Learning with Nets and Meshes

Don Sheehy

Thanks to my collaborators:

Benoit Hudson
Gary Miller
Todd Phillips
Steve Oudot

Point Clouds in low to medium dimensional ambient space.

Rule of thumb: $d!$ or 2^{d^2} is okay but n^d is *not*.

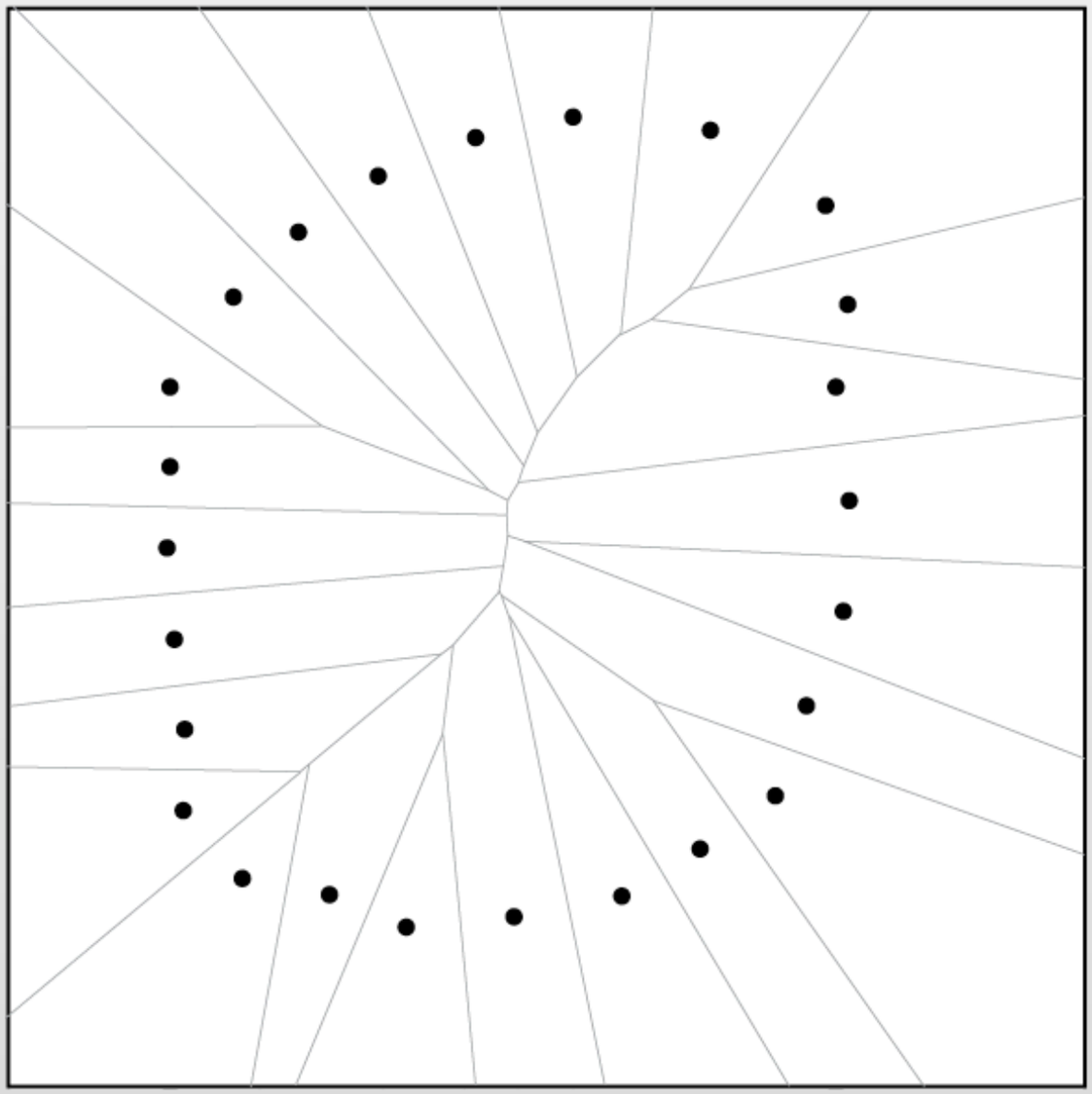
There are many geometric inference problems that could benefit from meshing.

Discretize Space
Approximate Functions
Adapt to density
Describe the space *around* the input.

Meshing

Input: $P \subset \mathbb{R}^d \qquad n = |P|$

Output: $M \supset P$ with a "nice" Voronoi diagram



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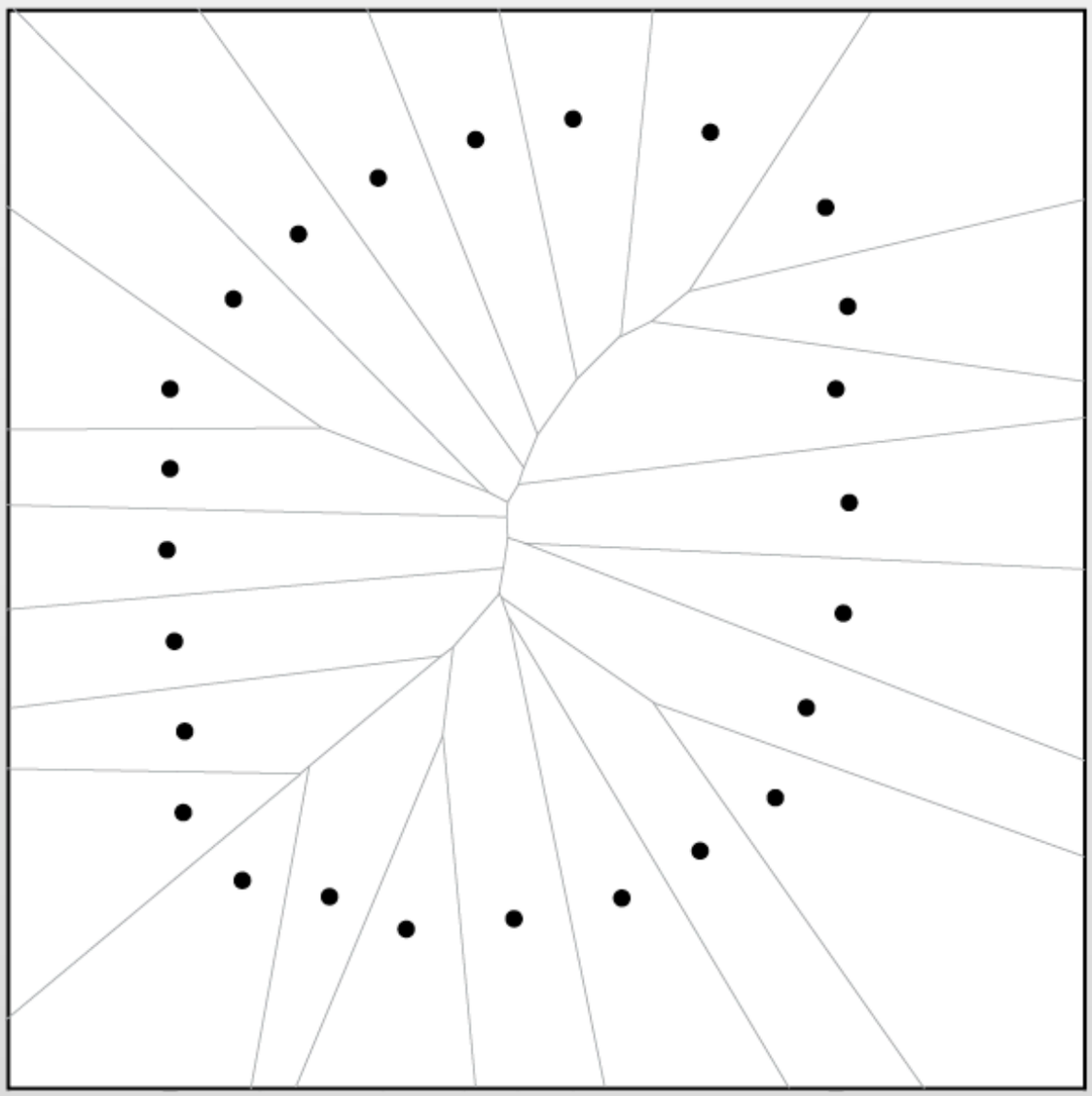
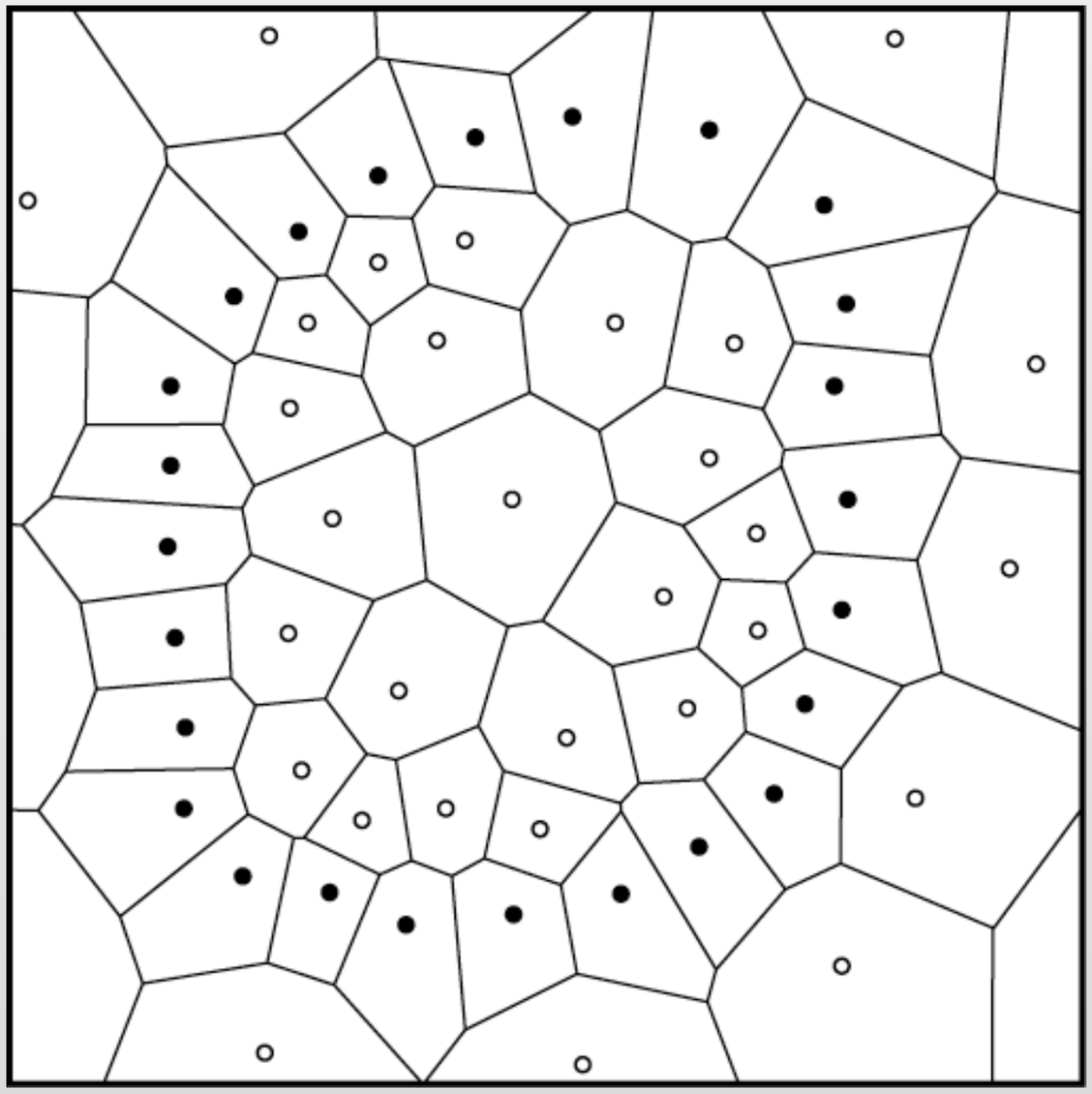
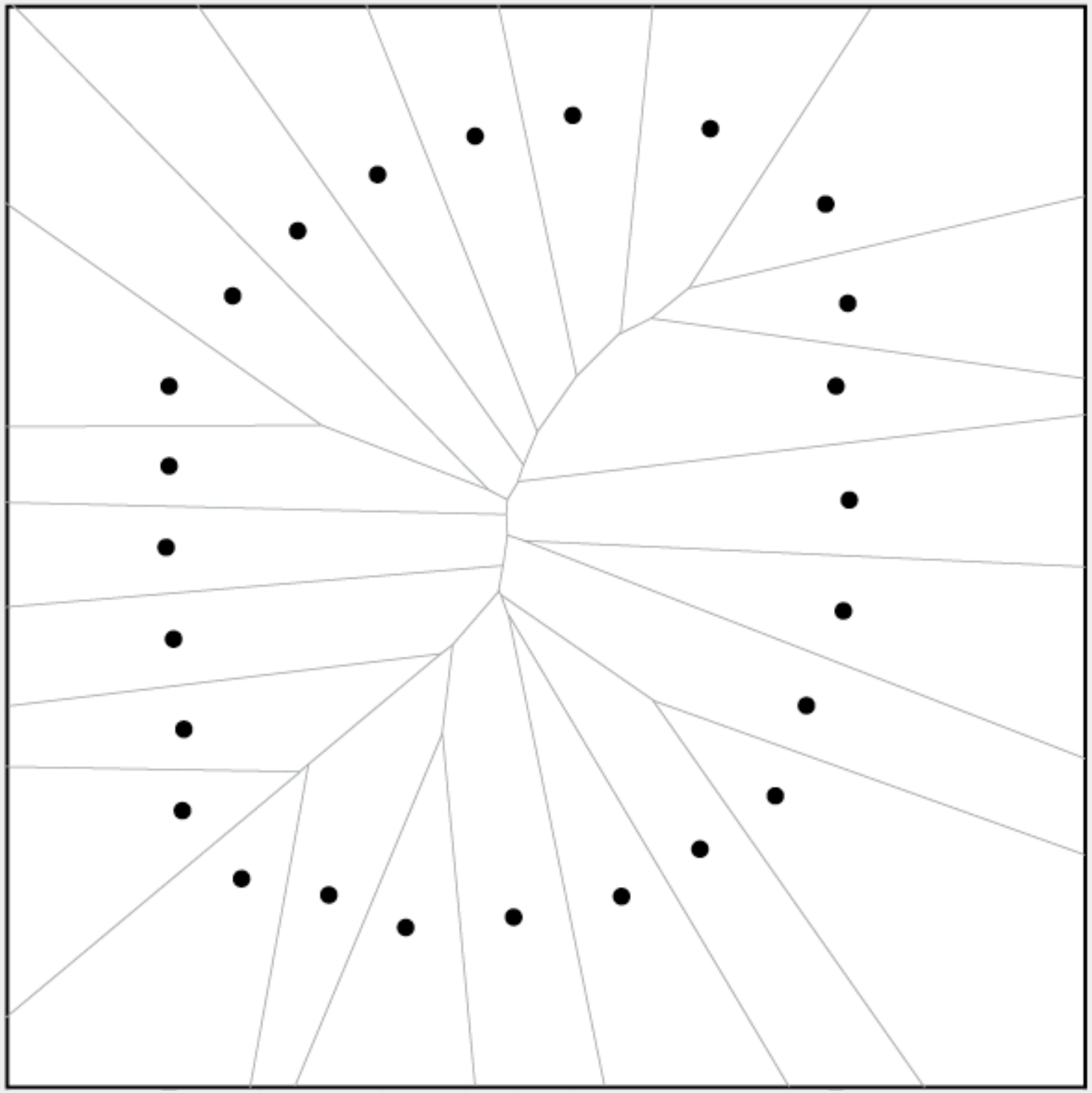




Points, offsets, homology, and persistence.



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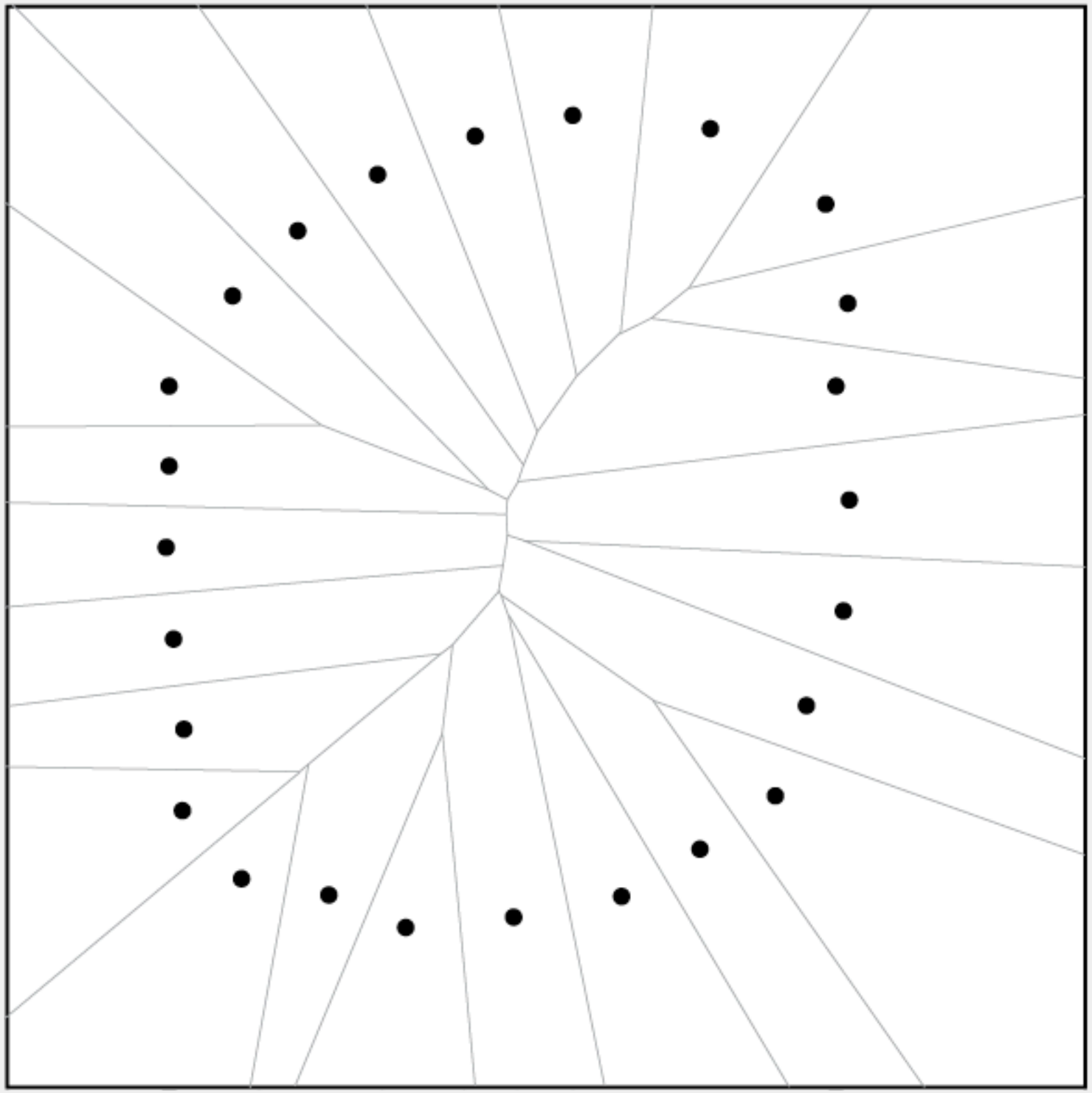

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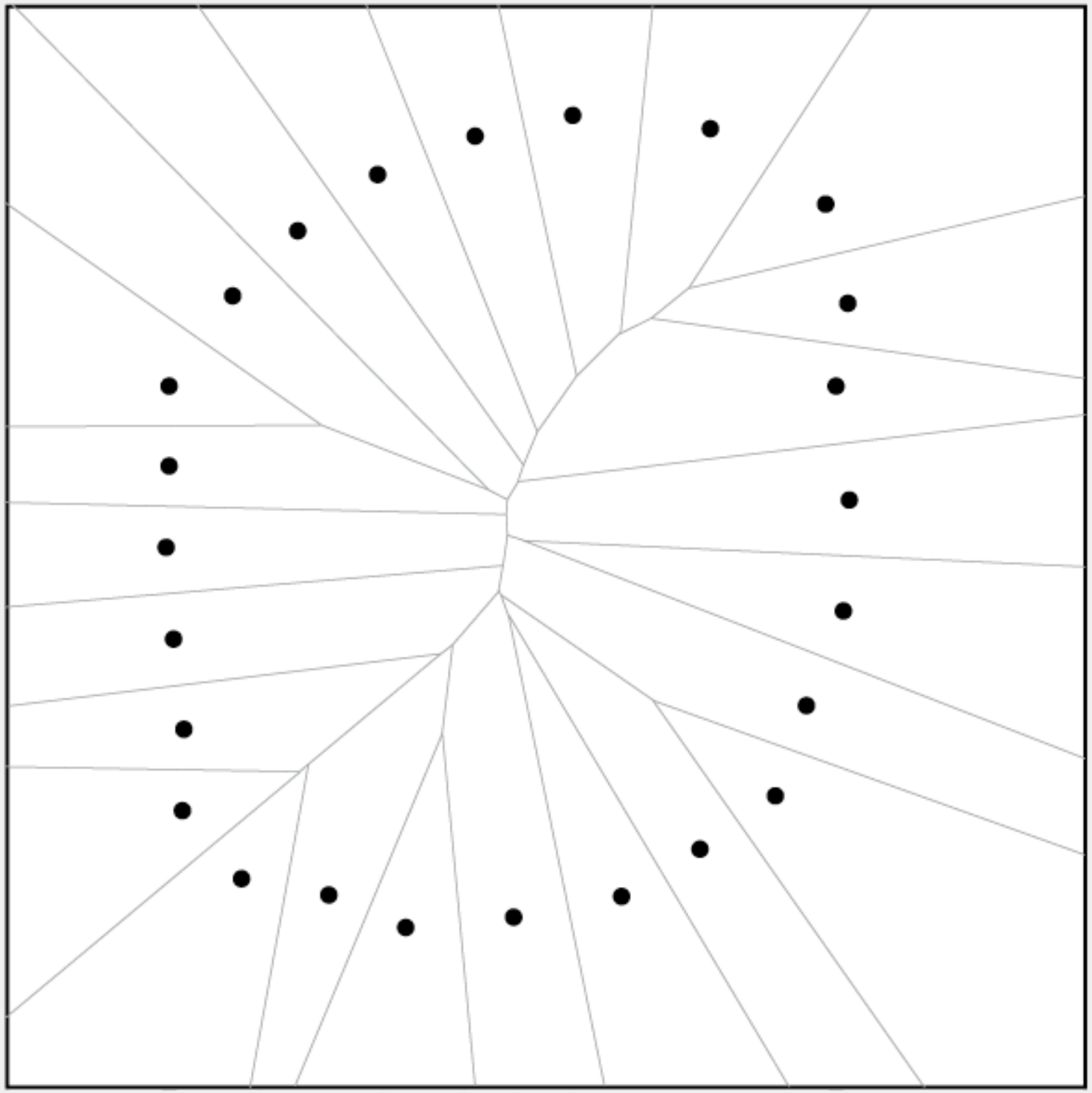

