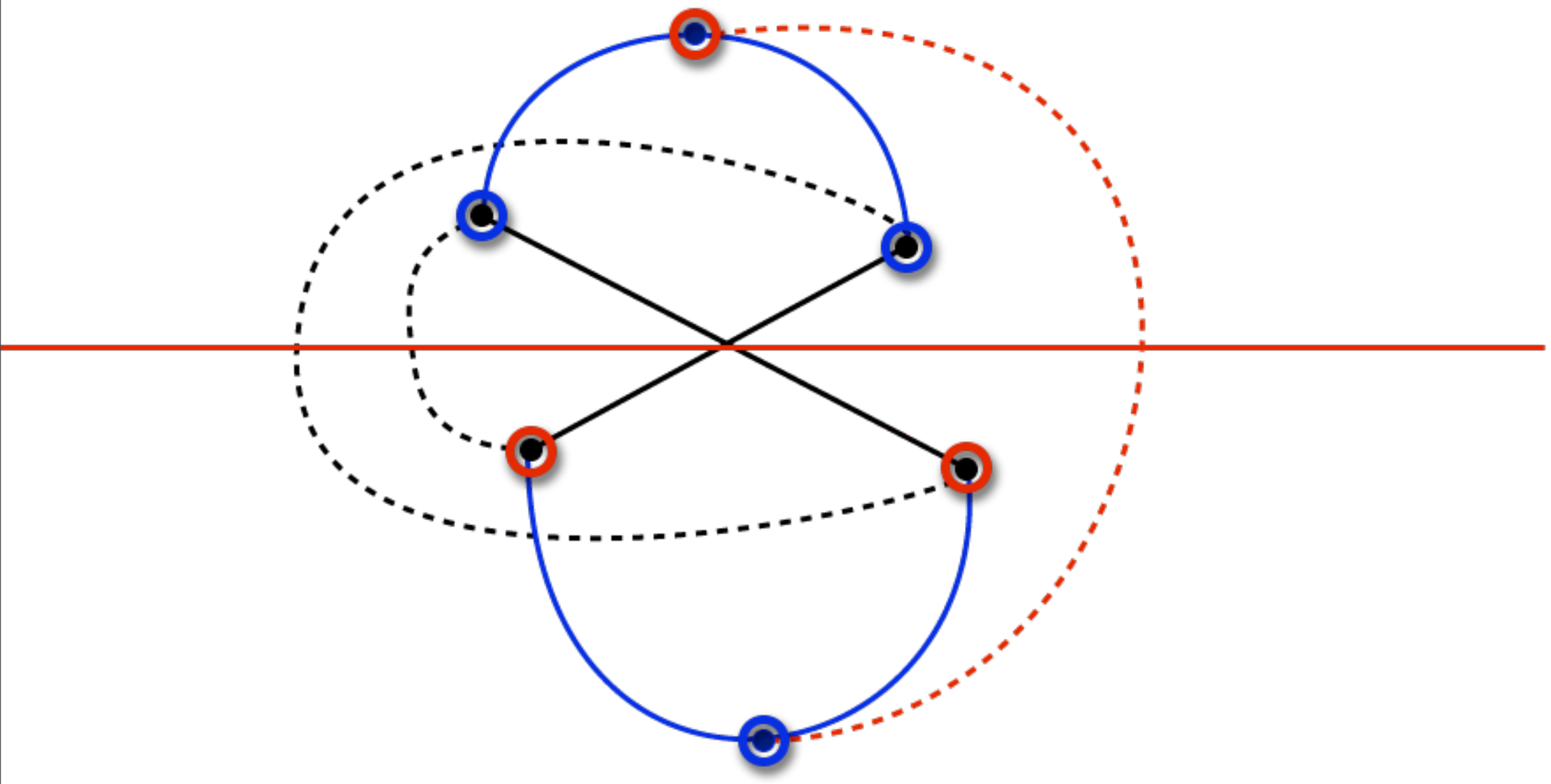
Tutte's Algorithm

No Crossings



Tutte's Algorithm

No Crossings

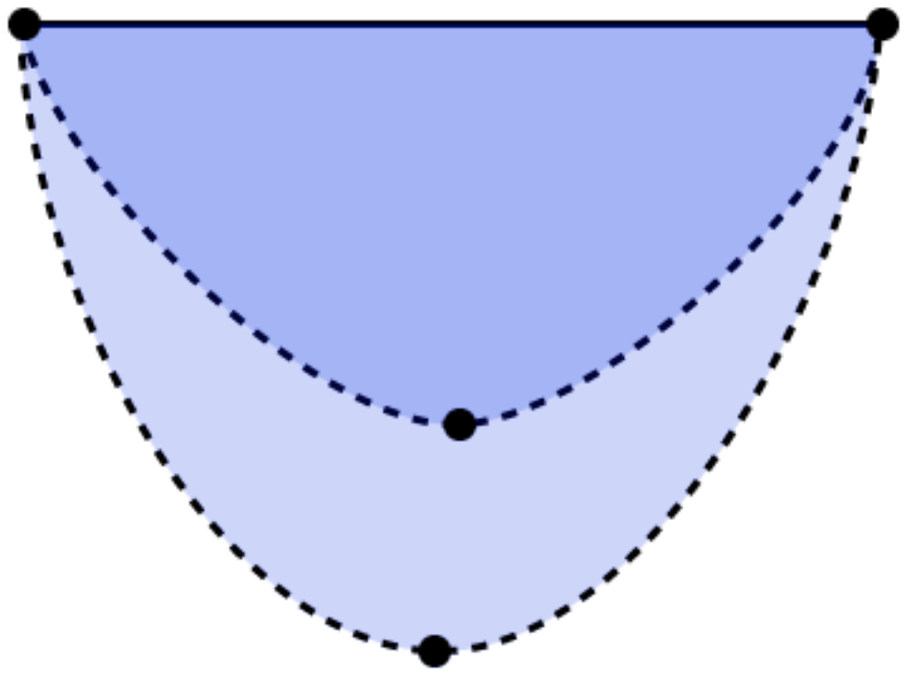


Tutte's Algorithm

No Overlaps

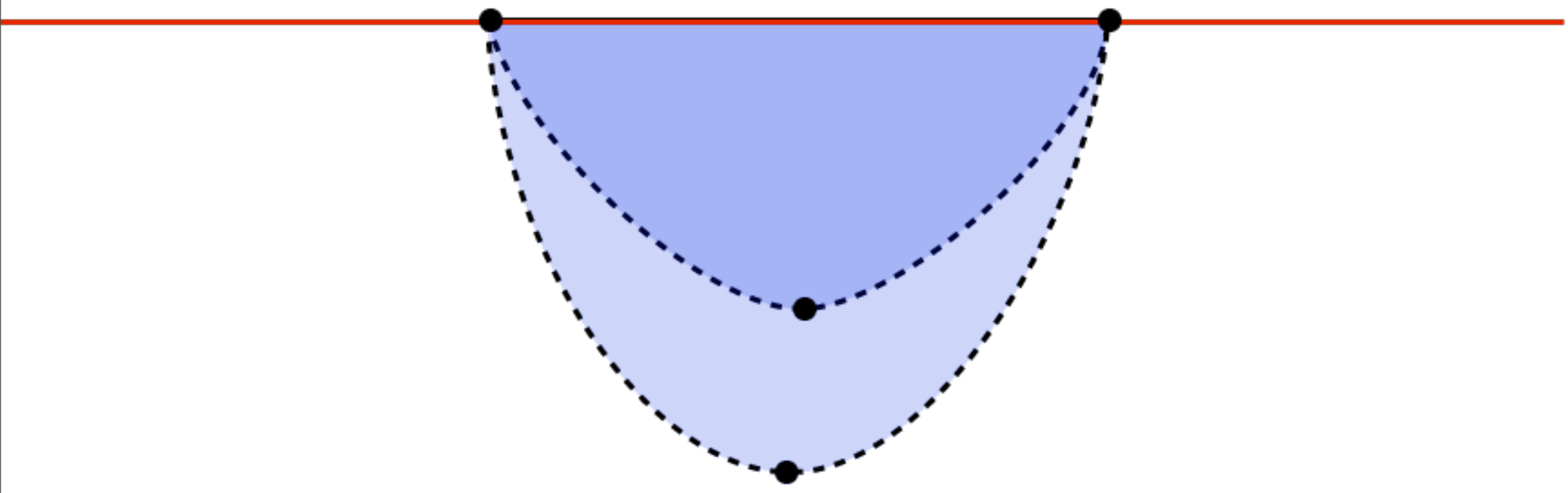
Tutte's Algorithm

No Overlaps



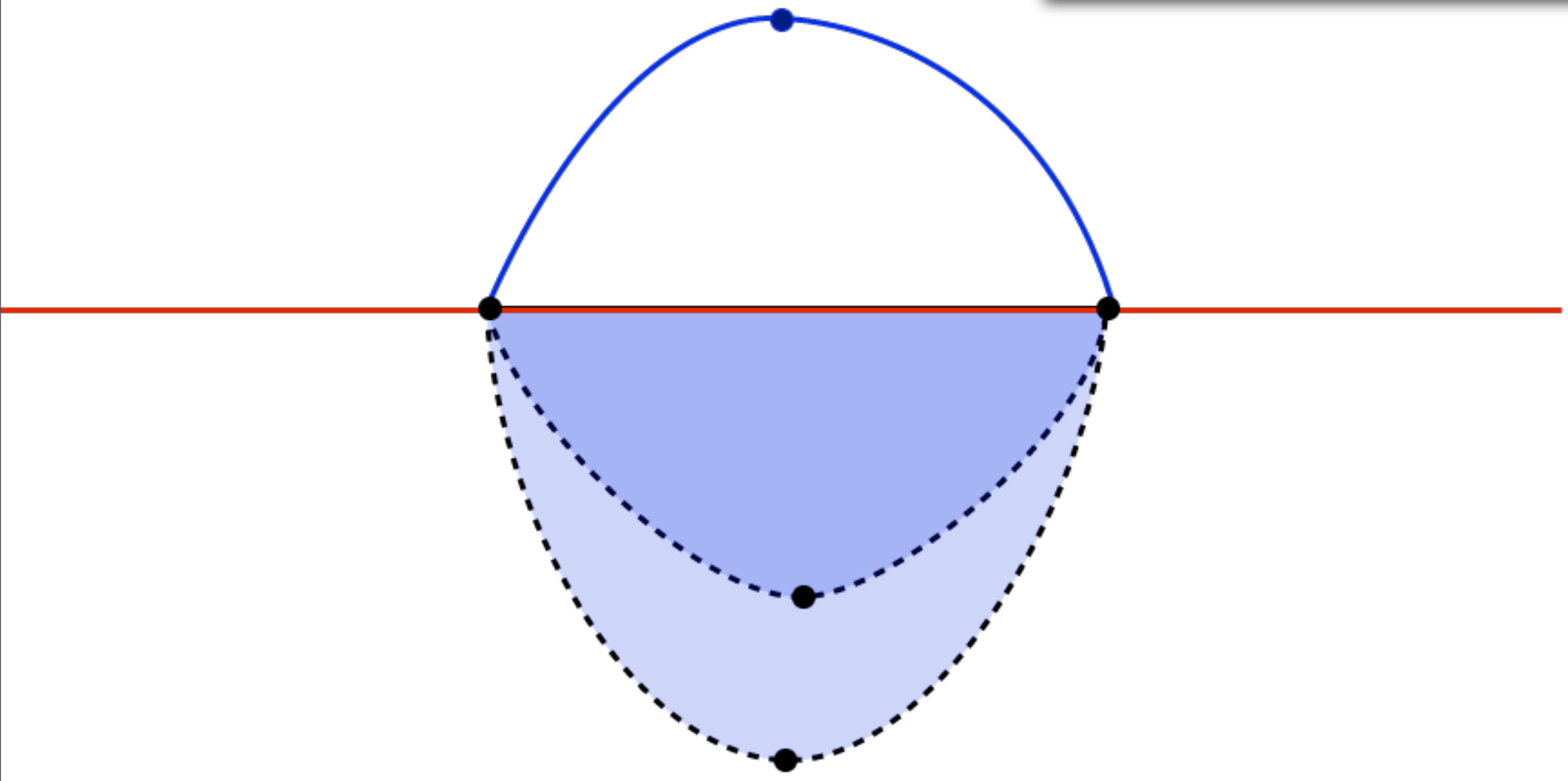
Tutte's Algorithm