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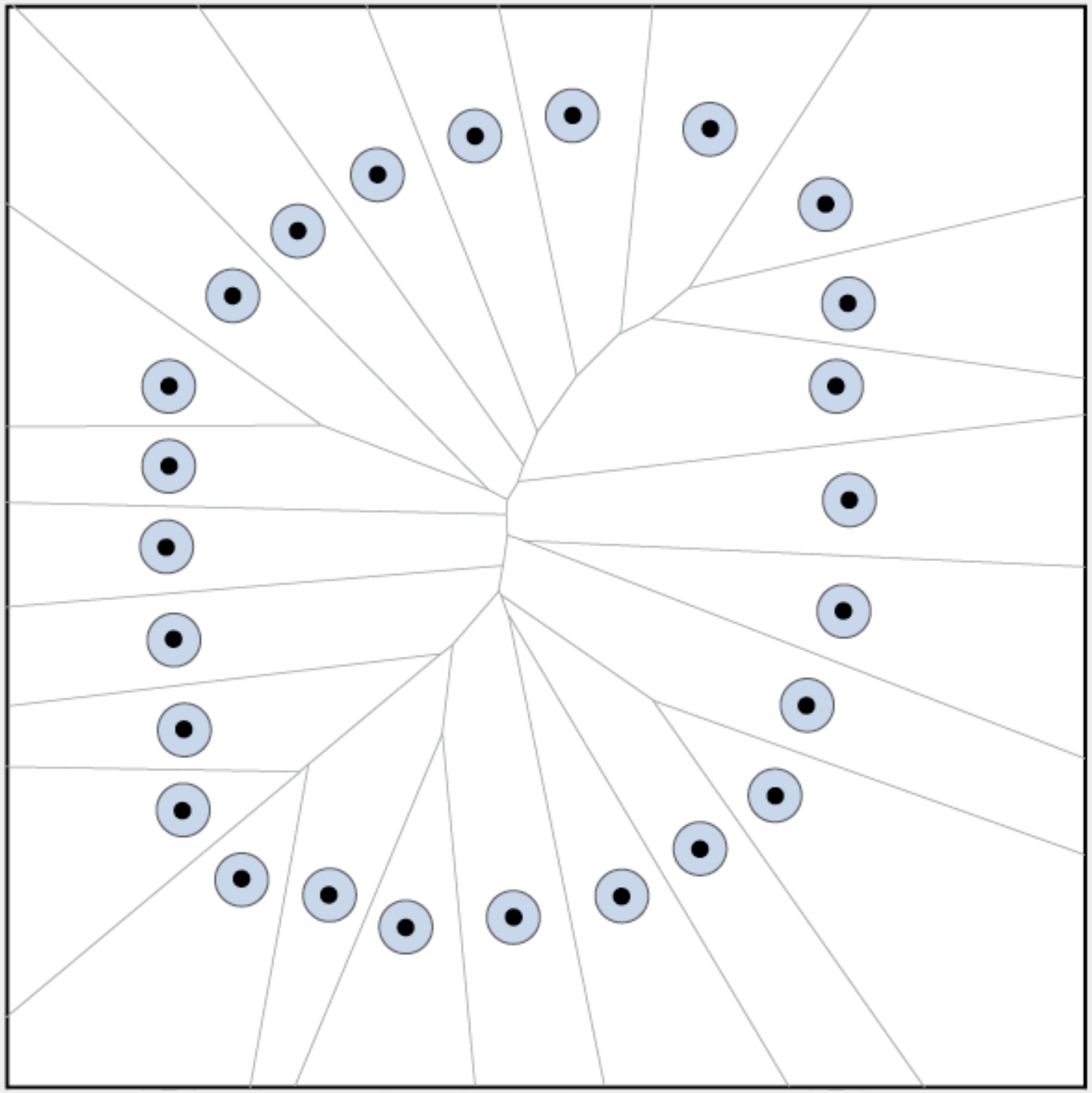

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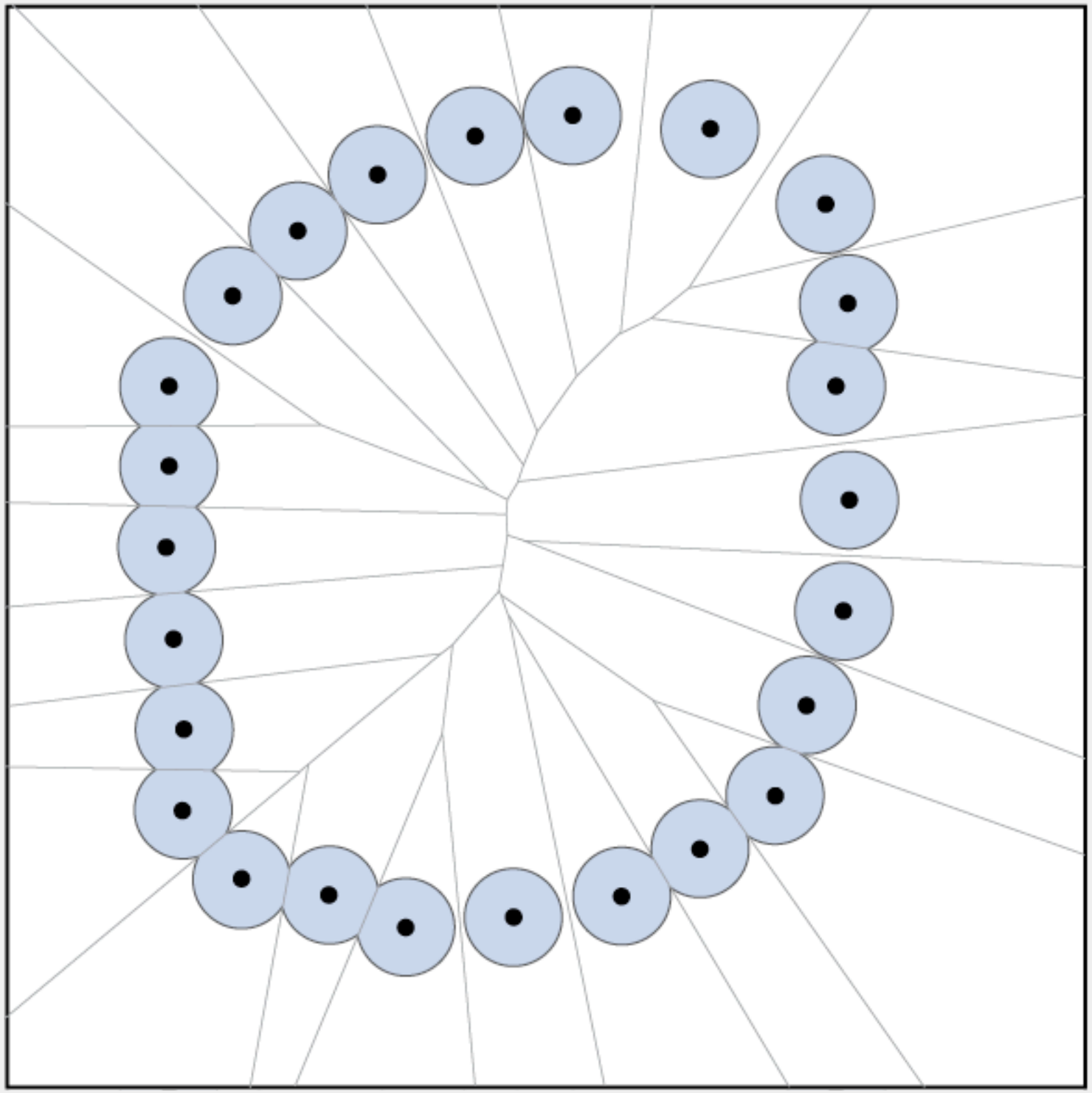

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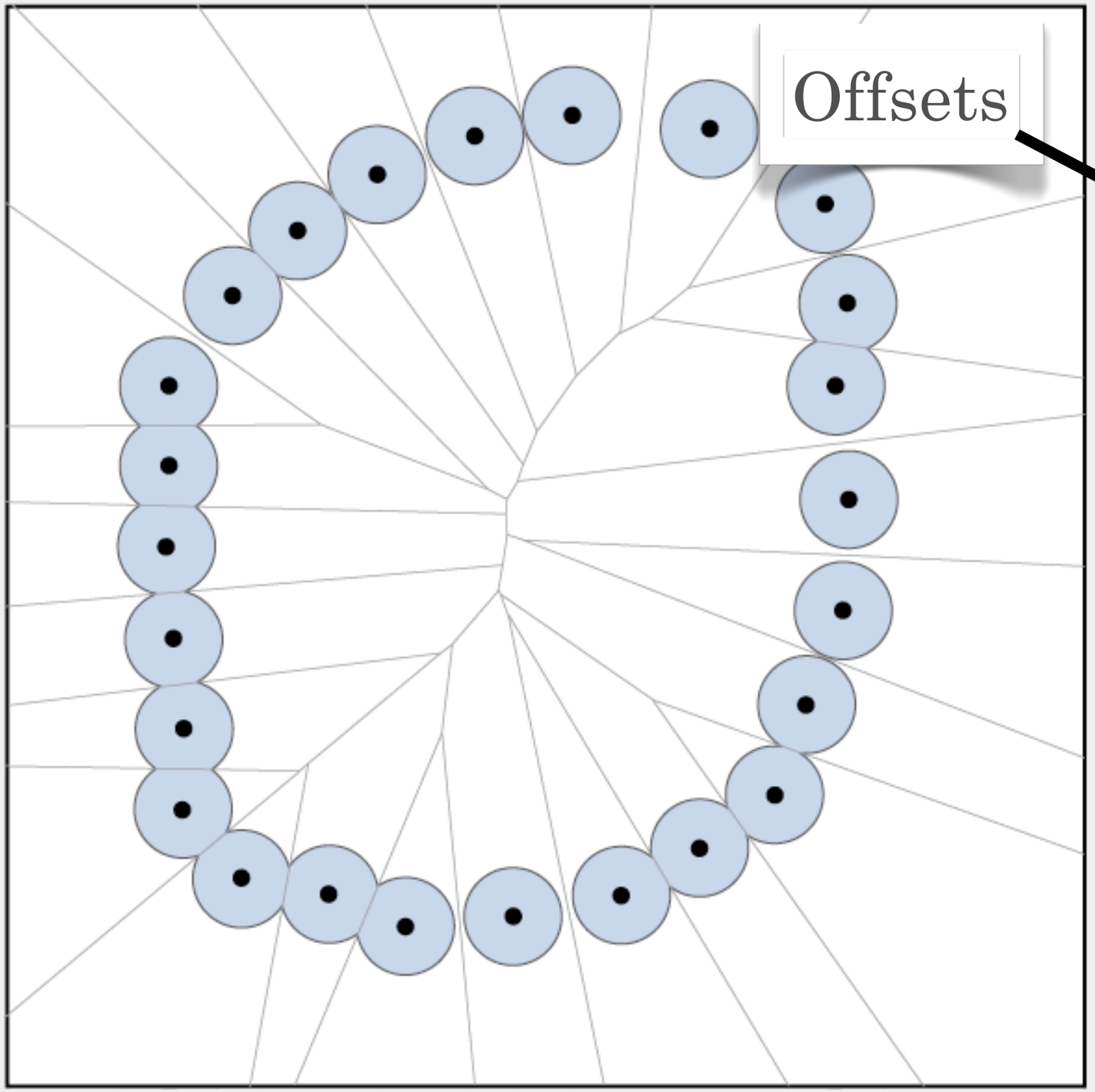

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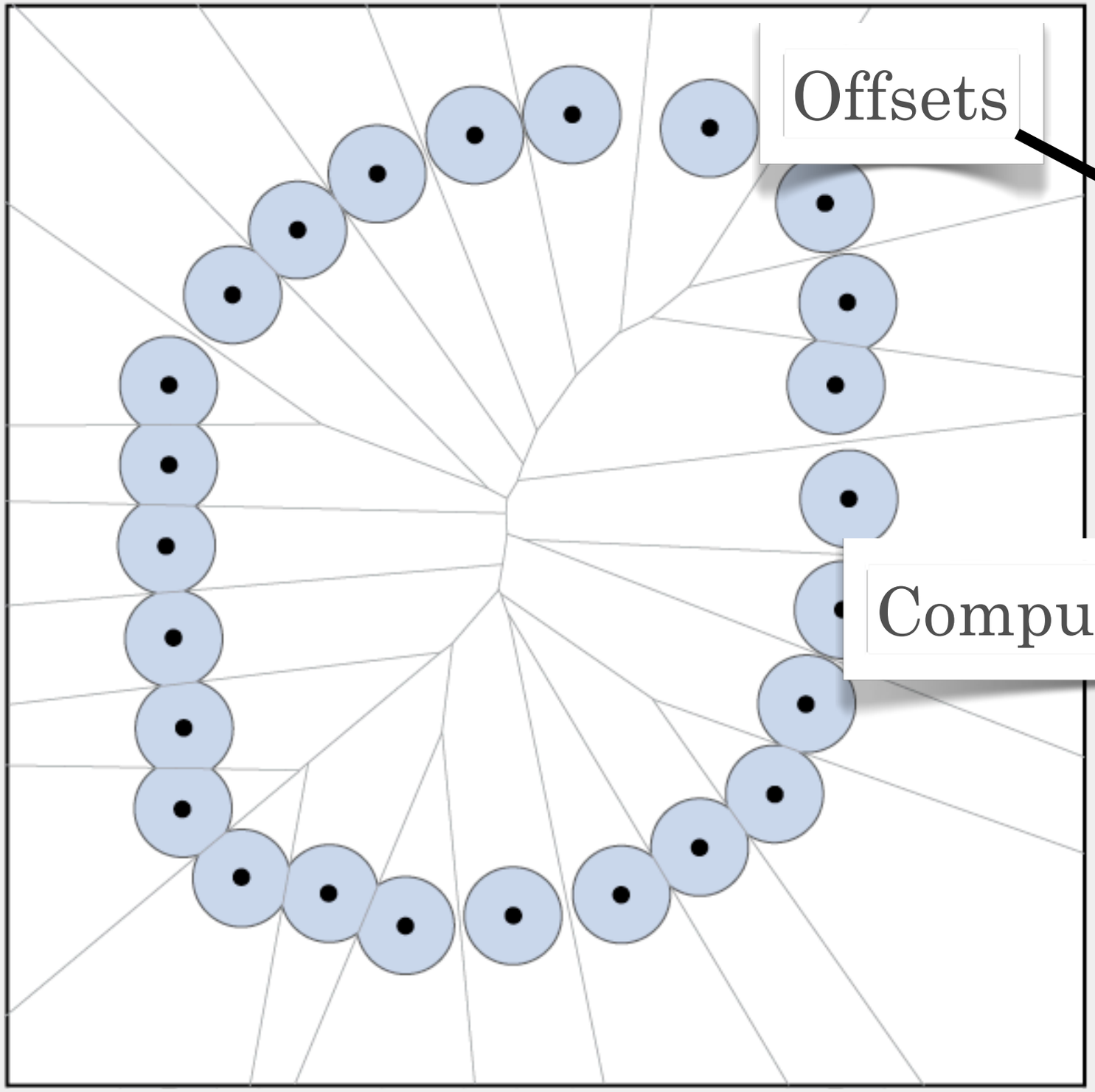

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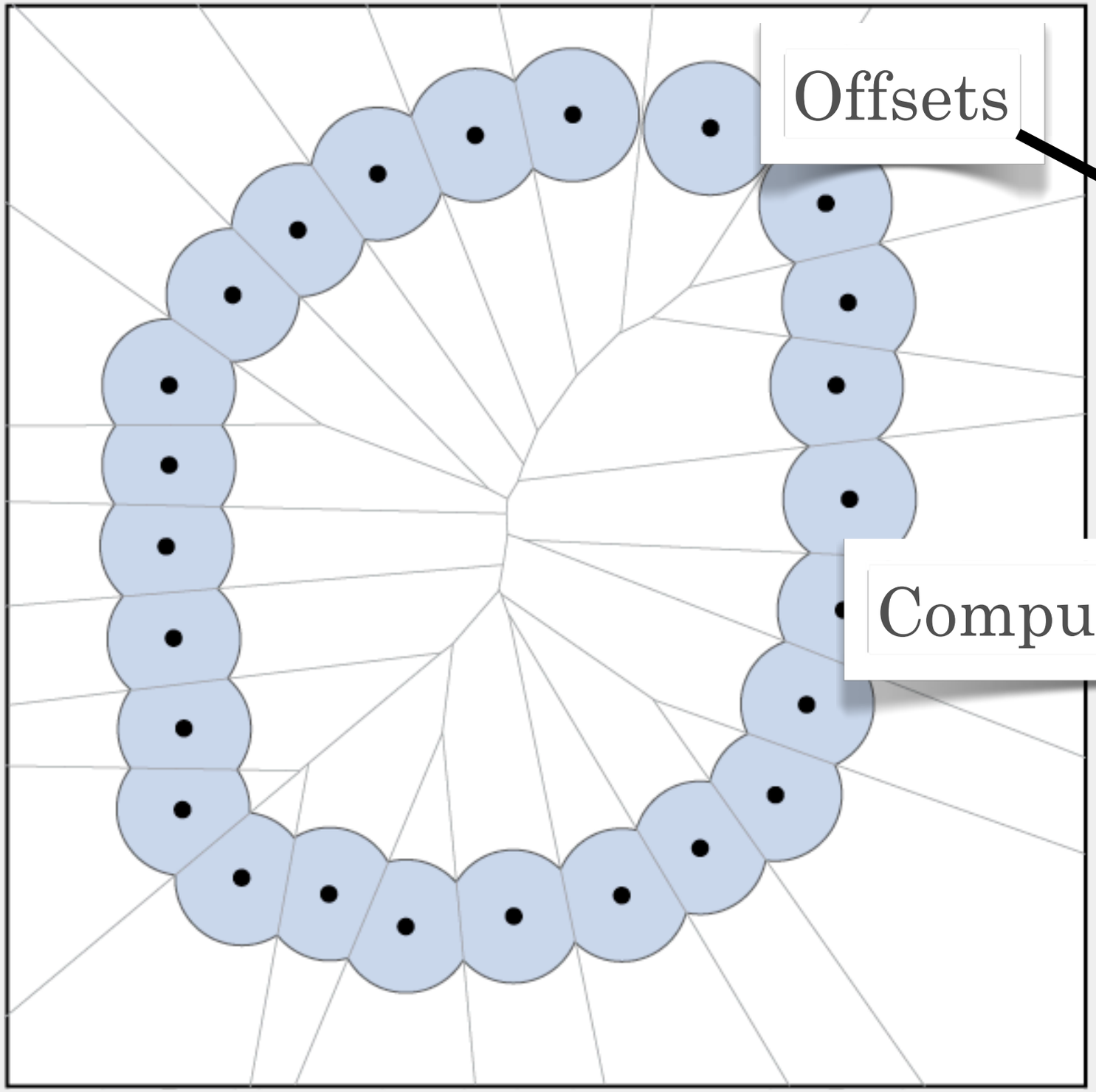

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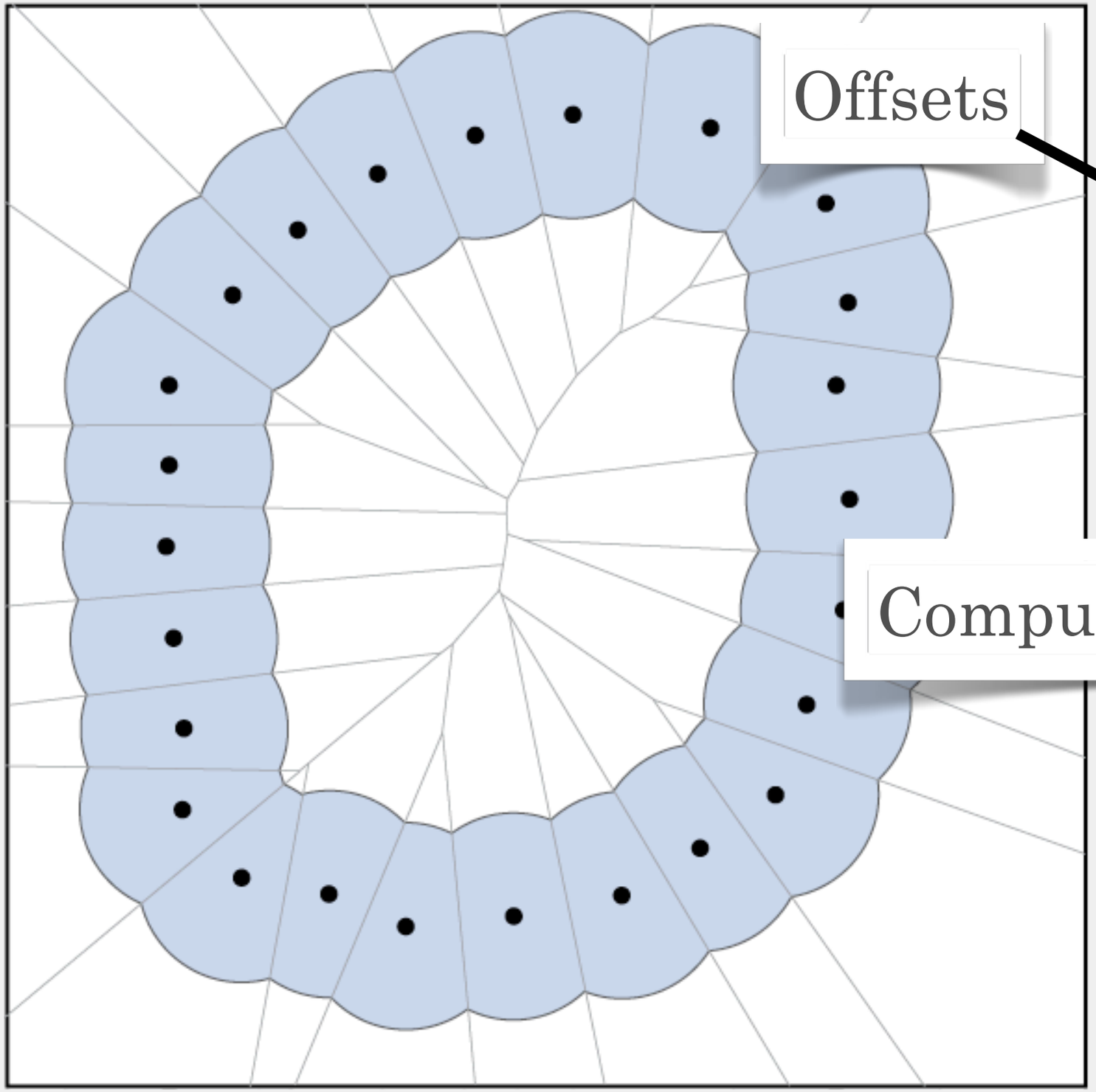

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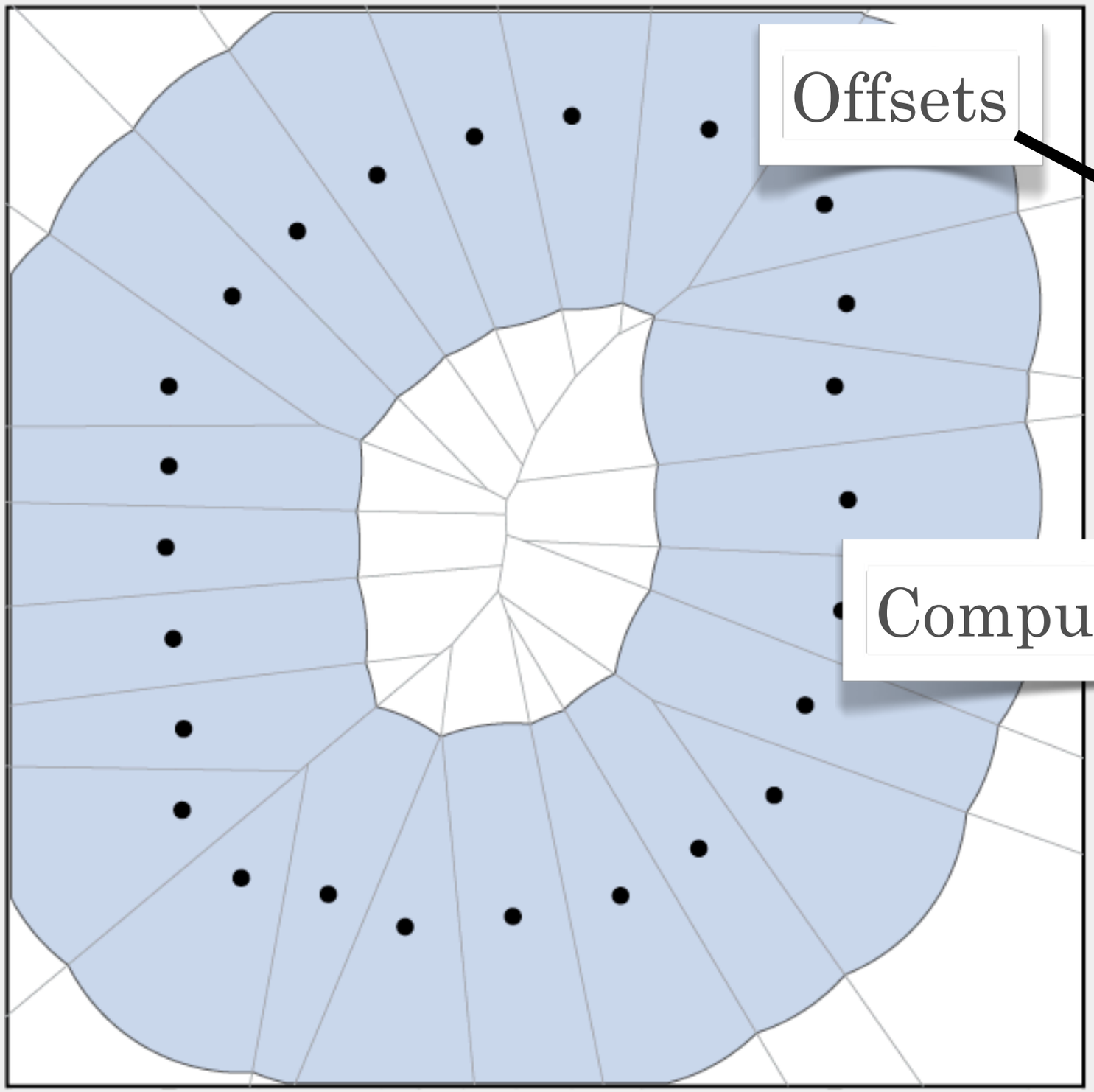

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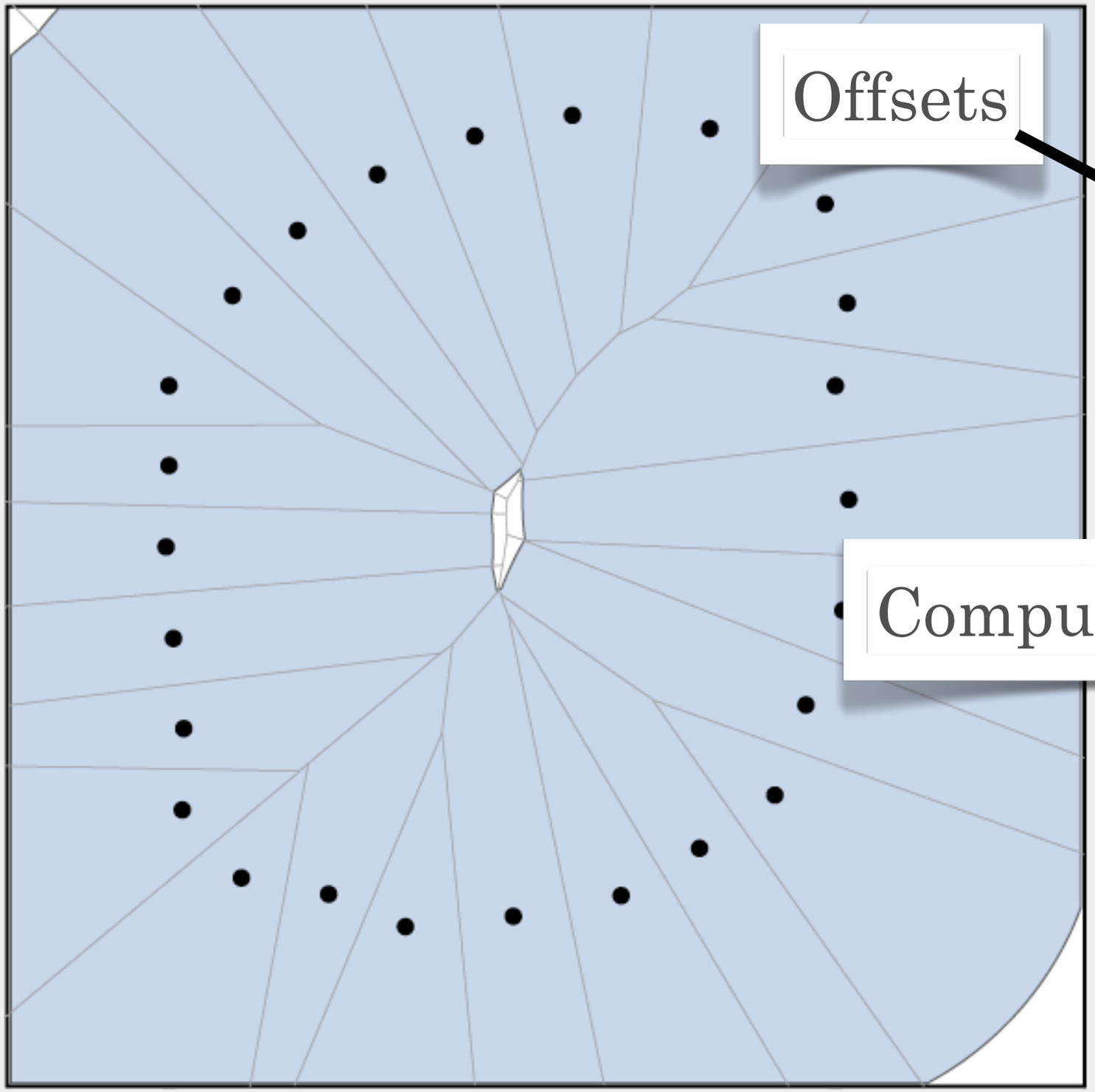

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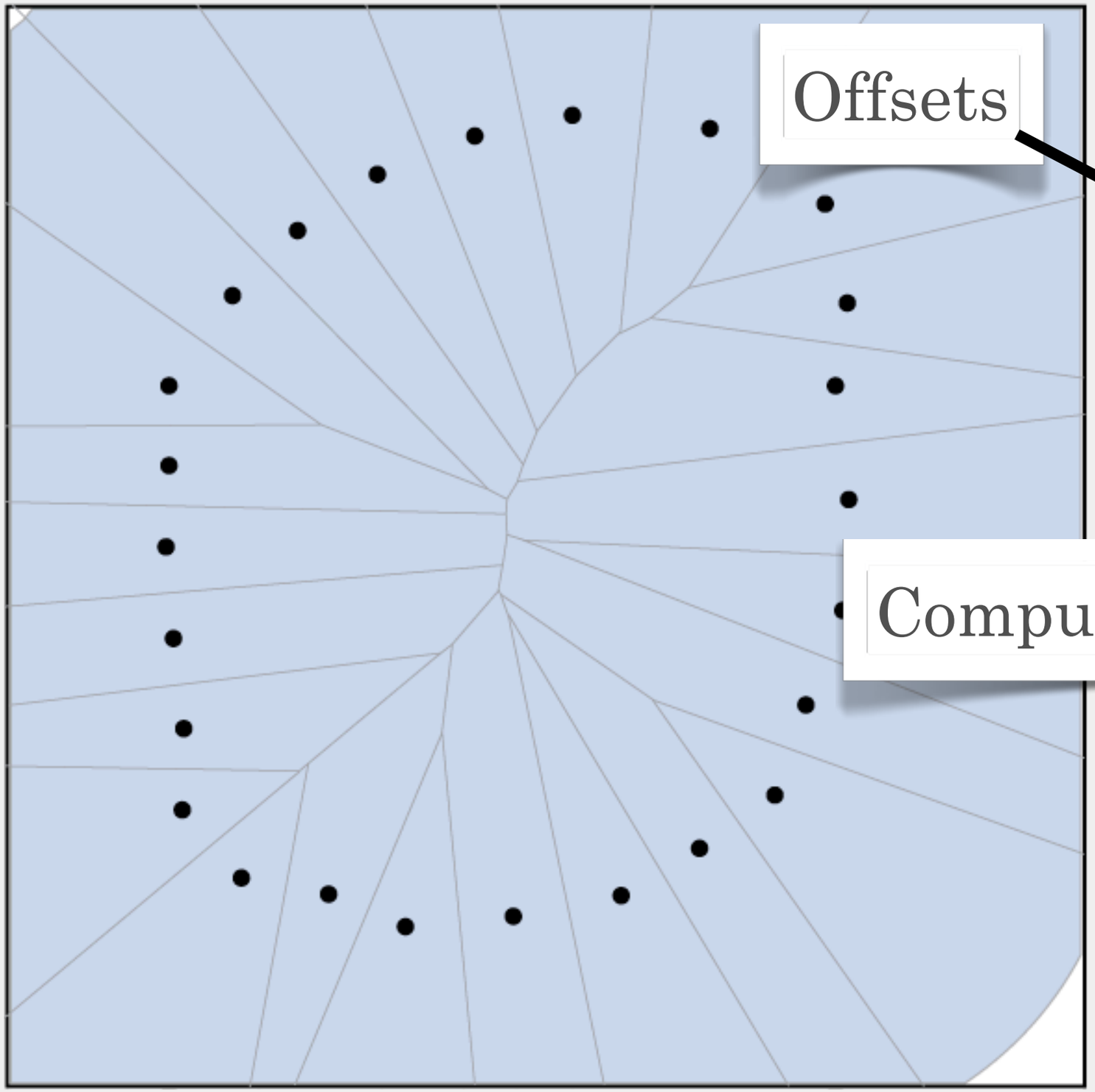

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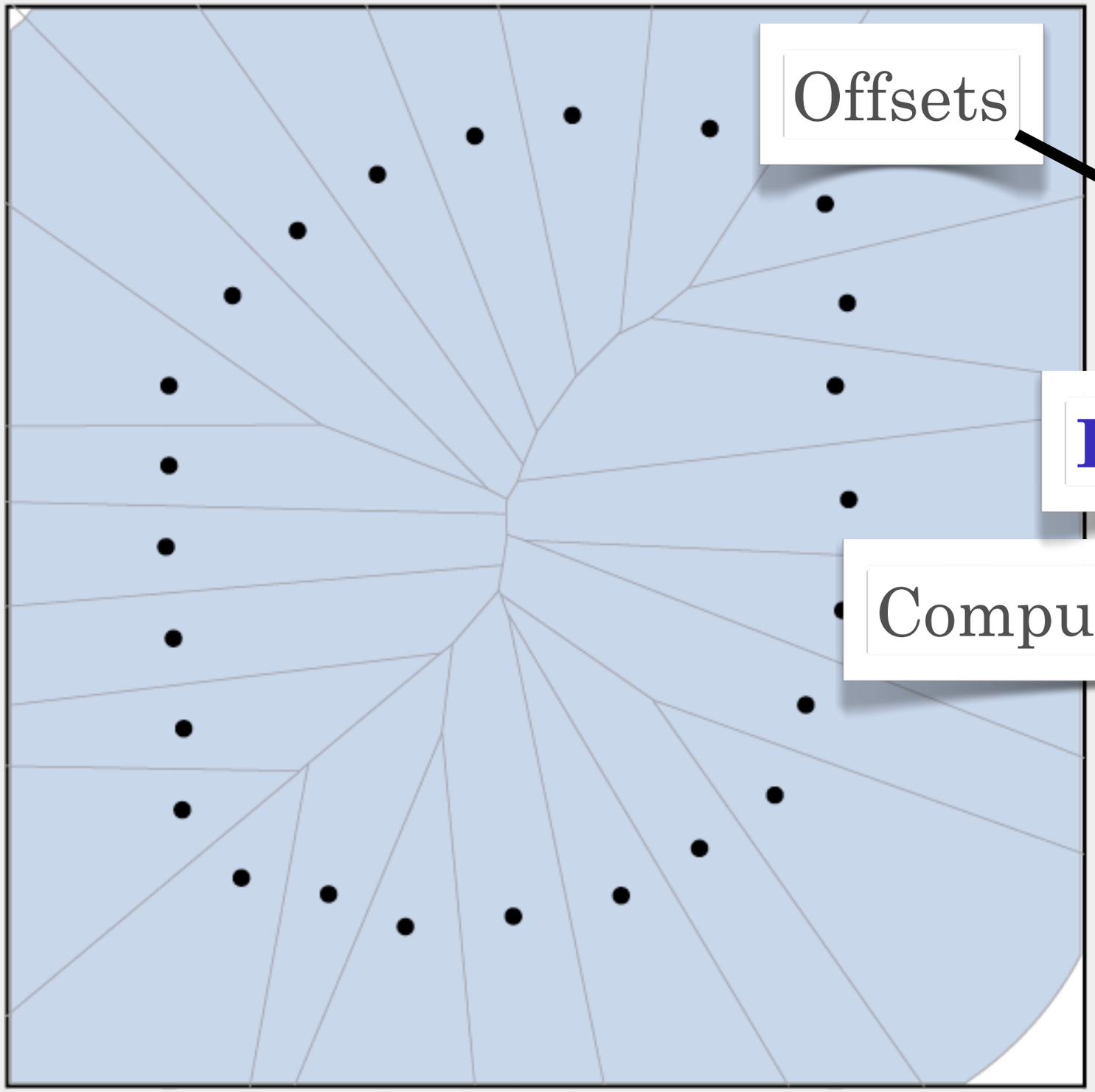

Offsets

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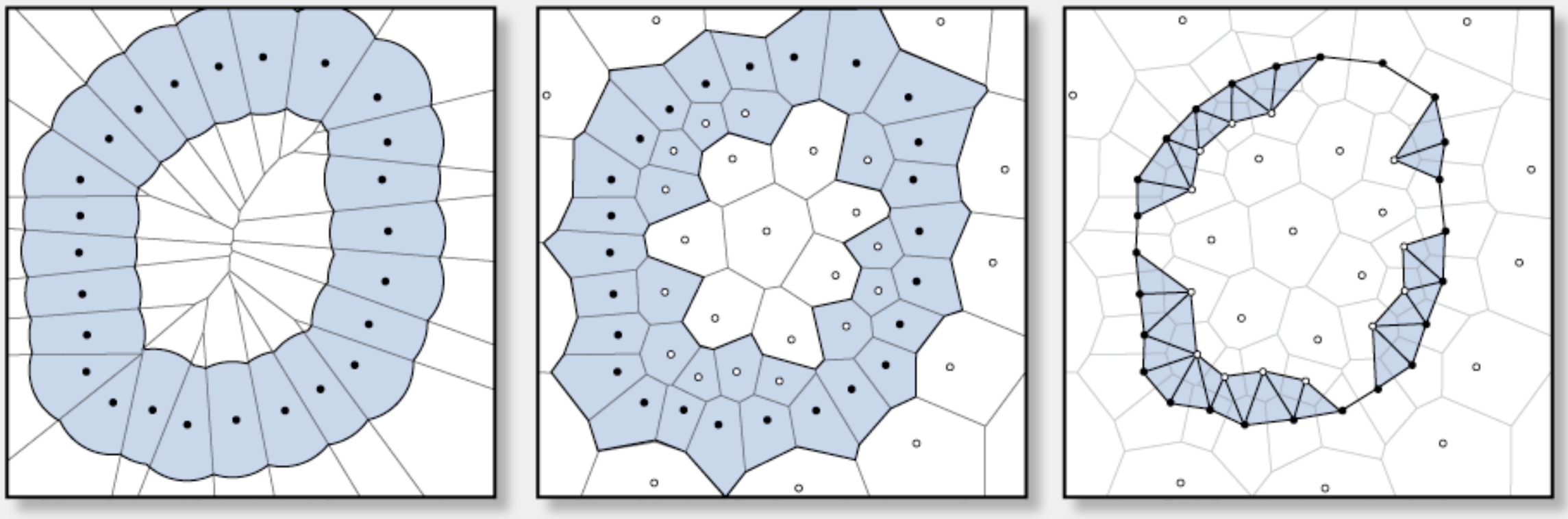
Persistent

Compute the **Homology**

Geometric Approximation

Topologically Equivalent



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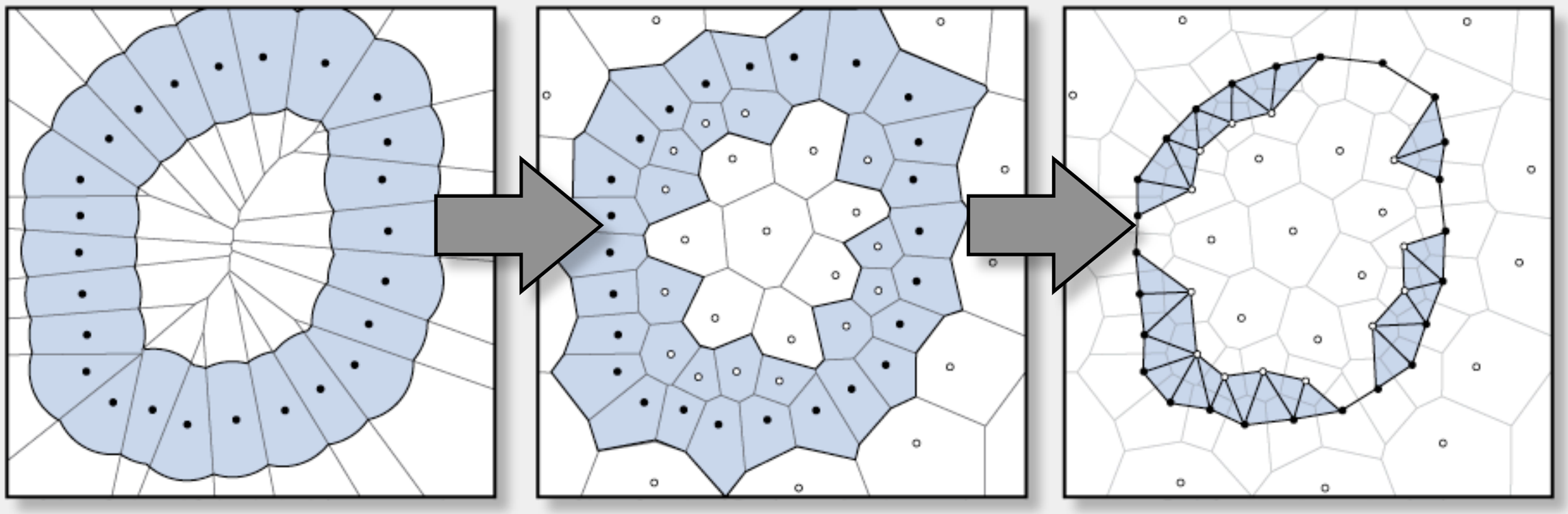