No Overlaps



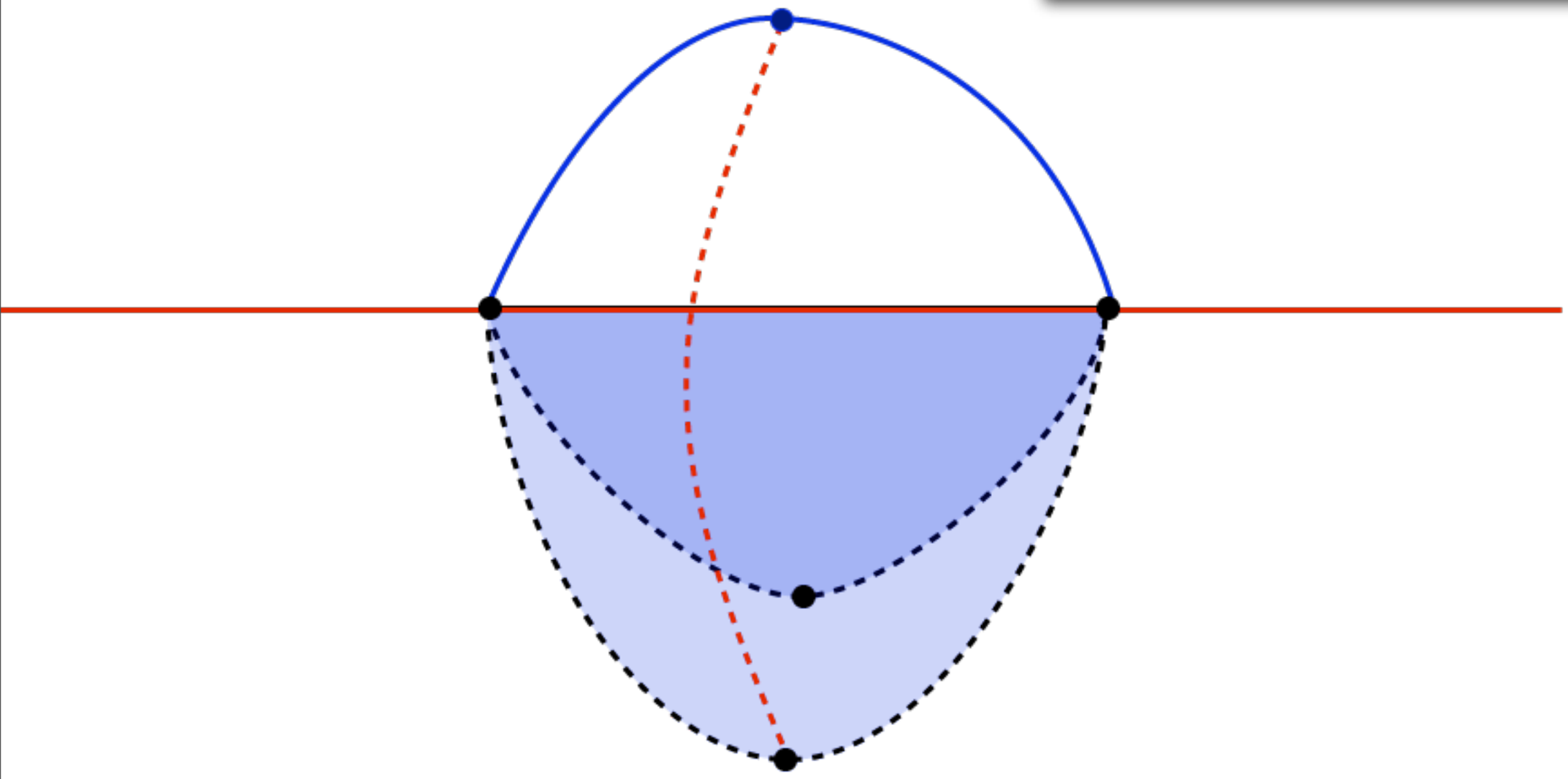
Tutte's Algorithm

No Overlaps



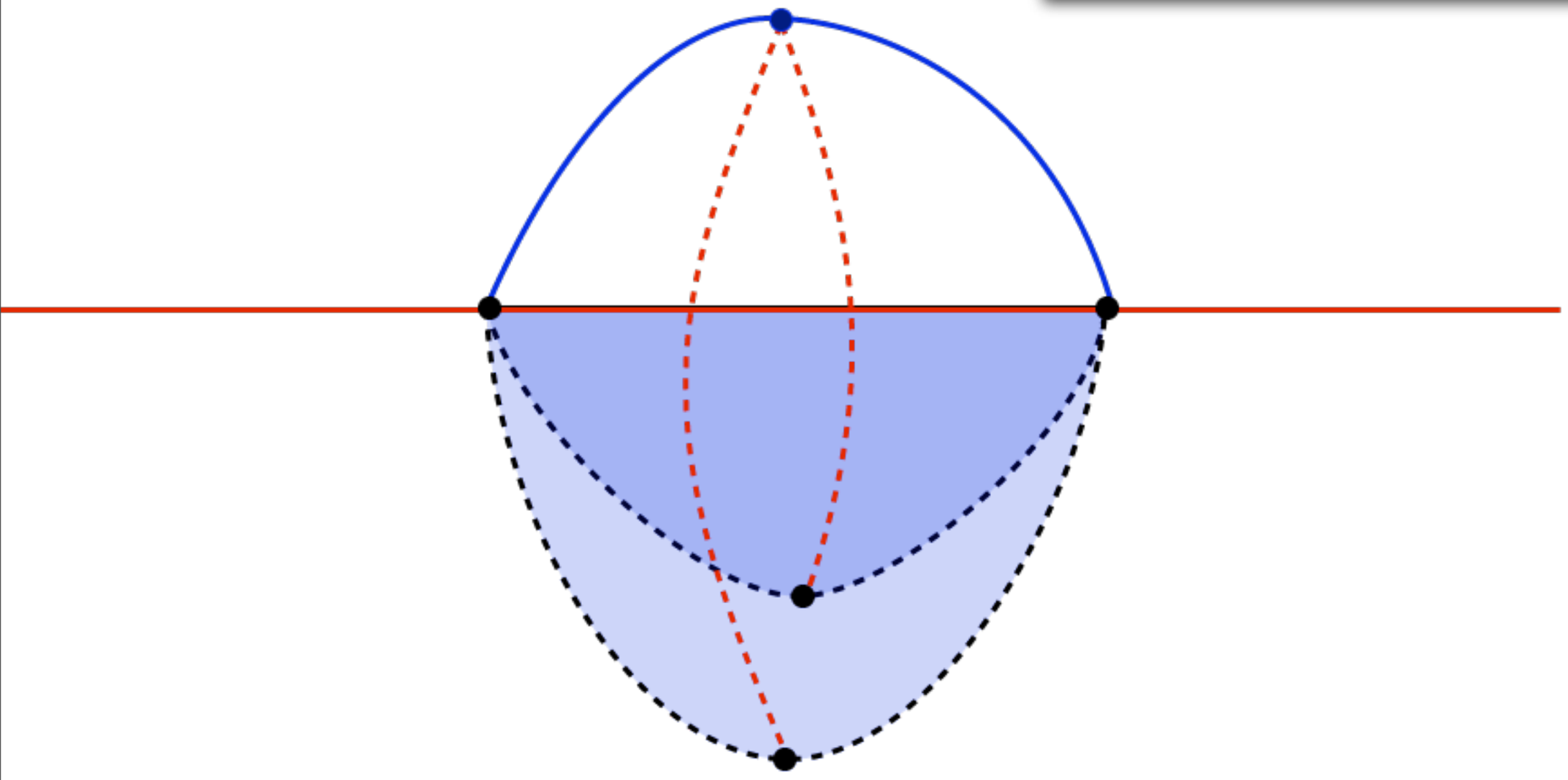
Tutte's Algorithm

No Overlaps



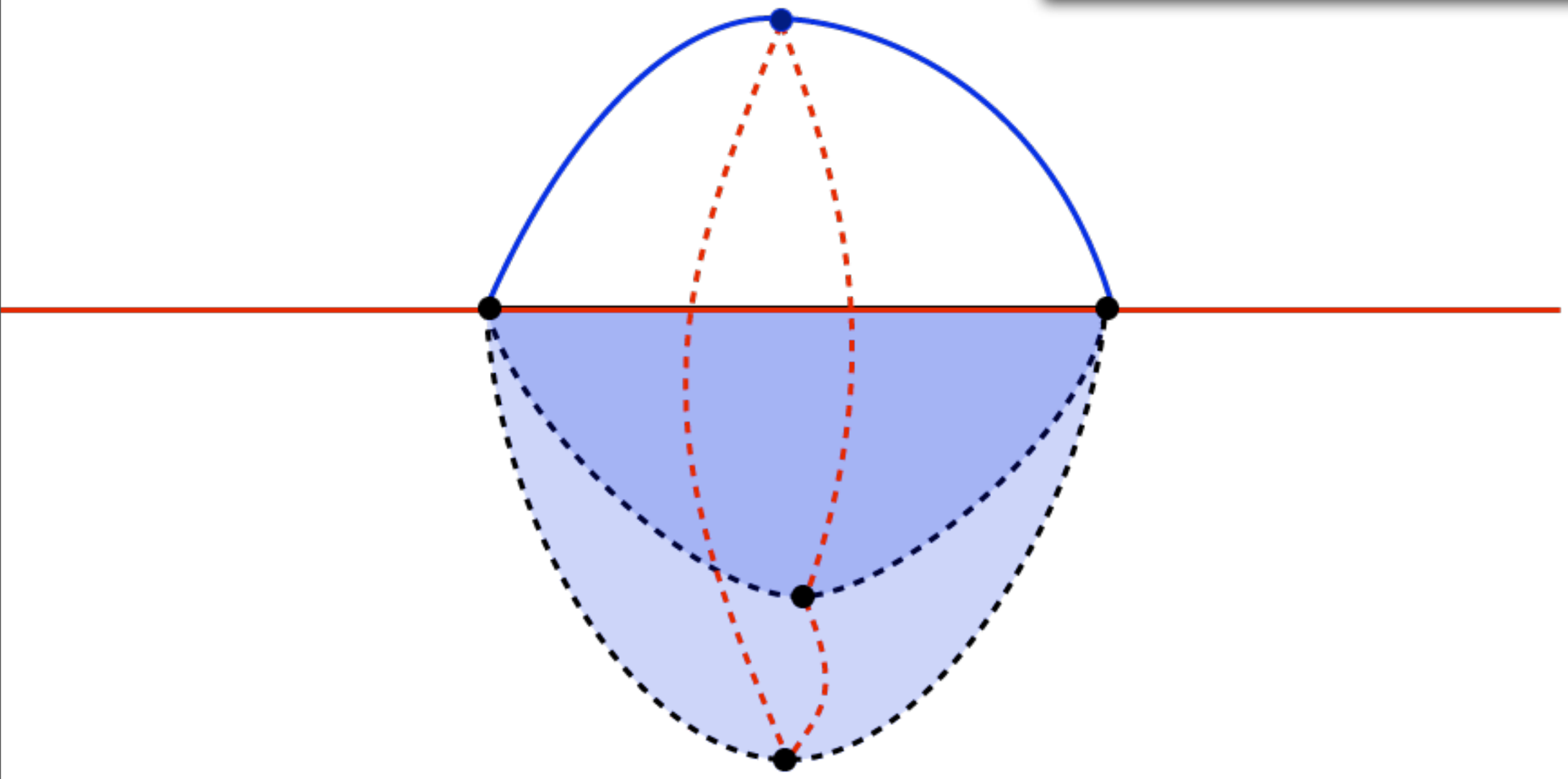
Tutte's Algorithm

No Overlaps



Tutte's Algorithm

No Overlaps



Tutte and Maxwell-Cremona

Tutte and Maxwell-Cremona

- ➡ Weirdness on the outer face.

Tutte and Maxwell-Cremona

- ➡ Weirdness on the outer face.
- ➡ Lifting still works, except outer face.

Tutte and Maxwell-Cremona

- ➡ Weirdness on the outer face.
- ➡ Lifting still works, except outer face.
- ➡ Lifting is convex.

Steinitz's Theorem

Steinitz's Theorem

A graph G is the 1-skeleton of a 3-polytope if and only if it is simple, planar, and 3-connected.

Steinitz's Theorem

Claim: If the graph has a triangle, then the Tutte embedding followed by the Maxwell-Cremona lifting gives the desired polytope.

Steinitz's Theorem

Claim: If the graph has a triangle, then the Tutte embedding followed by the Maxwell-Cremona lifting gives the desired polytope.

Fix the triangle as the outer face.

Steinitz's Theorem

Claim: If the graph has a triangle, then the Tutte embedding followed by the Maxwell-Cremona lifting gives the desired polytope.

Fix the triangle as the outer face.

After the lifting, the triangle must lie on a plane.