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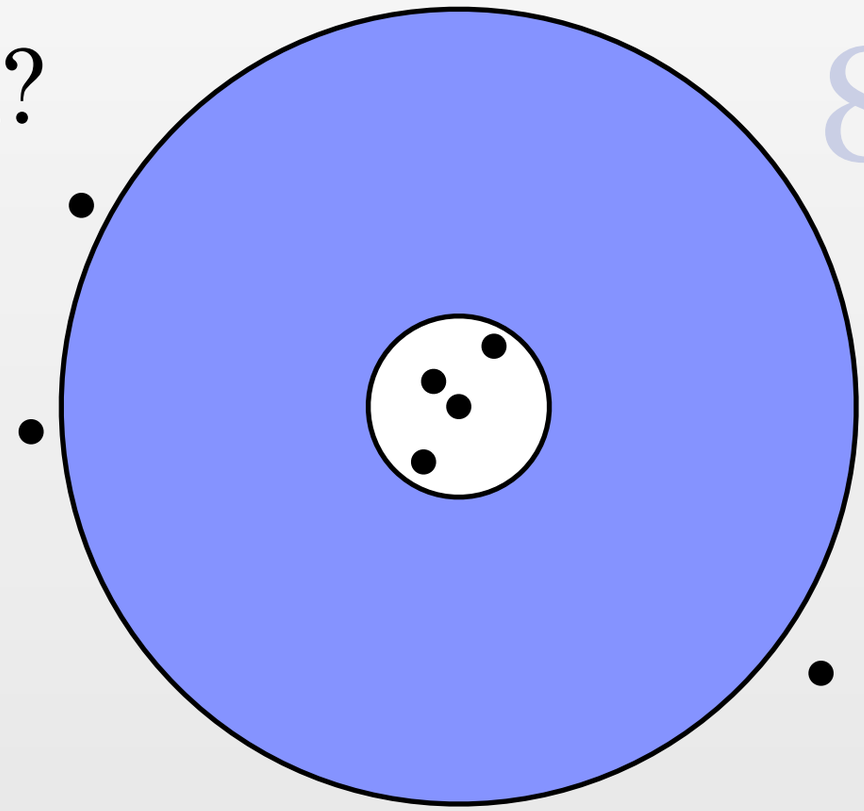

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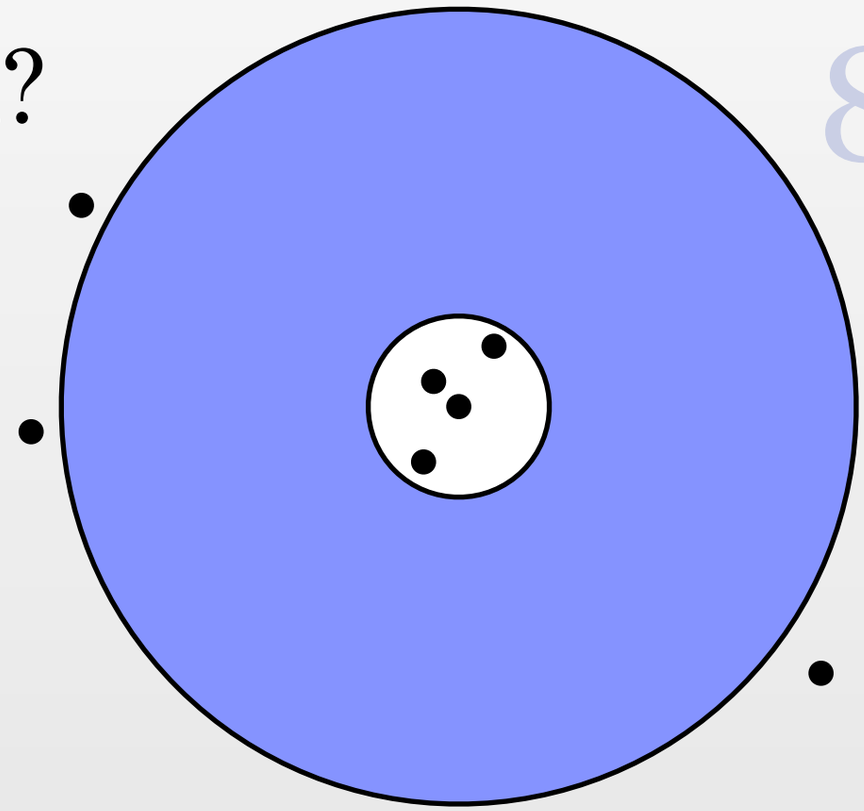

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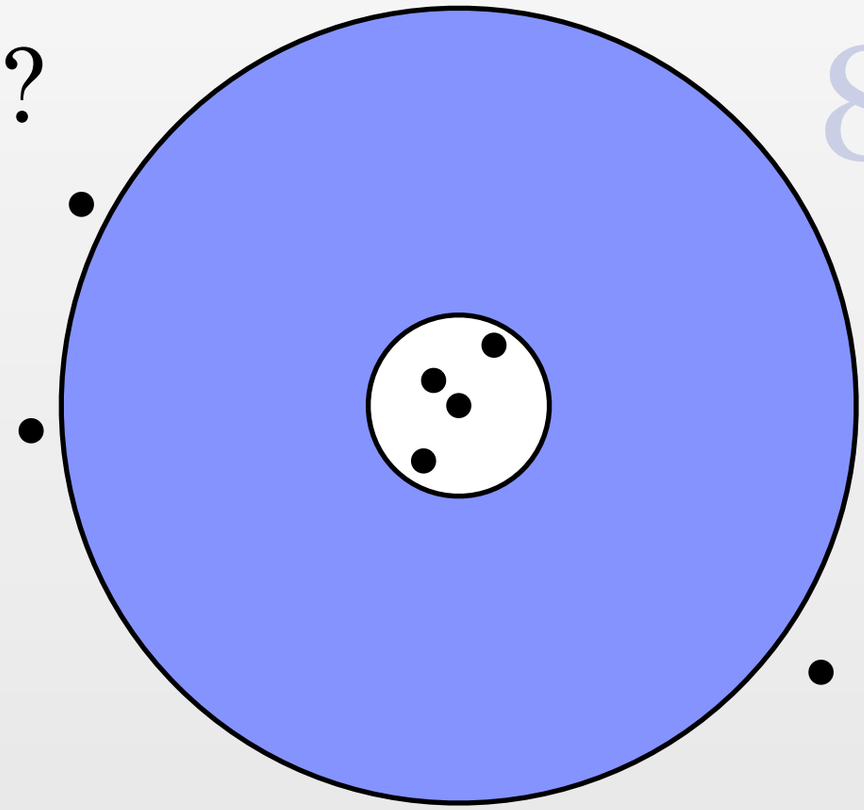

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How many simplices?

Only $O(|M|)$ simplices.
Compare to $|M|^{\lceil d/2 \rceil}$ for general Delaunay triangulations.
Constants depend on aspect ratio.

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Finishing post-process.

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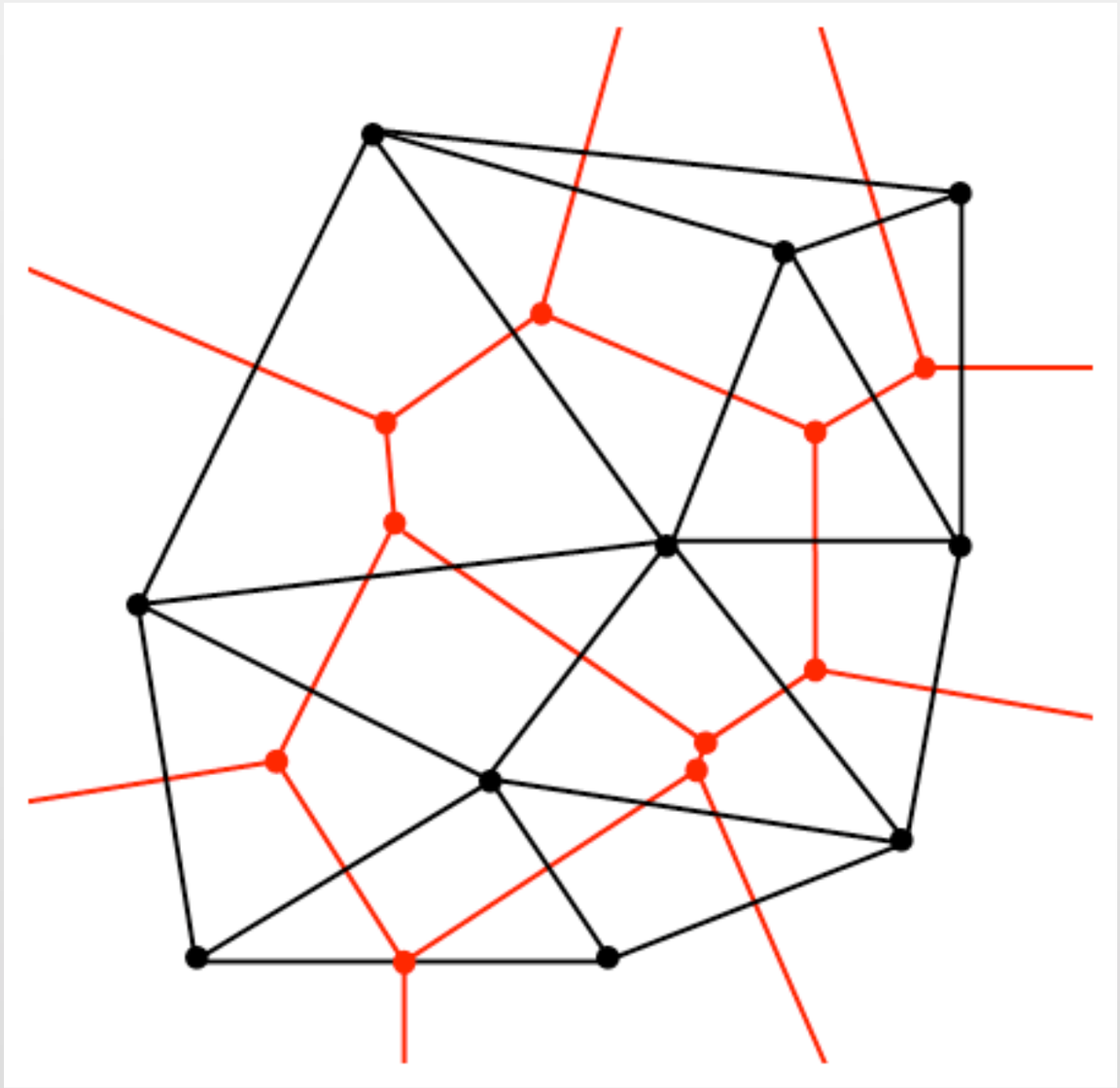

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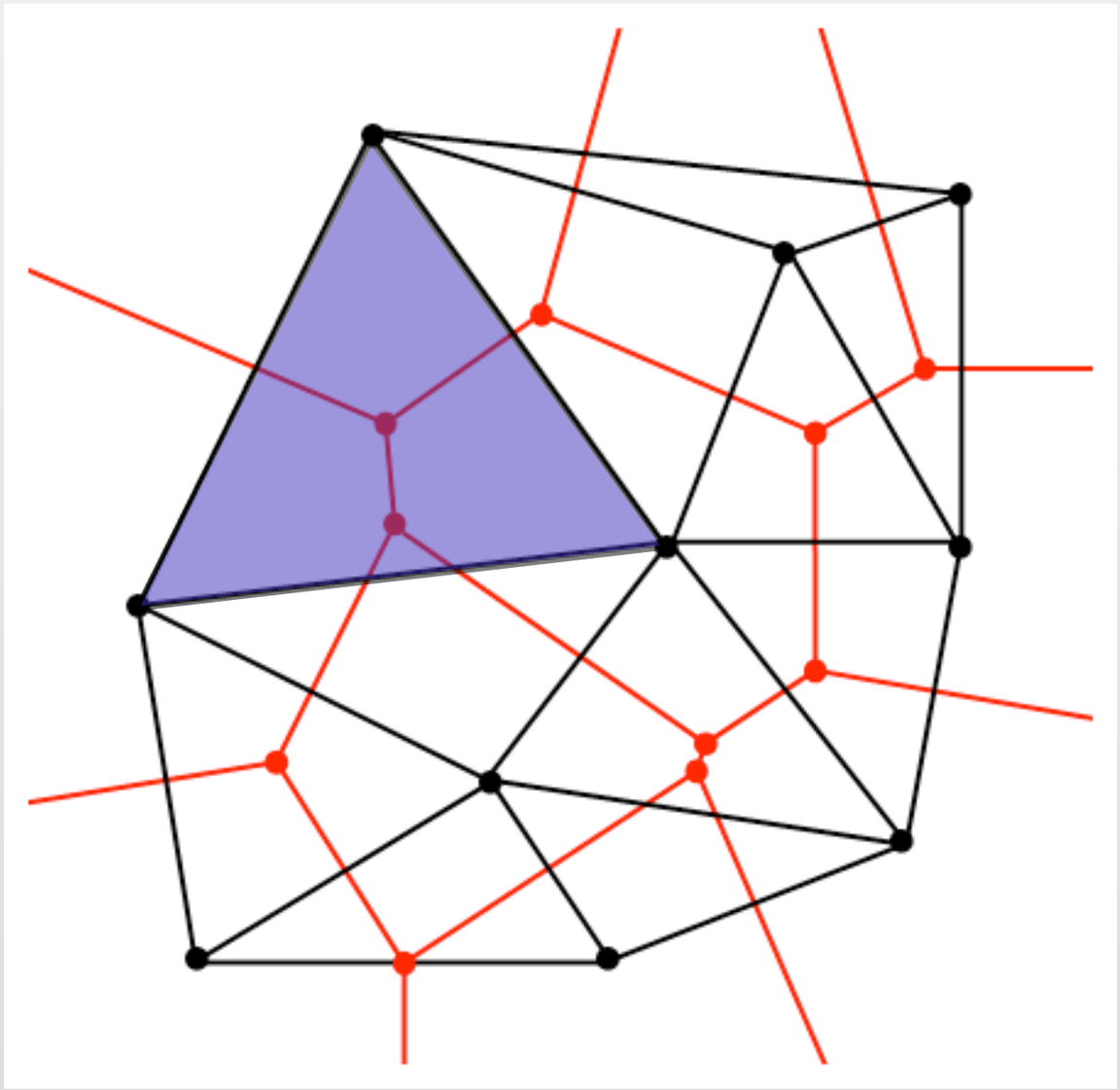

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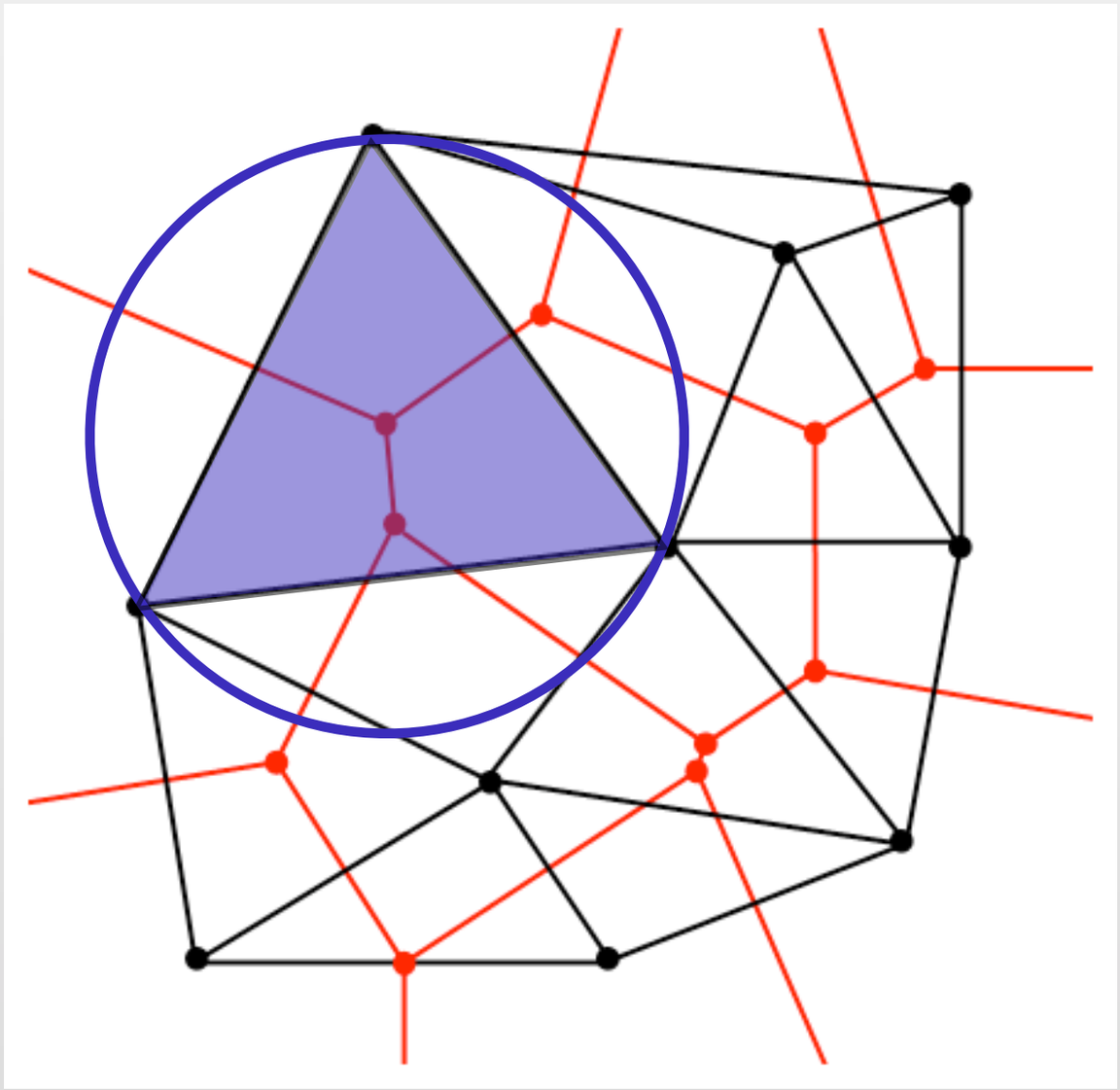

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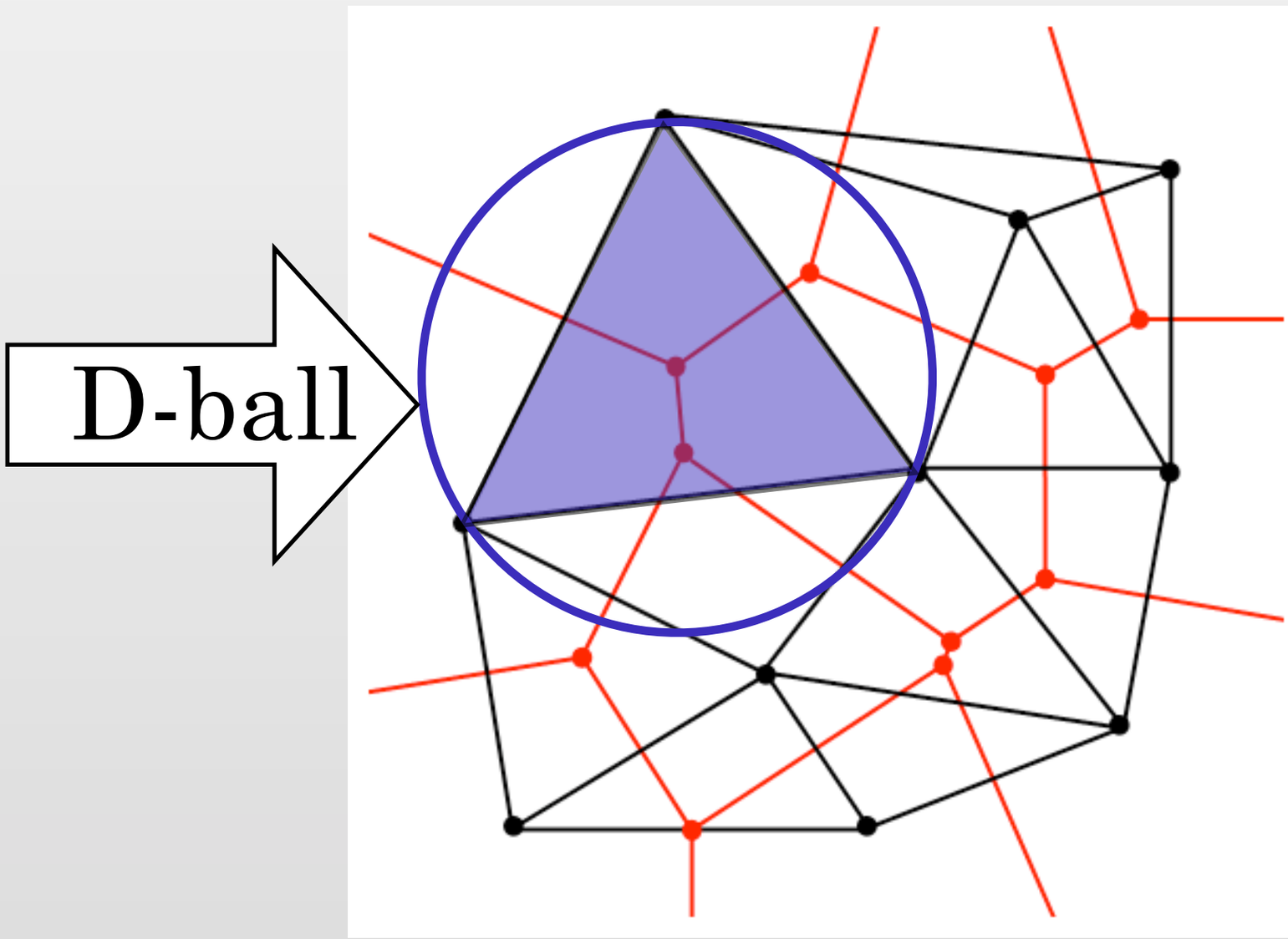
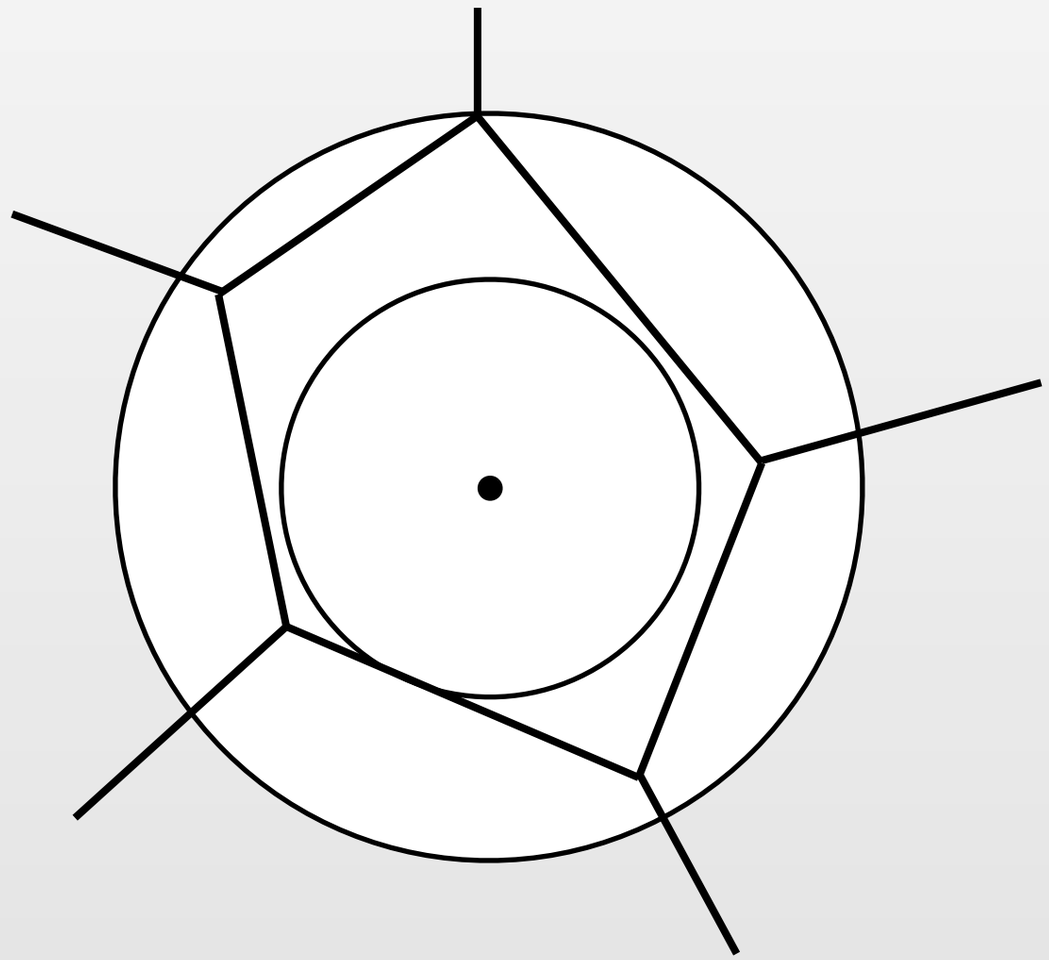


Quality Meshes



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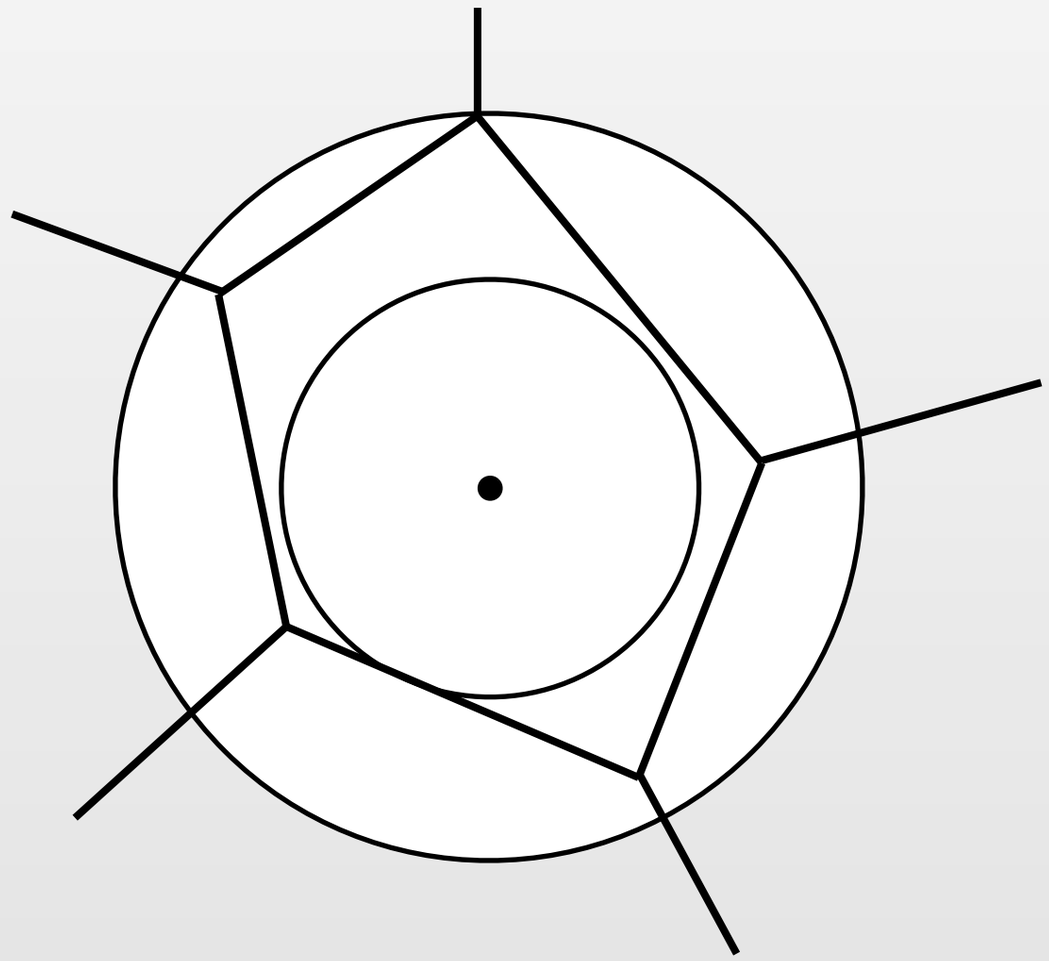

D-Balls are Constant Ply

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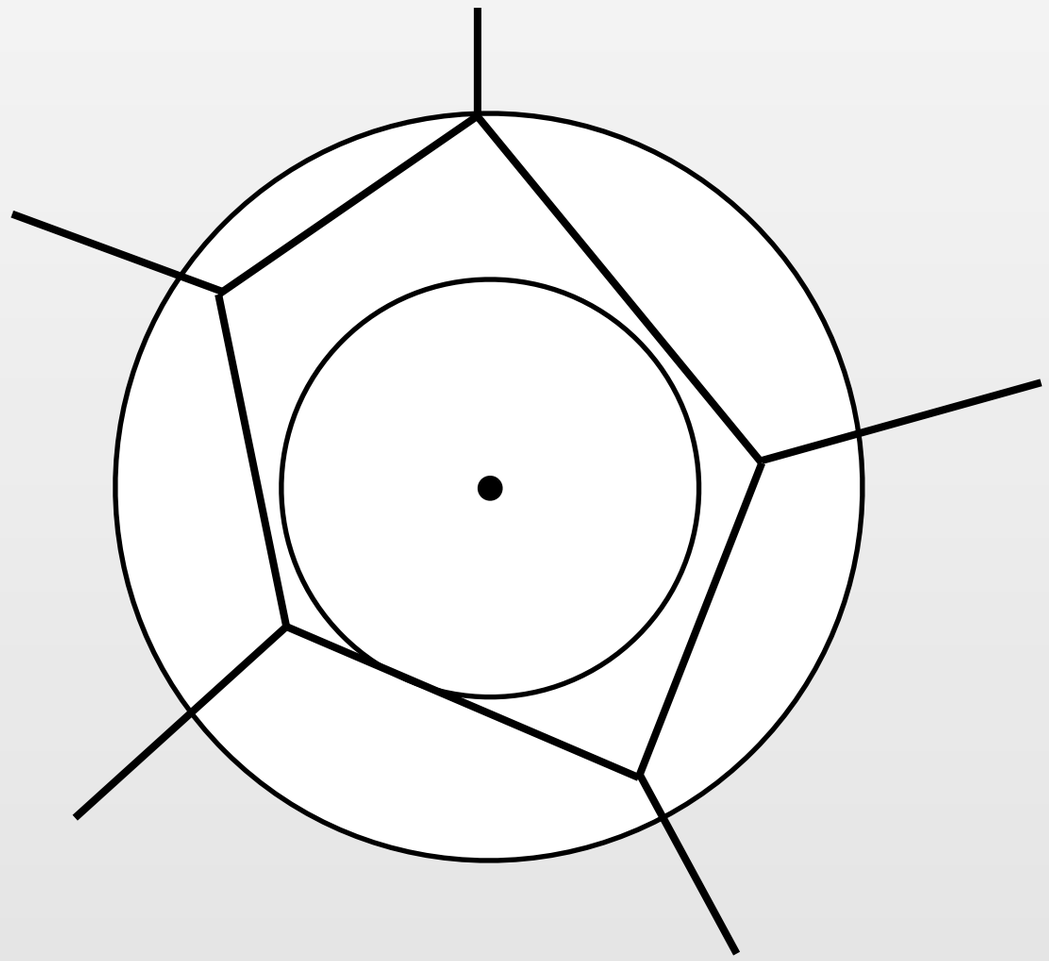

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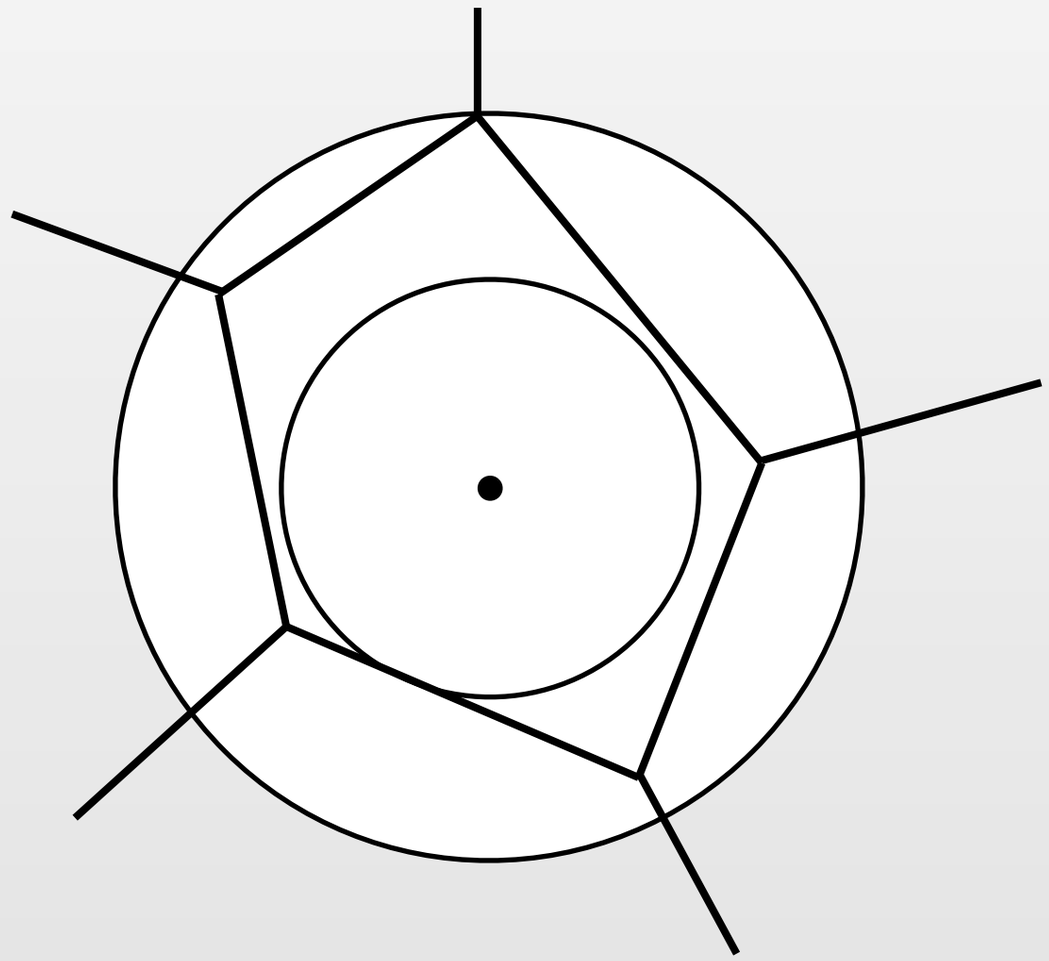

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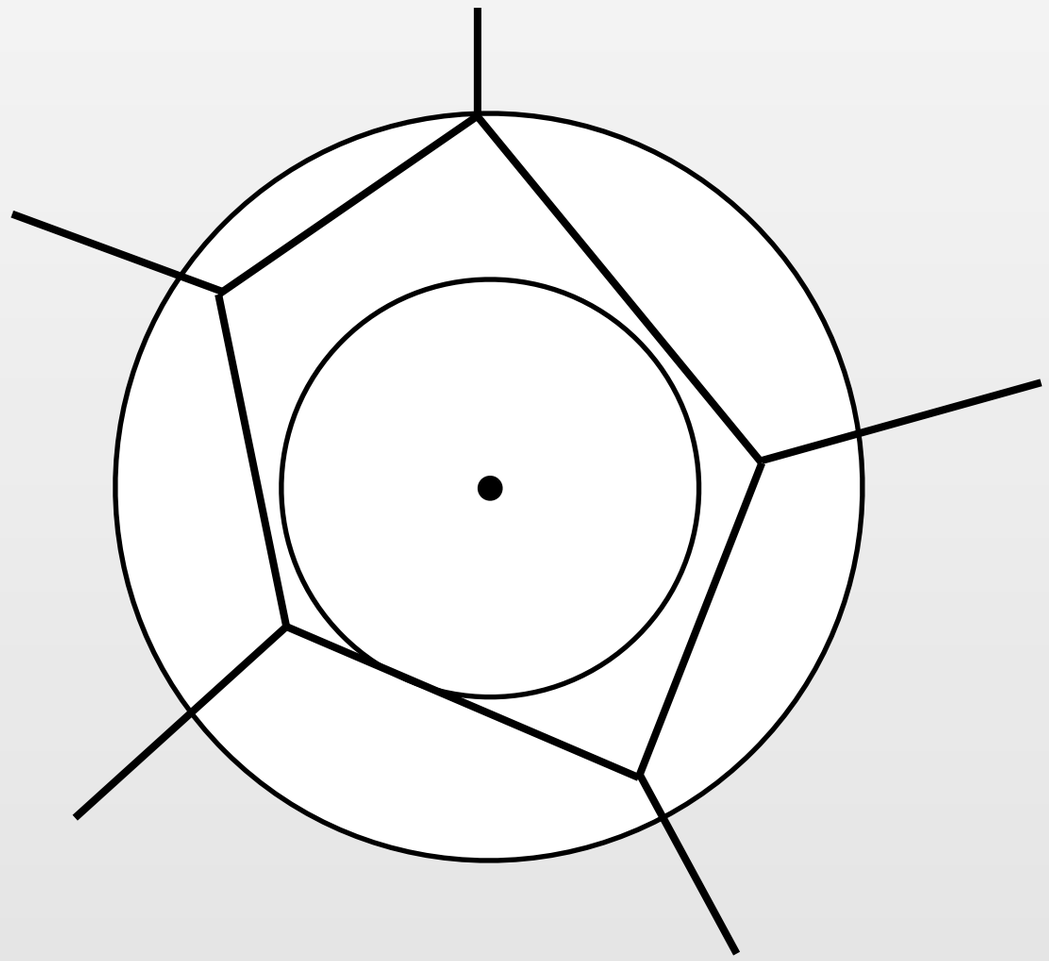

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Refine poor-quality cells by adding a Steiner point at its farthest vertex.

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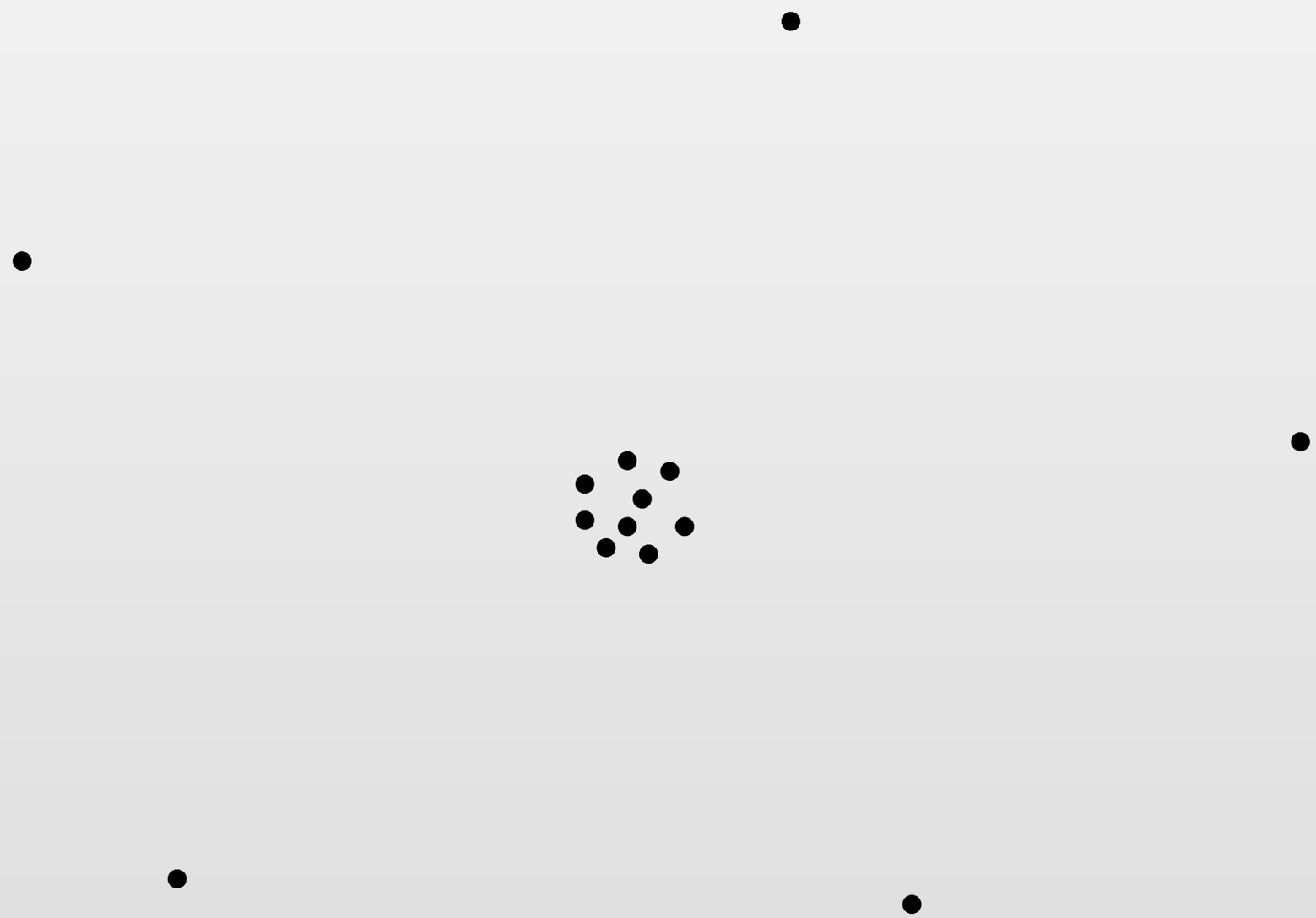