Steinitz's Theorem

Question: What if there is no triangle?

Steinitz's Theorem

Question: What if there is no triangle?

Answer: Dualize (the dual has a triangle)

Steinitz's Theorem

Lemma: Every 3-connected, planar graph has a triangle or a vertex of degree 3.

Steinitz's Theorem

$$|V| - |E| + |F| = 2$$

Lemma: Every 3-connected, planar graph has a triangle or a vertex of degree 3.

Steinitz's Theorem

$$|V| - |E| + |F| = 2$$

$$|E| = \frac{1}{2} \sum_{v \in V} \delta(v)$$

Lemma: Every 3-connected, planar graph has a triangle or a vertex of degree 3.

Steinitz's Theorem

$$|V| - |E| + |F| = 2$$

$$|E| = \frac{1}{2} \sum_{v \in V} \delta(v)$$

$$|E| = \frac{1}{2} \sum_{f \in F} |f|$$

Lemma: Every 3-connected, planar graph has a triangle or a vertex of degree 3.

Steinitz's Theorem

Lemma: Every 3-connected, planar graph has a triangle or a vertex of degree 3.

$$|V| - |E| + |F| = 2$$

$$|E| = \frac{1}{2} \sum_{v \in V} \delta(v)$$

$$|E| = \frac{1}{2} \sum_{f \in F} |f|$$

$$\forall v \quad \delta(v) \geq 4 \Rightarrow |E| \geq 2|V|$$ (No degree 3)

Steinitz's Theorem

$$|V| - |E| + |F| = 2$$

$$|E| = \frac{1}{2} \sum_{v \in V} \delta(v)$$

$$|E| = \frac{1}{2} \sum_{f \in F} |f|$$

Lemma: Every 3-connected, planar graph has a triangle or a vertex of degree 3.

$$\forall v \quad \delta(v) \geq 4 \Rightarrow |E| \geq 2|V| \qquad \text{(No degree 3)}$$

$$\forall f \quad |f| \geq 4 \Rightarrow |E| \geq 2|F| \qquad \text{(No triangles)}$$

Steinitz's Theorem

$$|V| - |E| + |F| = 2$$

$$|E| = \frac{1}{2}\sum_{v \in V} \delta(v)$$

$$|E| = \frac{1}{2}\sum_{f \in F} |f|$$

Lemma: Every 3-connected, planar graph has a triangle or a vertex of degree 3.

$$\forall v \quad \delta(v) \geq 4 \Rightarrow |E| \geq 2|V| \qquad \text{(No degree 3)}$$

$$\forall f \quad |f| \geq 4 \Rightarrow |E| \geq 2|F| \qquad \text{(No triangles)}$$

$$\frac{|E|}{2} - |E| + \frac{|E|}{2} \geq 2$$

Steinitz's Theorem

$$|V| - |E| + |F| = 2$$

$$|E| = \frac{1}{2}\sum_{v \in V} \delta(v)$$

$$|E| = \frac{1}{2}\sum_{f \in F} |f|$$

Lemma: Every 3-connected, planar graph has a triangle or a vertex of degree 3.

$$\forall v \quad \delta(v) \geq 4 \Rightarrow |E| \geq 2|V| \qquad \text{(No degree 3)}$$

$$\forall f \quad |f| \geq 4 \Rightarrow |E| \geq 2|F| \qquad \text{(No triangles)}$$

$$\frac{|E|}{2} - |E| + \frac{|E|}{2} \geq 2$$

$$0 \geq 2$$

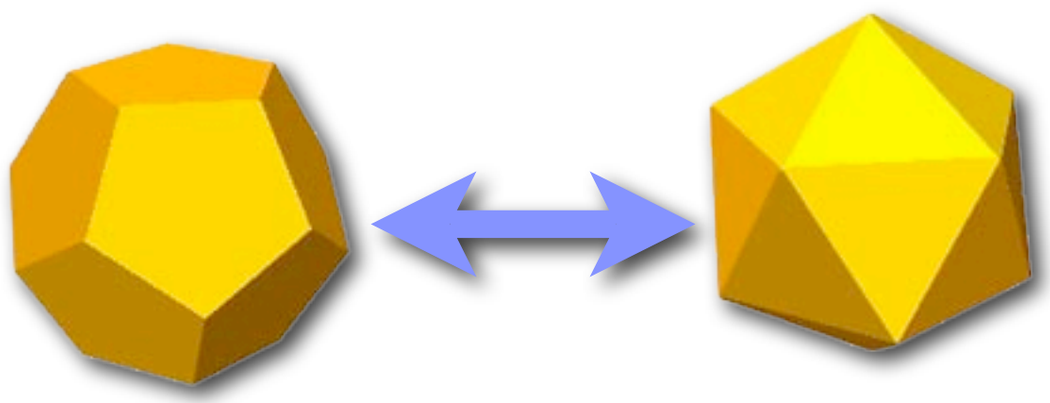
Steinitz's Theorem

So, with the Tutte embedding and the Maxwell-Cremona Correspondence, we can construct a polytope with 1-skeleton isomorphic to *either* the graph *or* its dual.