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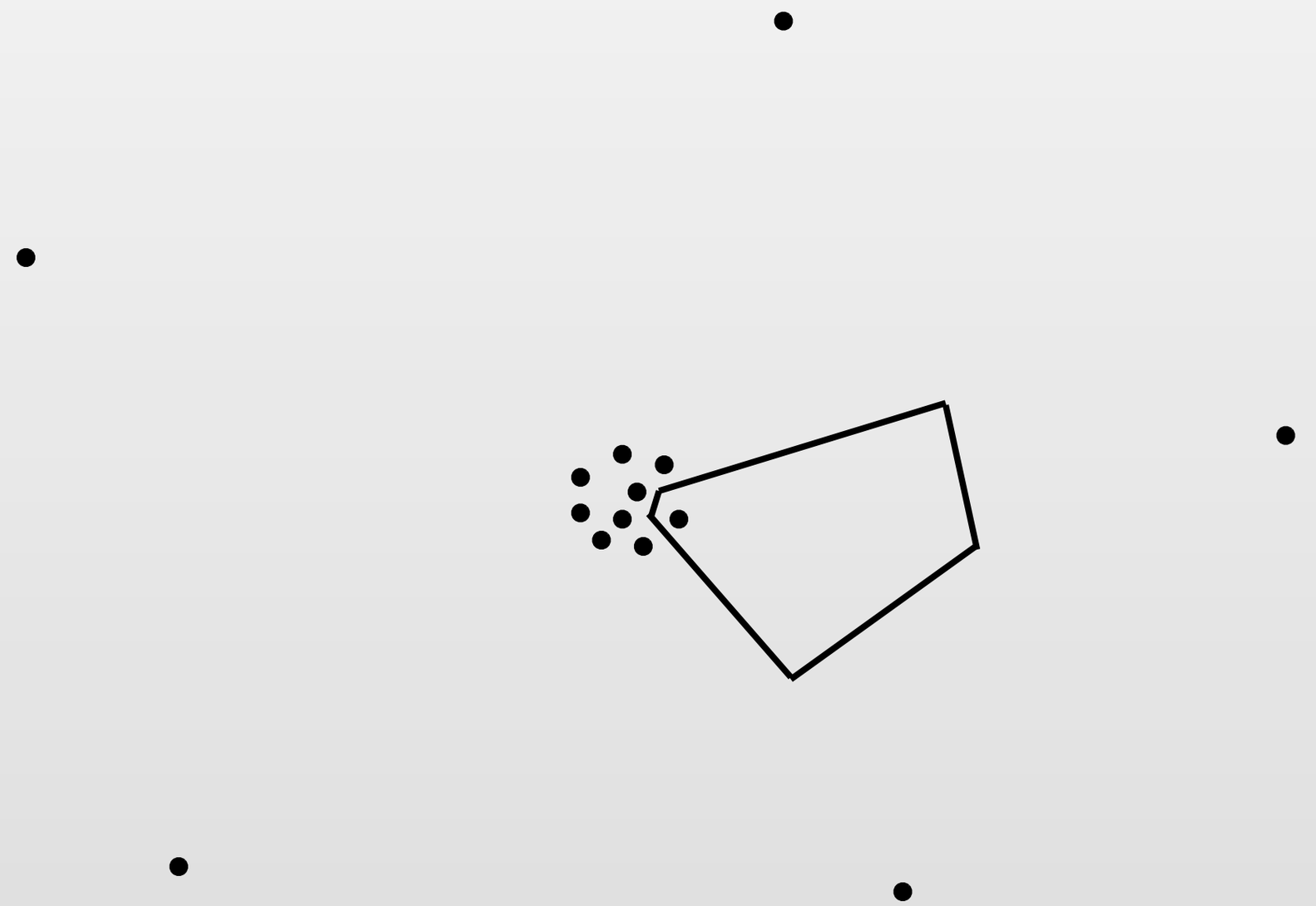

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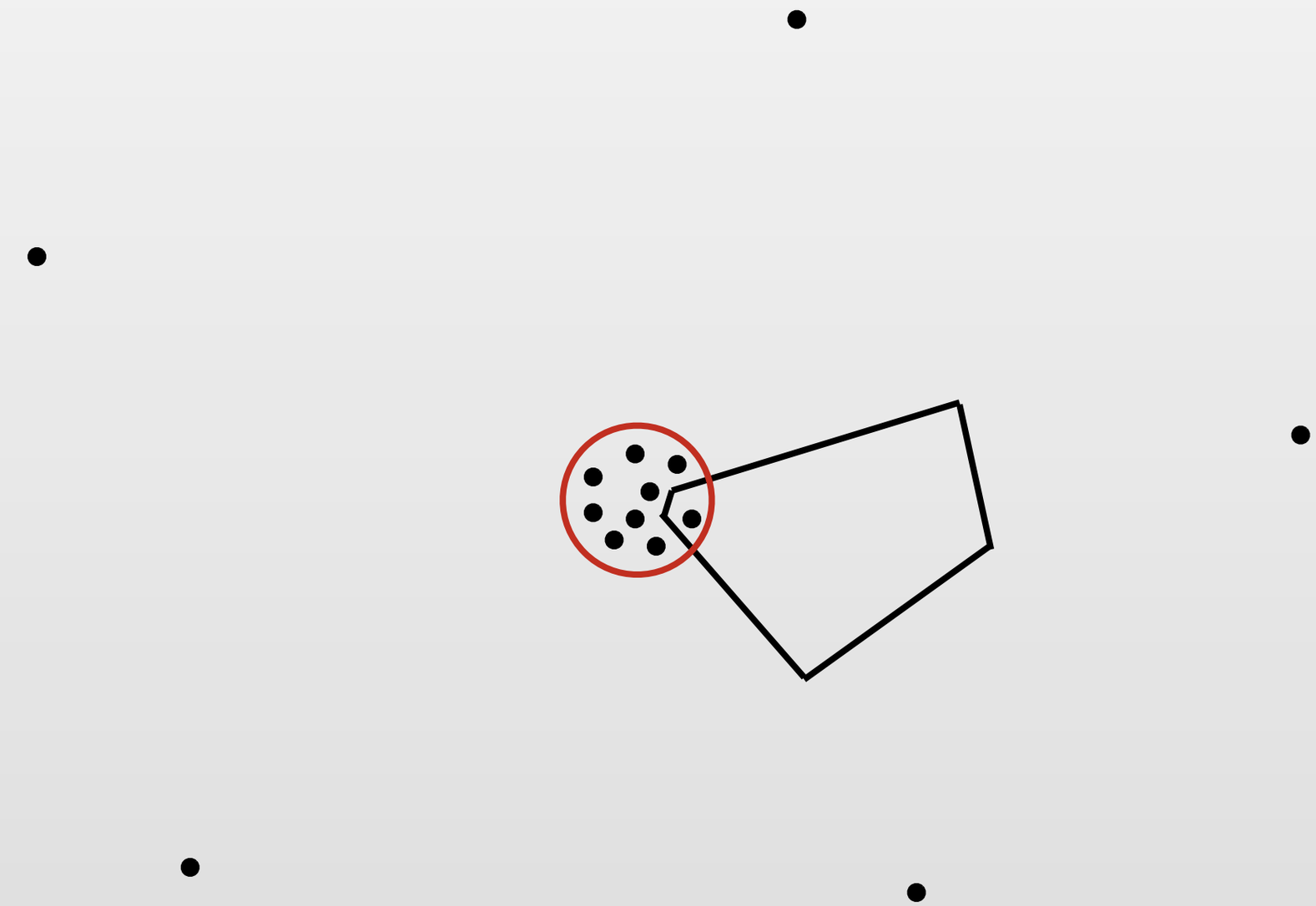

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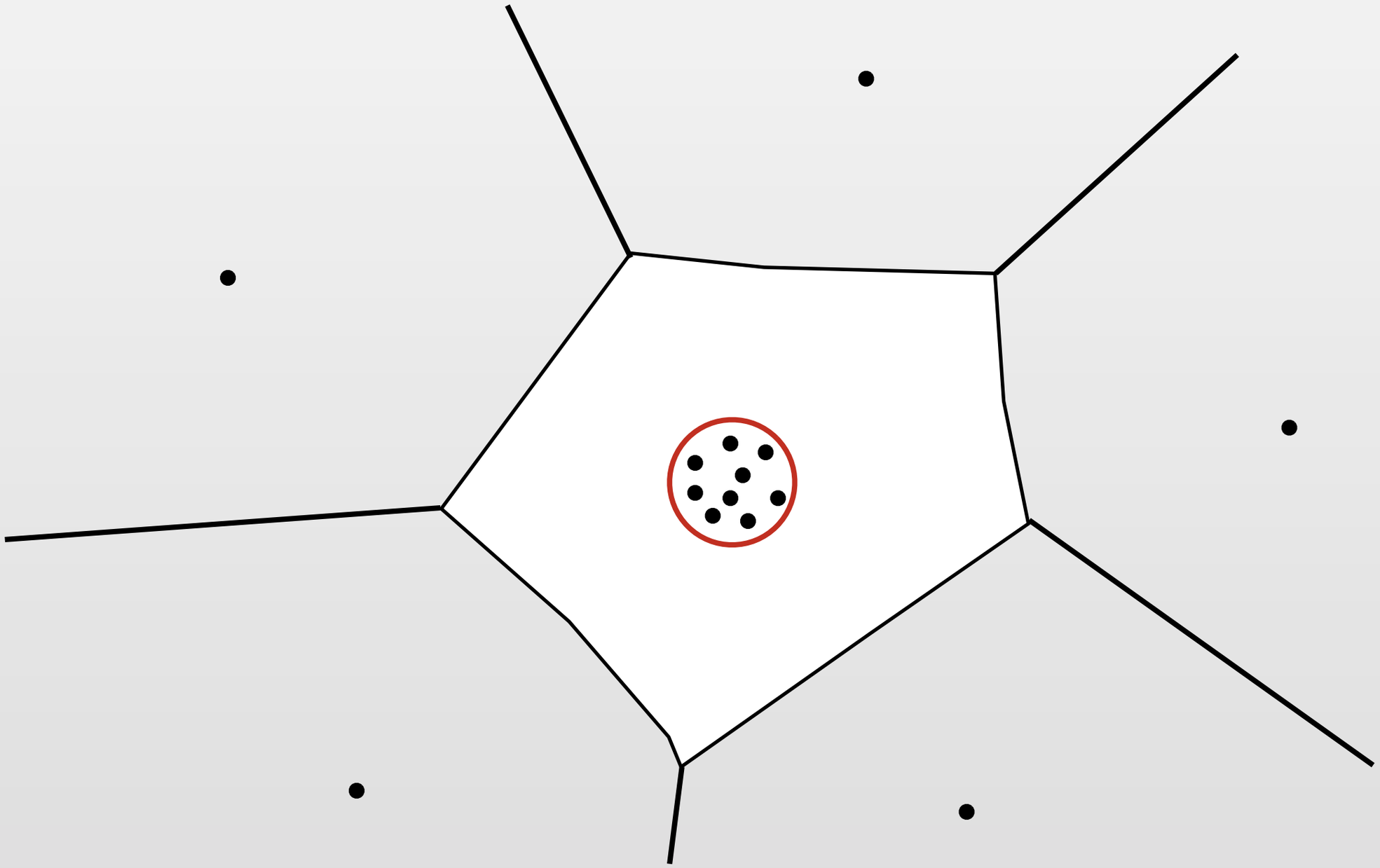

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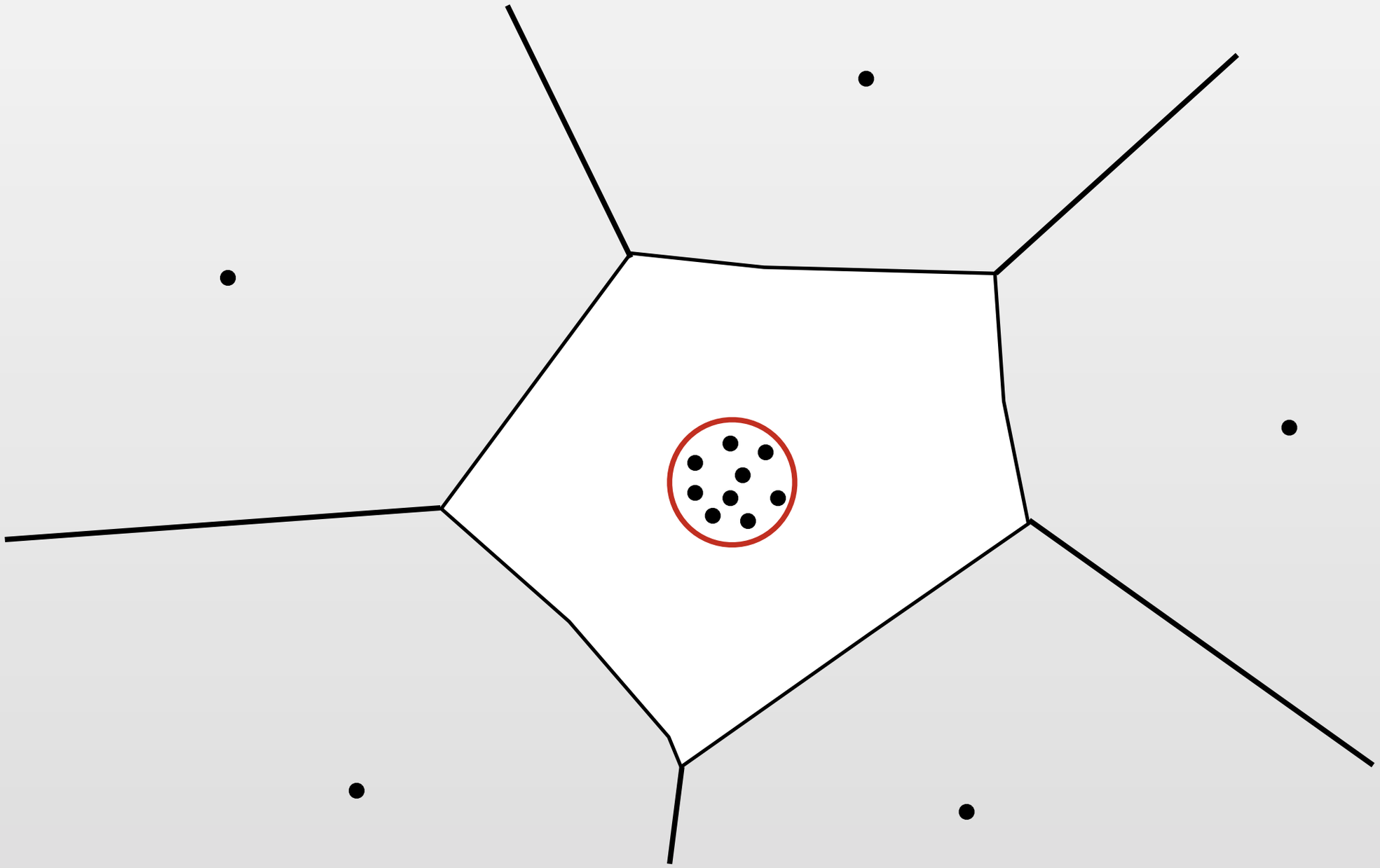

Inside the cage: Old definition of quality.

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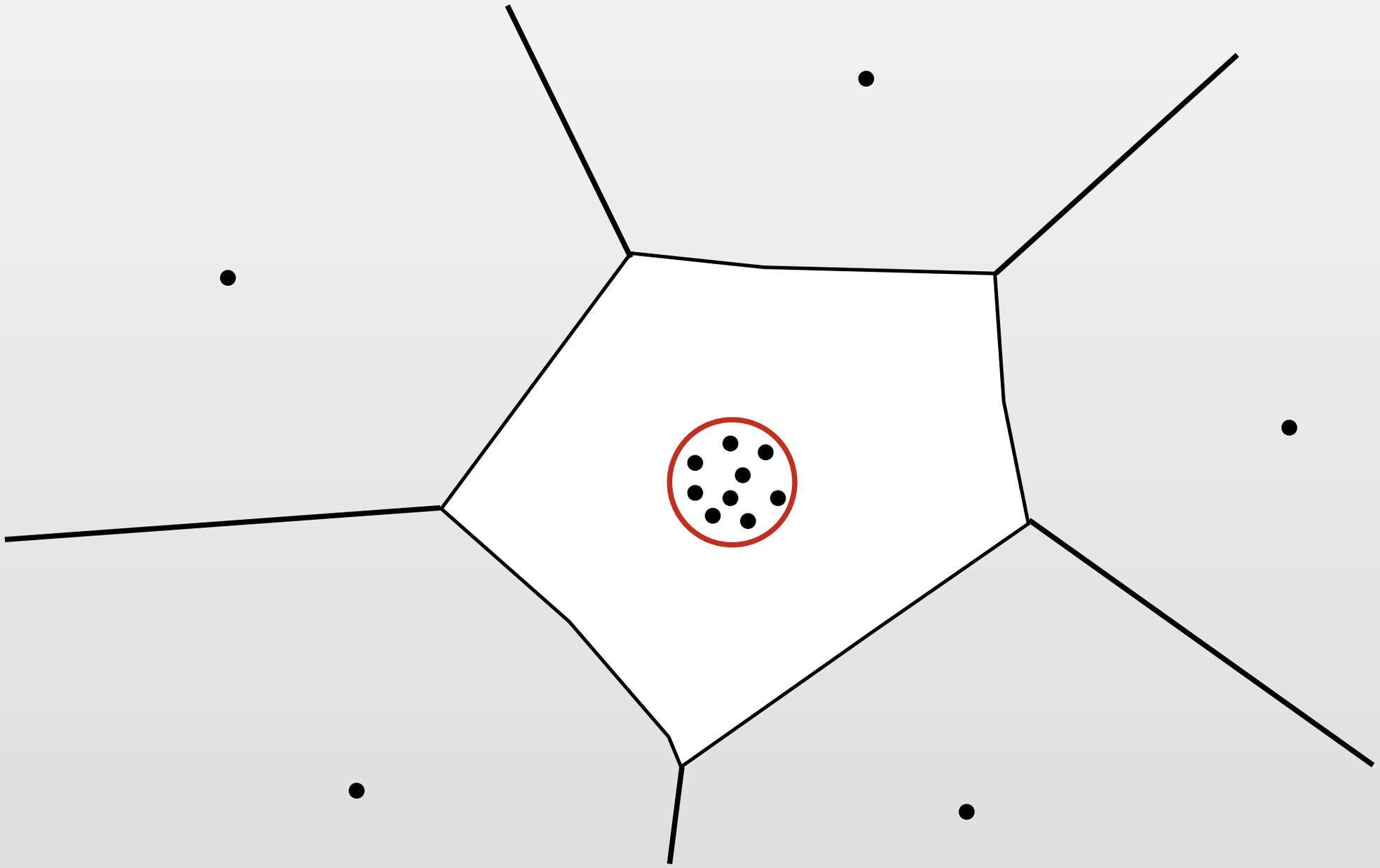

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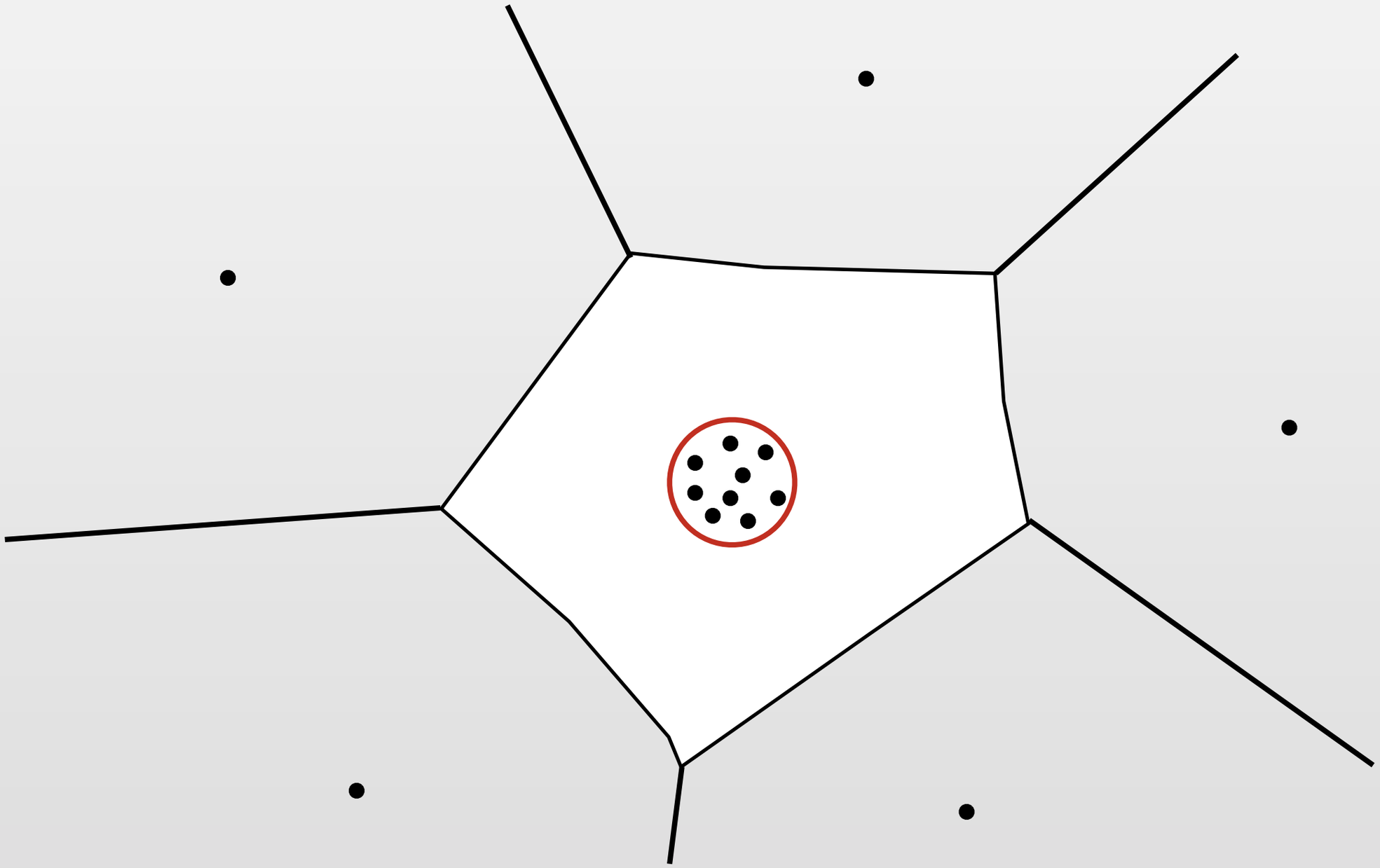

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All the same properties as quality meshes: ply, degree, etc.