Steinitz's Theorem

So, with the Tutte embedding and the Maxwell-Cremona Correspondence, we can construct a polytope with 1-skeleton isomorphic to *either* the graph *or* its dual.

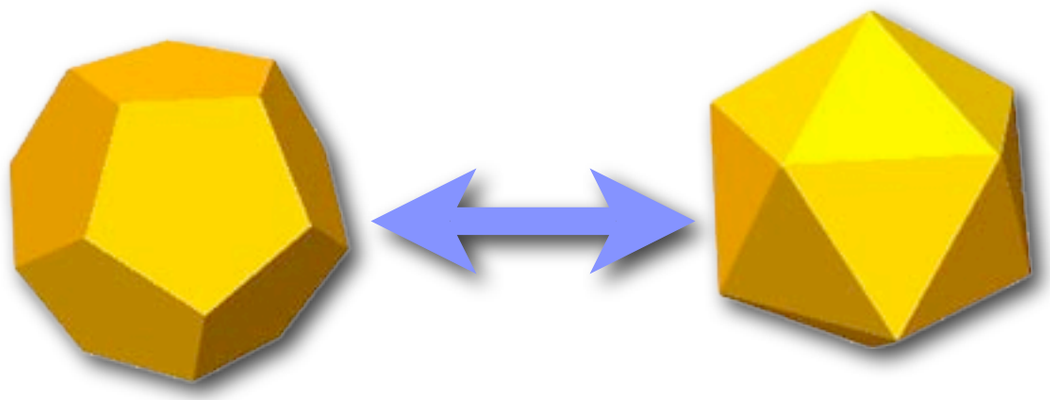
If we have the dual, *polarize*.



Steinitz's Theorem

So, with the Tutte embedding and the Maxwell-Cremona Correspondence, we can construct a polytope with 1-skeleton isomorphic to *either* the graph *or* its dual.

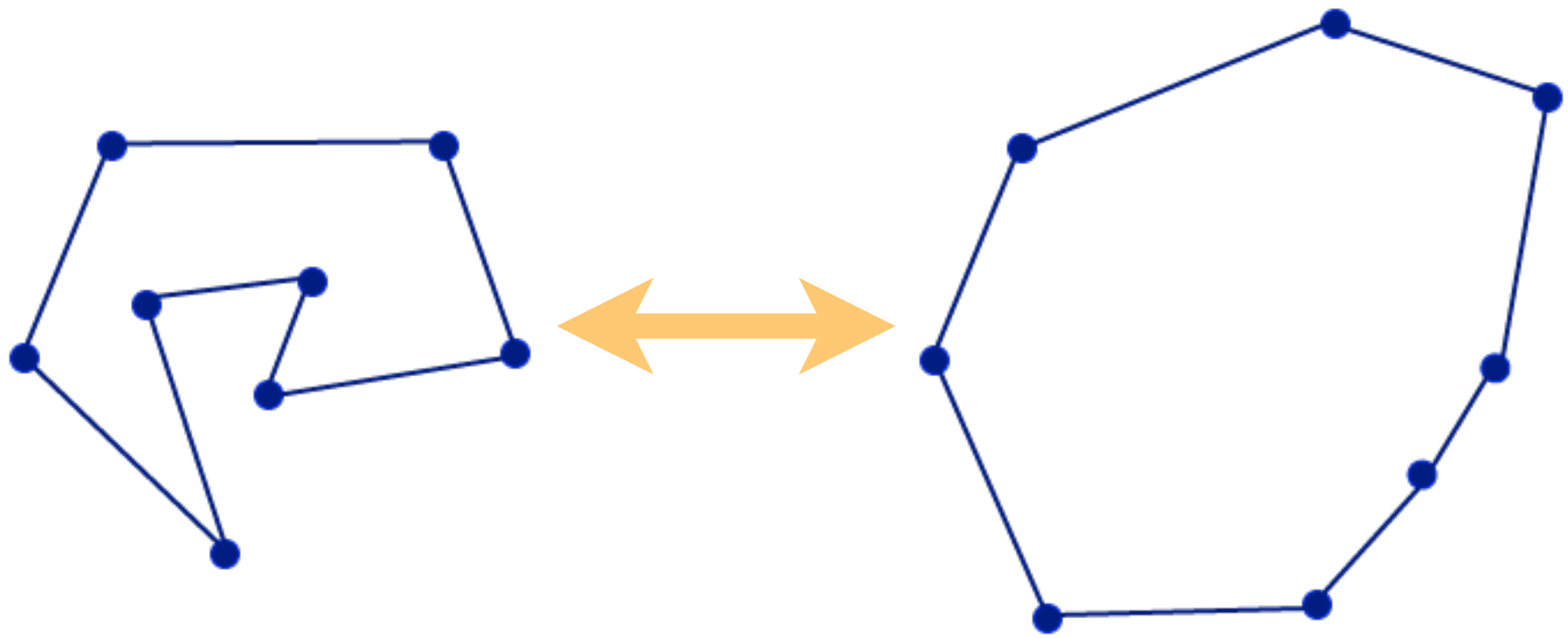
If we have the dual, *polarize*.