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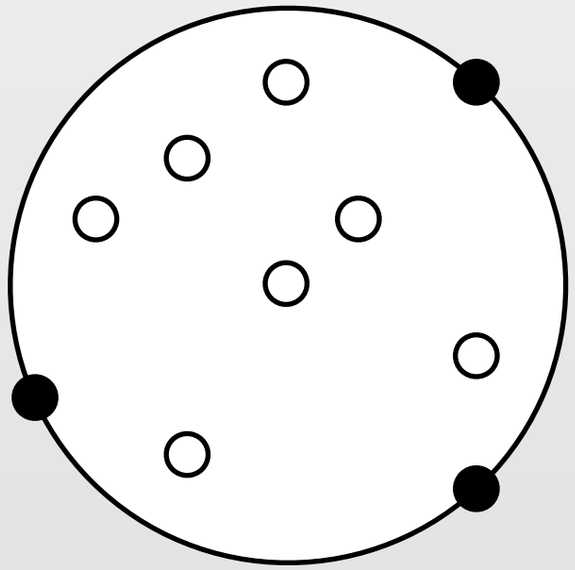
This is how we avoid the worst-case Voronoi bounds.

Idea: Store the uninserted points in the D-balls.
When the balls change, make local updates.



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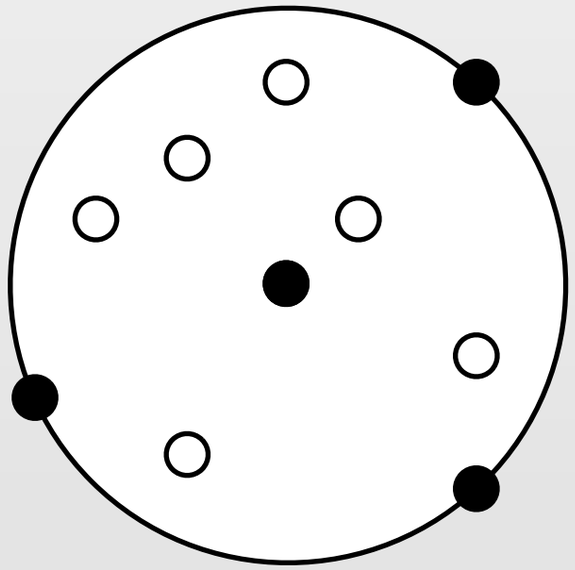

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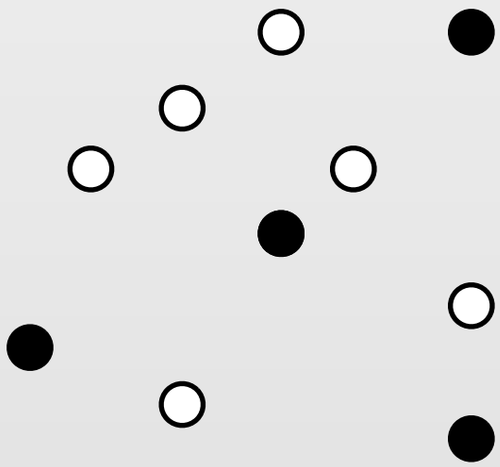

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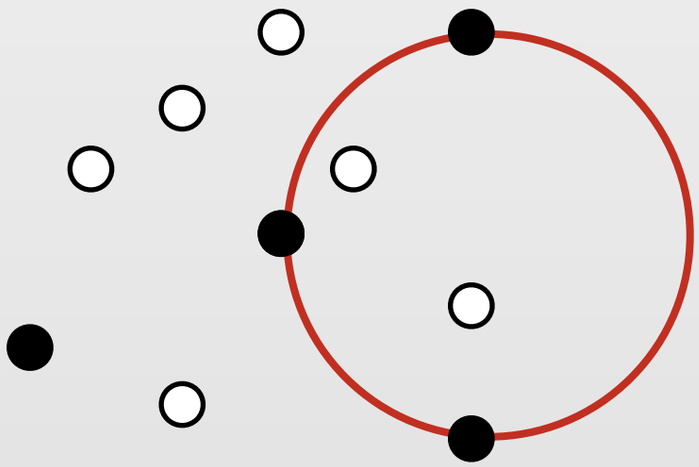

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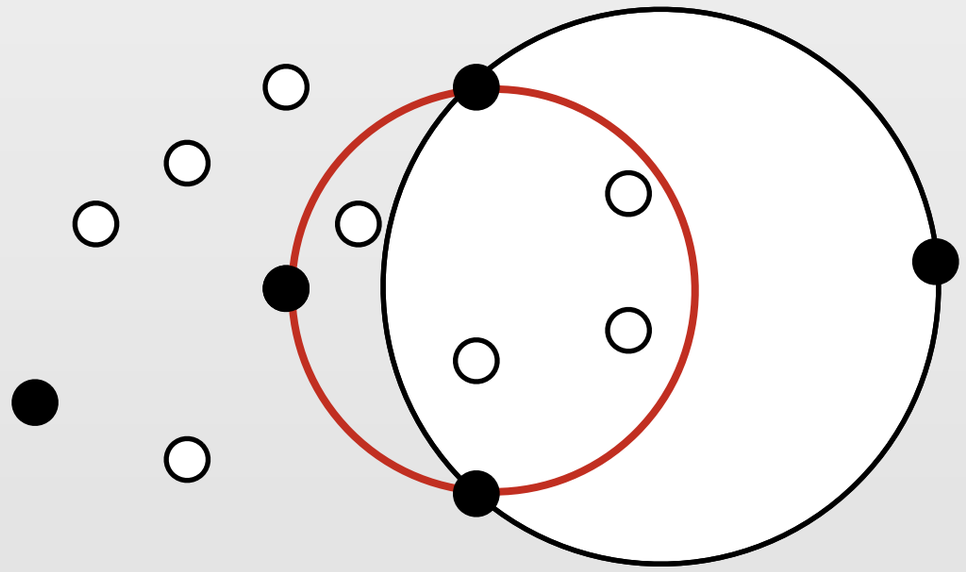

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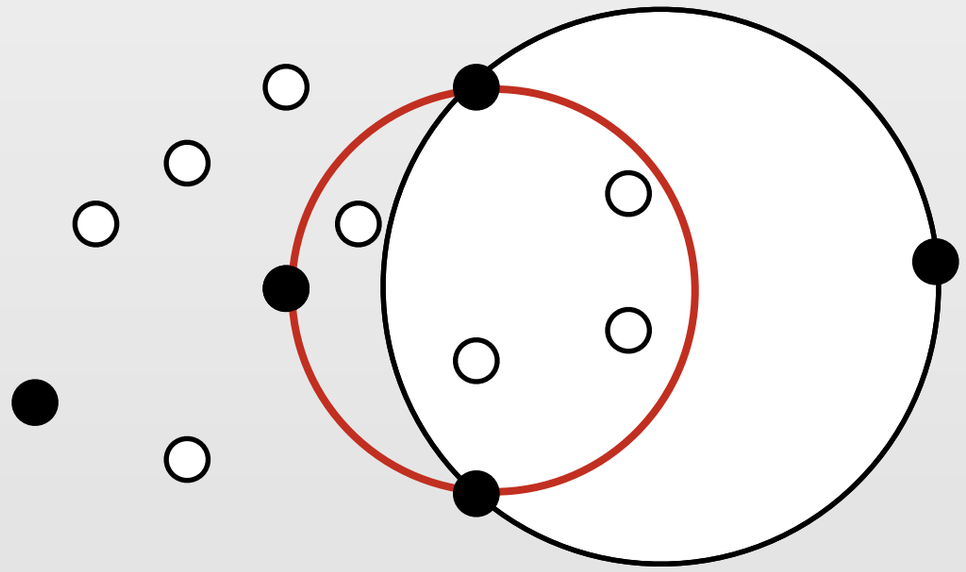

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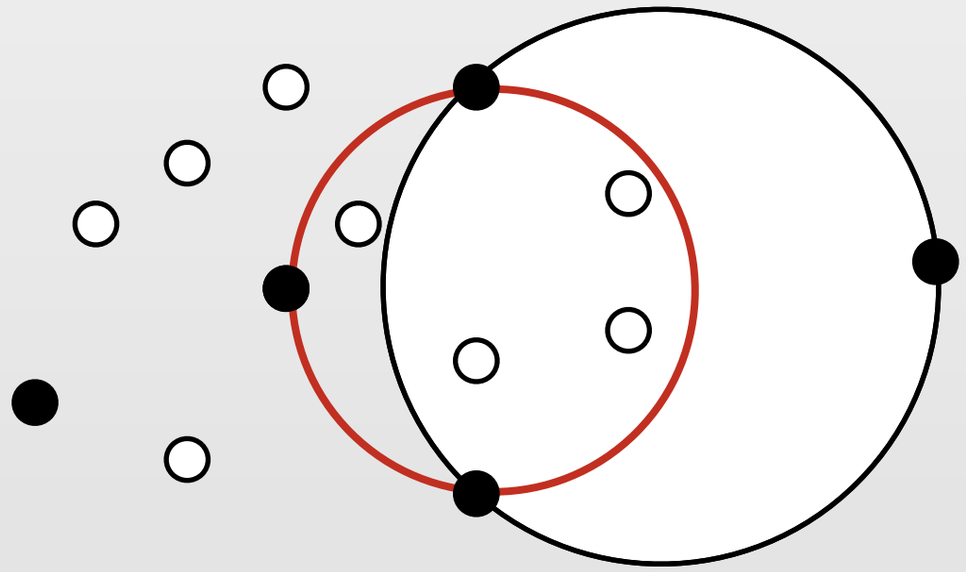

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Geometric Divide and Conquer: $O(n \log \Delta)$

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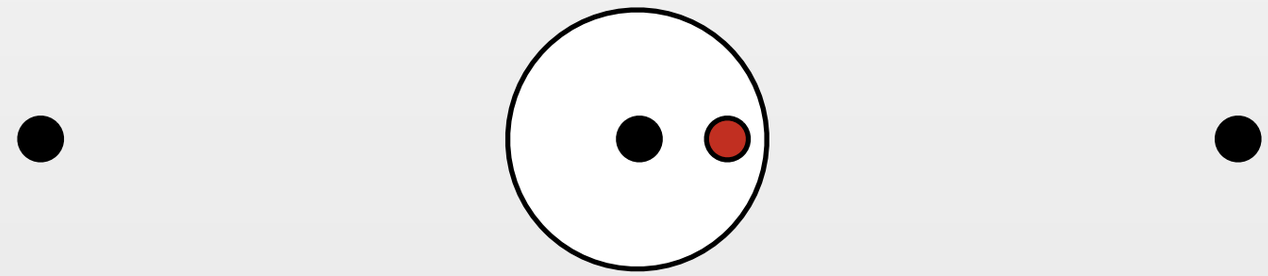

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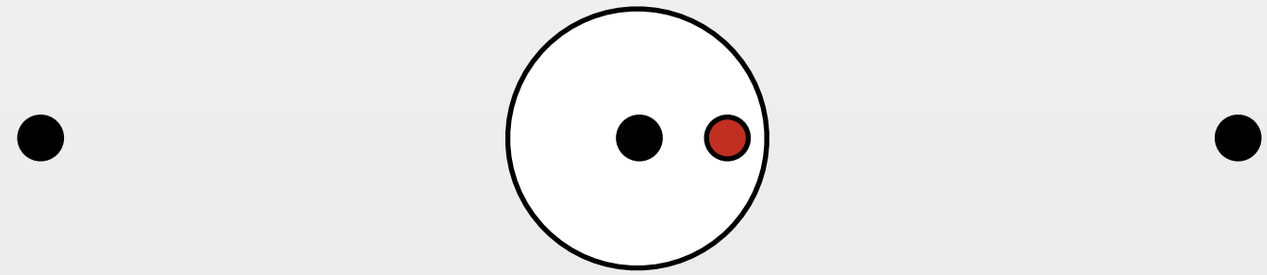

Goal: Combinatorial Divide and Conquer

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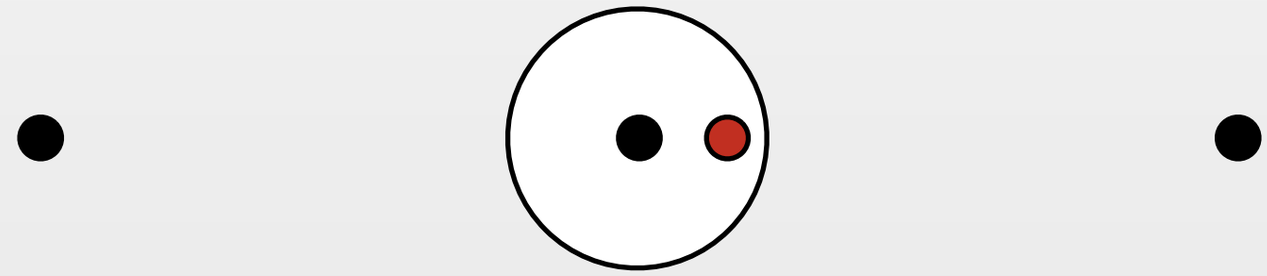

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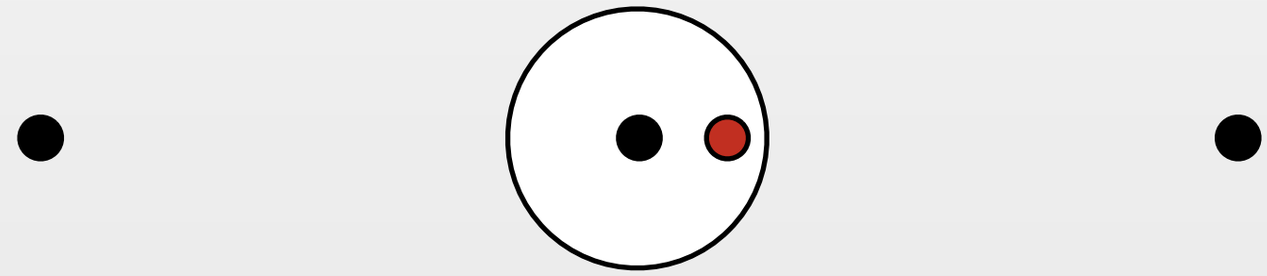

Goal: Combinatorial Divide and Conquer

Old: Progress was decreasing the radius by a factor of 2.

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Range Nets

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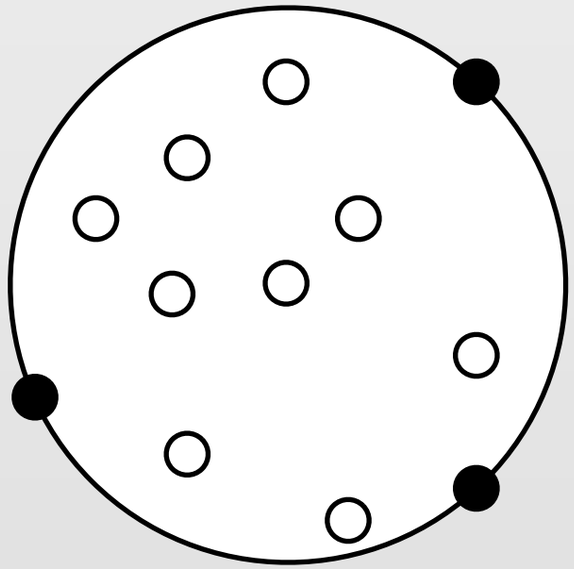
Theorem: [Chazelle & Matousek 96] For ε, d fixed constants, ε-nets of size *O(1)* can be computed in *O(n)* deterministic time.

Ordering the inputs

For each D-Ball, select a 1/2d-net of the points it contains.
Take the union of these nets and call it a round.
Insert these.
Repeat.



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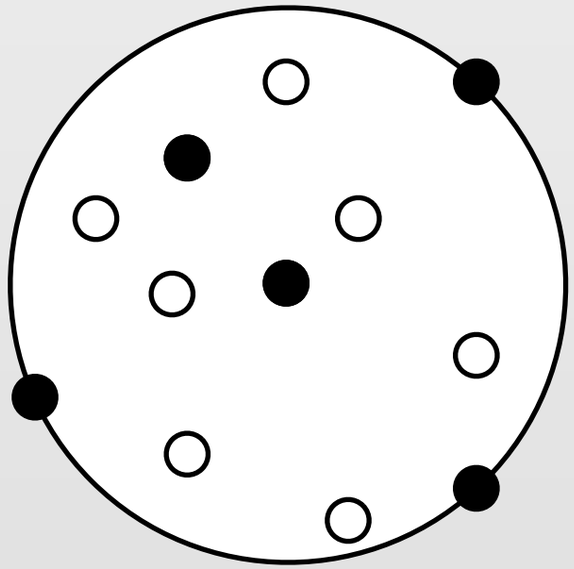

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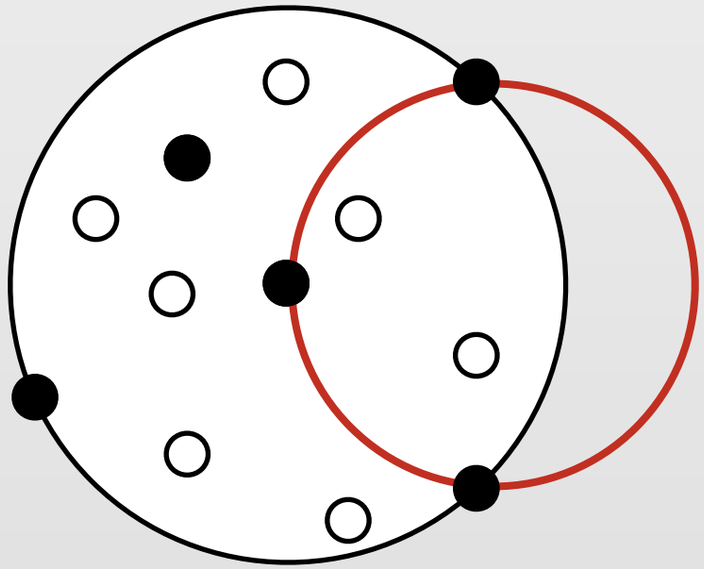

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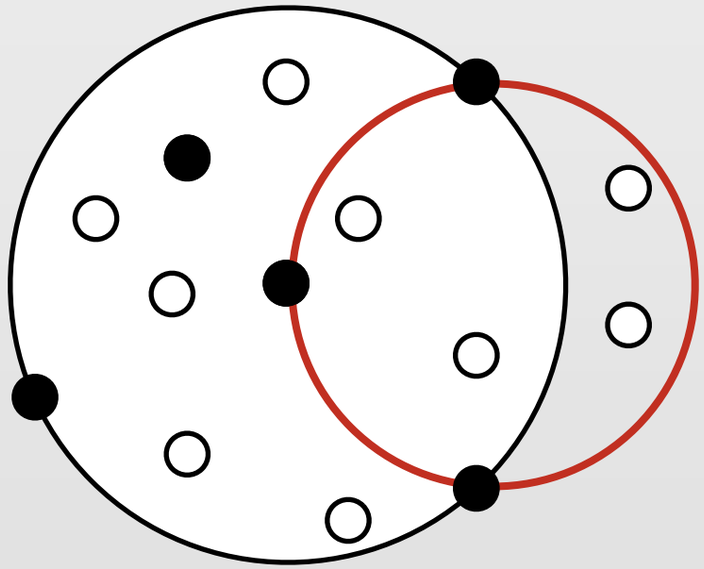

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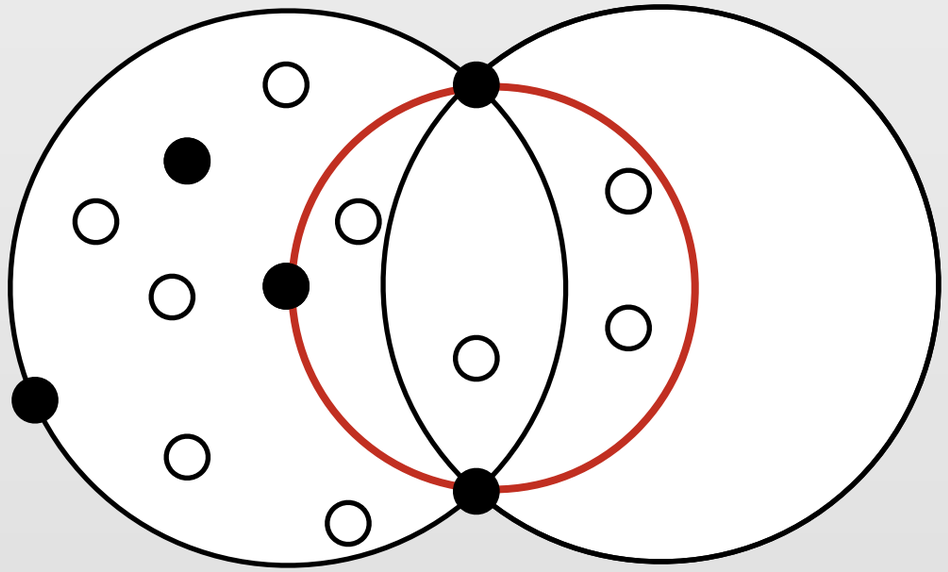