[Eades, Garvan 1995]

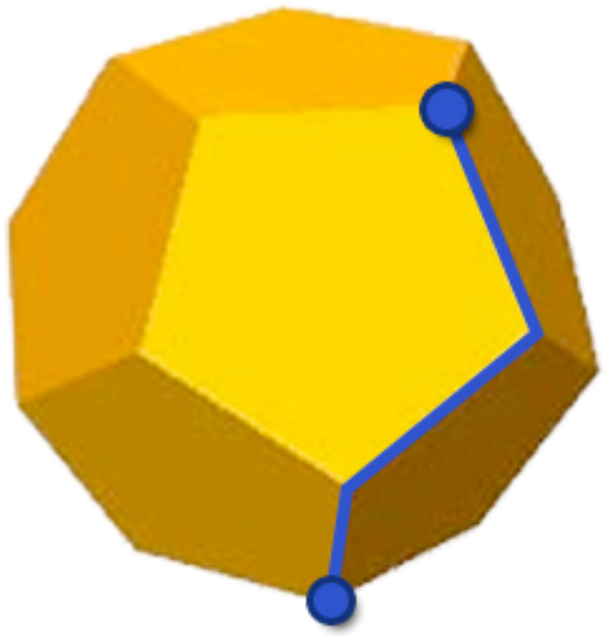
A Tour of Other Stuff

Rigidity and Unfolding



[Connelly, Demaine, Rote, 2000]

Greedy Routing

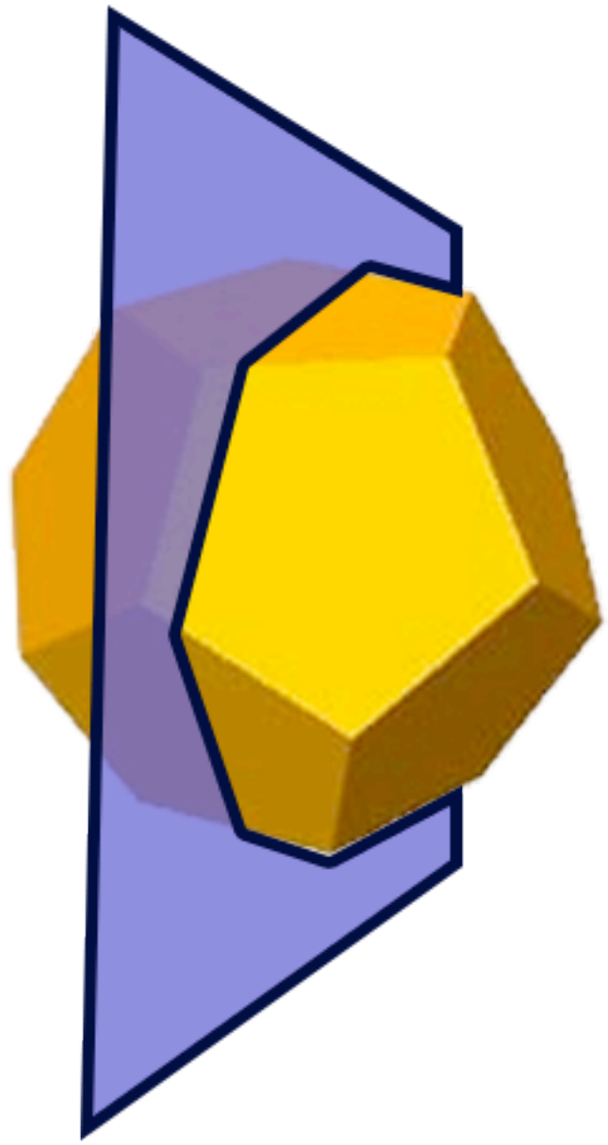


[Morin, 2001]

[Papadimitriou, Ratajczak, 2004]

Robust Geometric Computing

[Hopcroft and Kahn 1992]



Spectral Embedding

Correspondence between Colin de Verdiere matrices and Steinitz representations [Lovasz, 2000]

Spectral Embedding

Correspondence between Colin de Verdiere matrices and Steinitz representations

[Lovasz, 2000]

The construction will start with the polar polytope. Let $G^* = (V^*, E^*)$ denote the dual graph of G.

Lemma 4 *We can assign a vector w_f to each $f \in V^*$ so that whenever $ij \in E$ and fg is corresponding edge of G^*, then*

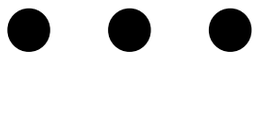
$$w_f - w_g = M_{ij}(u_i \times u_j). \tag{2}$$

Proof. Let $v_{fg} = M_{ij}(u_i \times u_j)$. It suffices to show that the vectors v_{fg} sum to 0 over the edges of any cycle in G^*. Since G^* is a planar graph, it suffices to verify this for the facets of G^*. Expressing this in terms of the edges of G, it suffices to show that

$$\sum_{j \in N(i)} M_{ij}(u_i \times u_j) = 0$$

(where, as usual, $N(i)$ denotes the set of neighbors of i). But this follows from (1) upon multiplying by u_i, taking into account that $u_i \times u_i = 0$ and $M_{ij} = 0$ for $j \notin N(i) \cup \{i\}$. □

It's Maxwell-Cremona



Thank you.

Thank you.

Questions?