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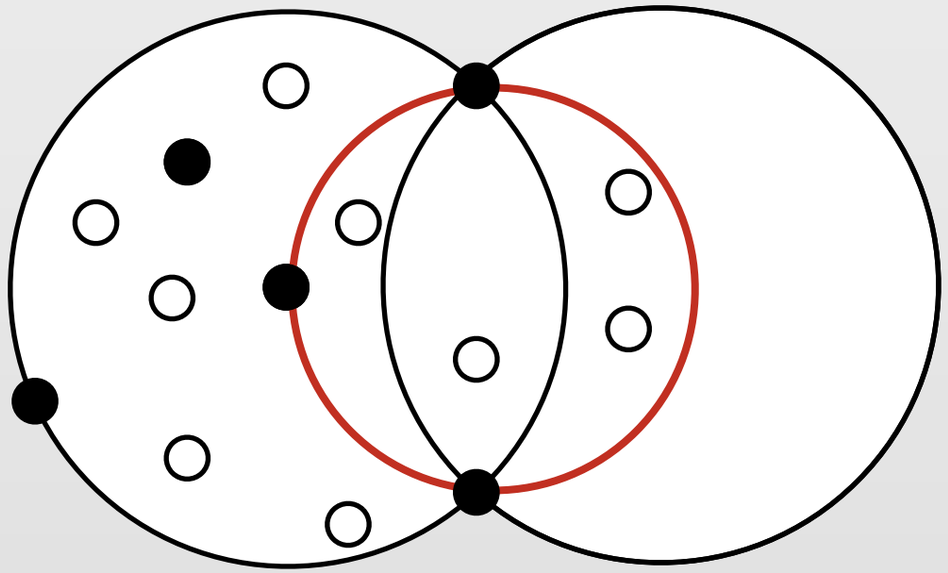

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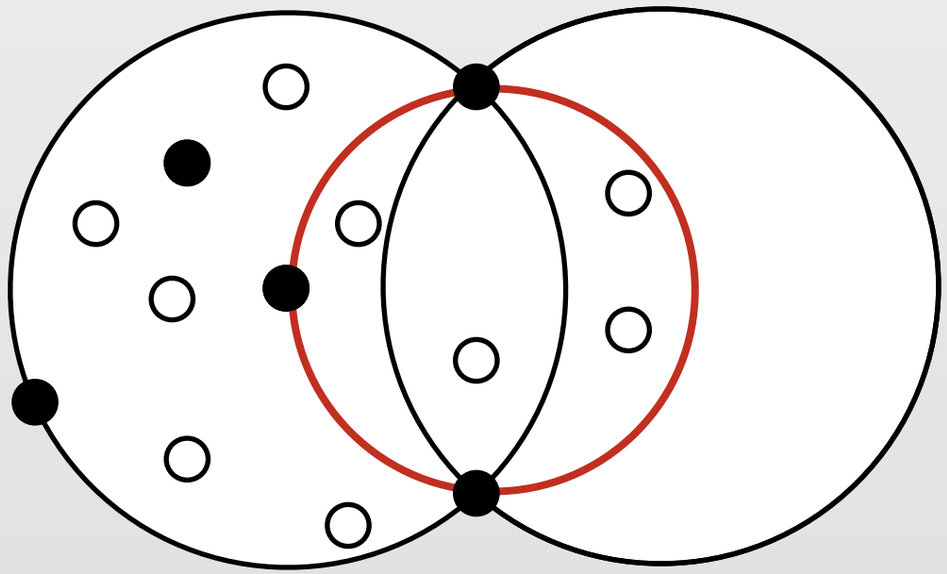

log(n) Rounds

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Watch an uninserted point x.
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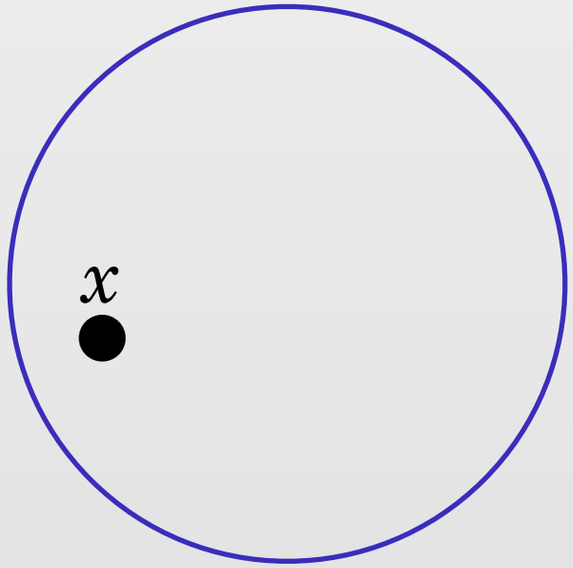

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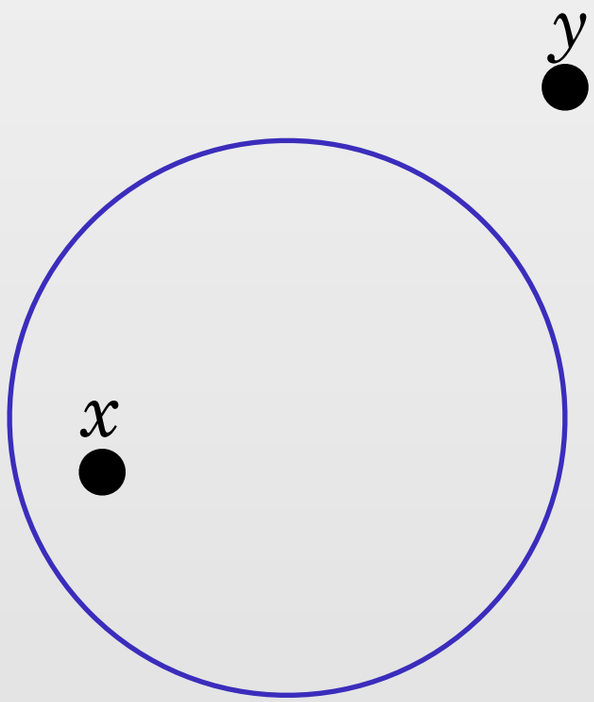

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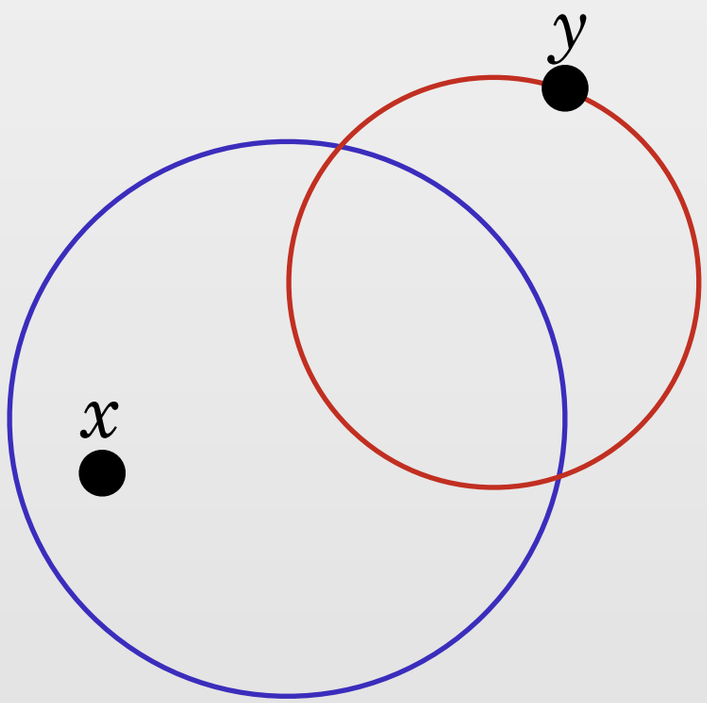

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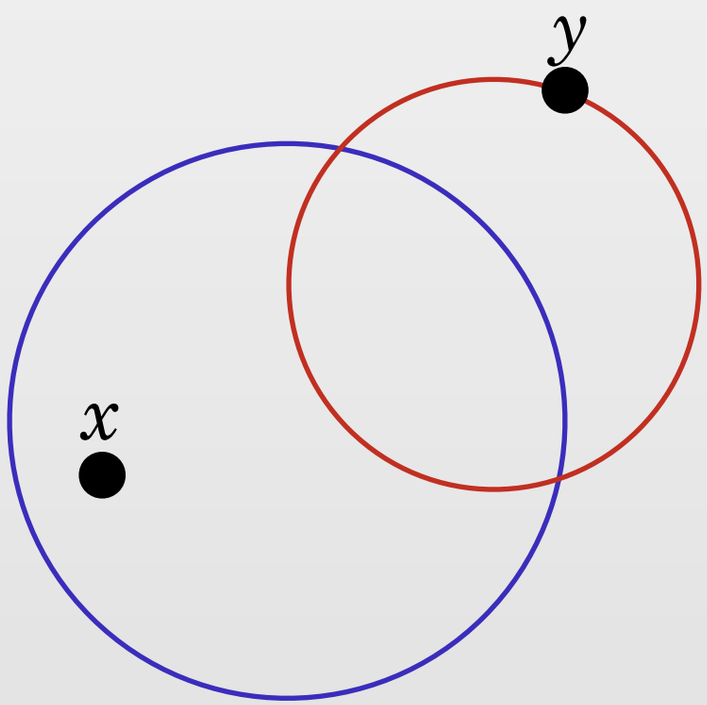

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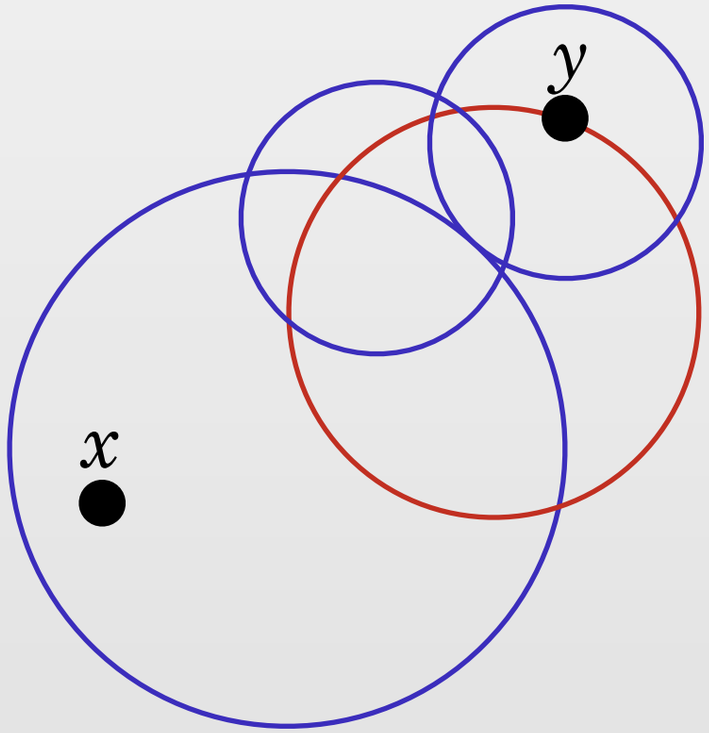

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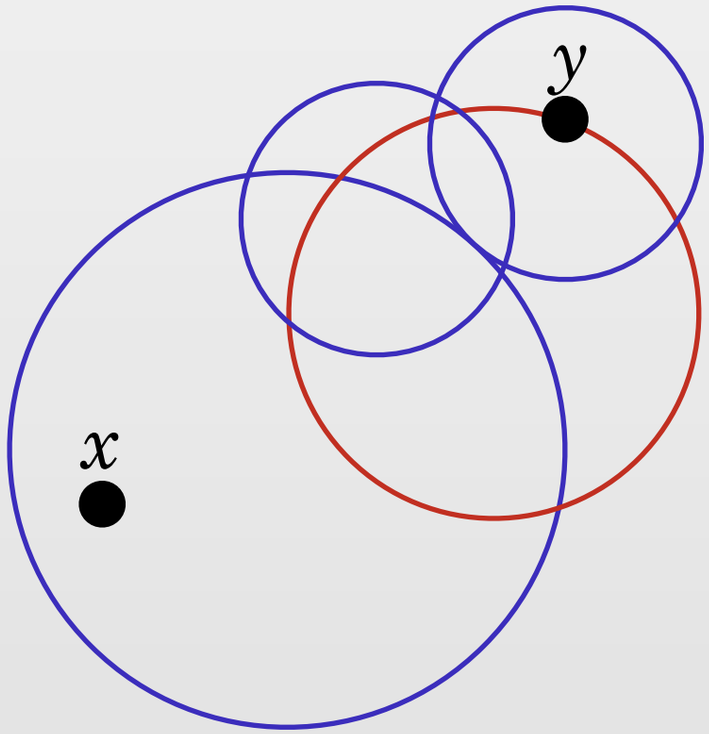

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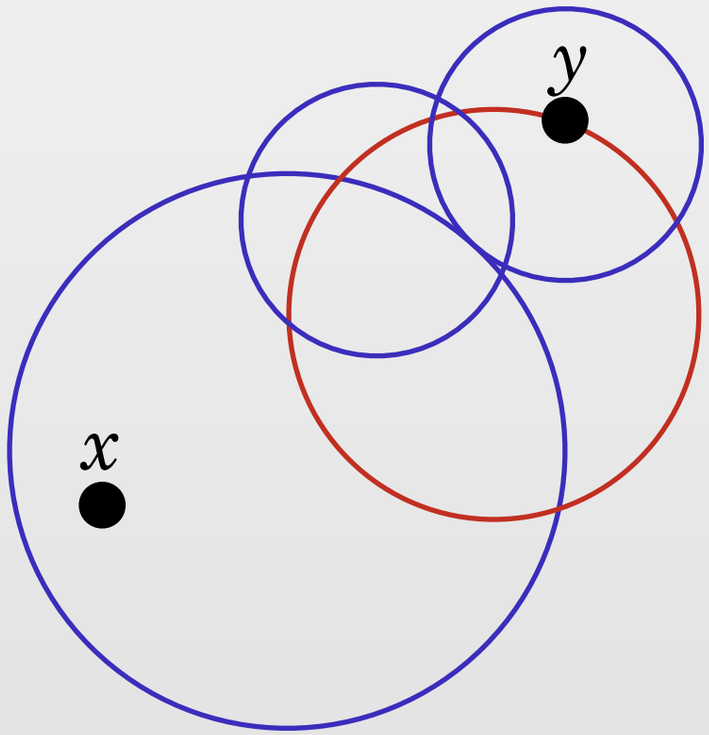

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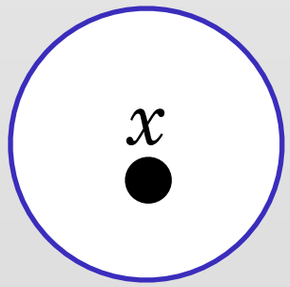

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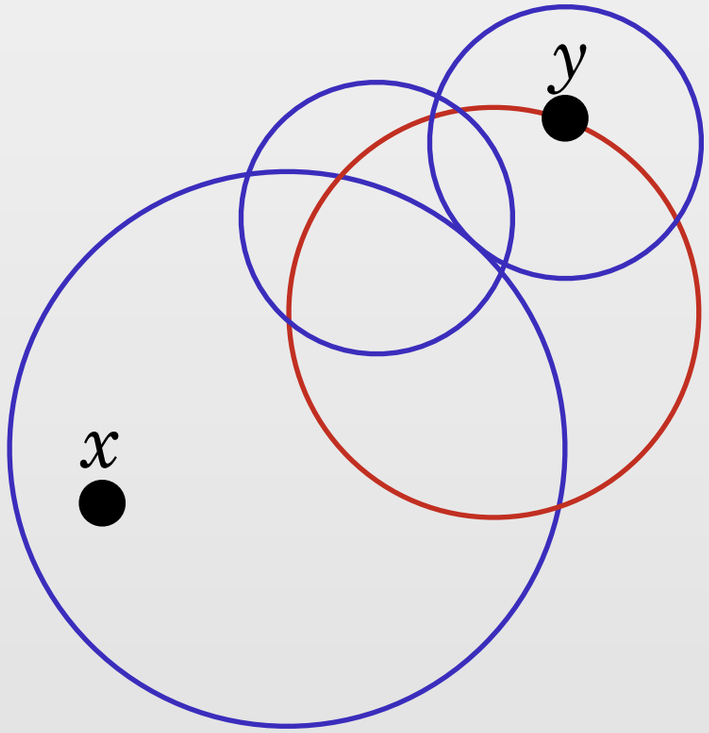

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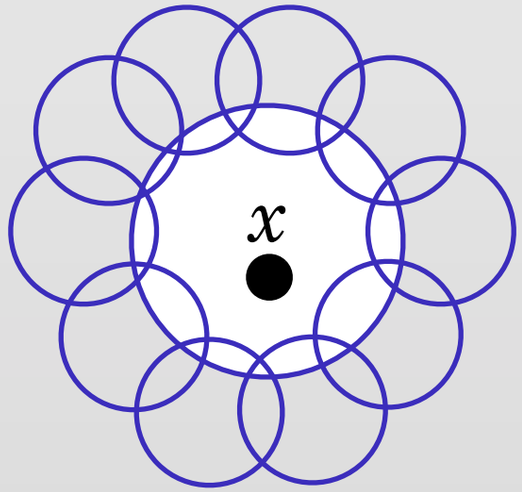

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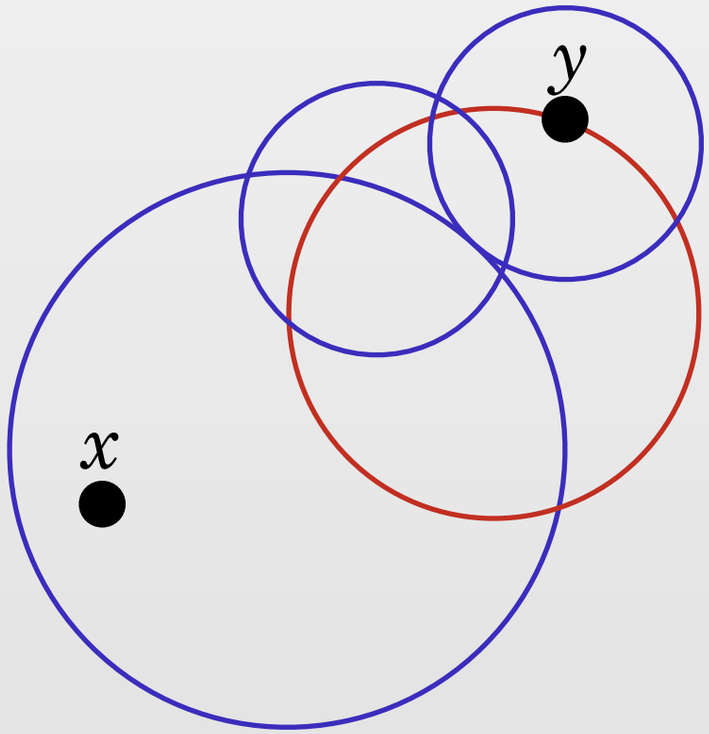

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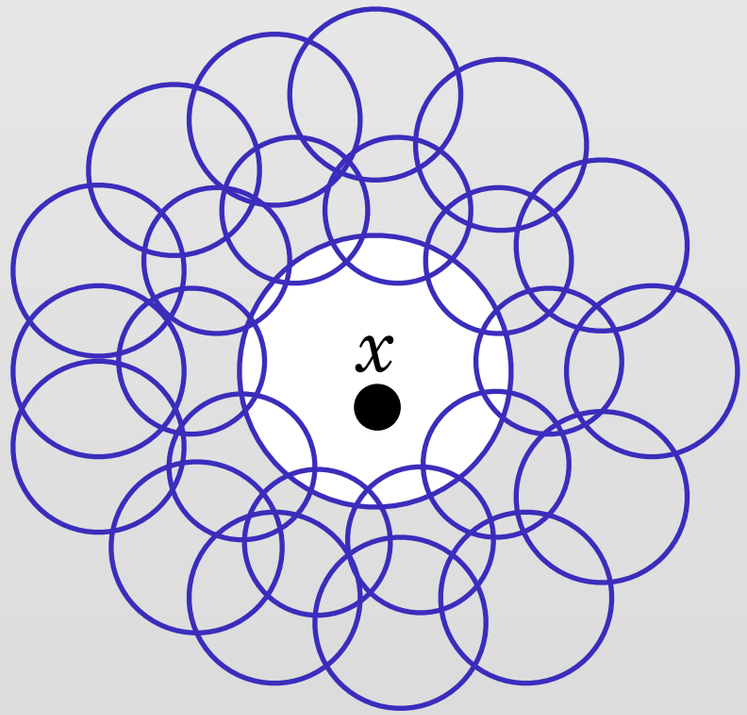

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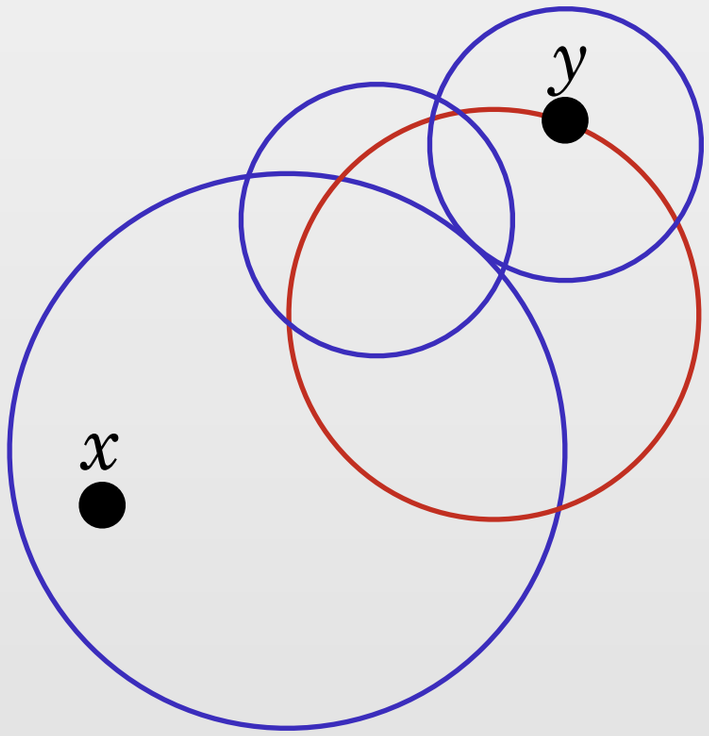

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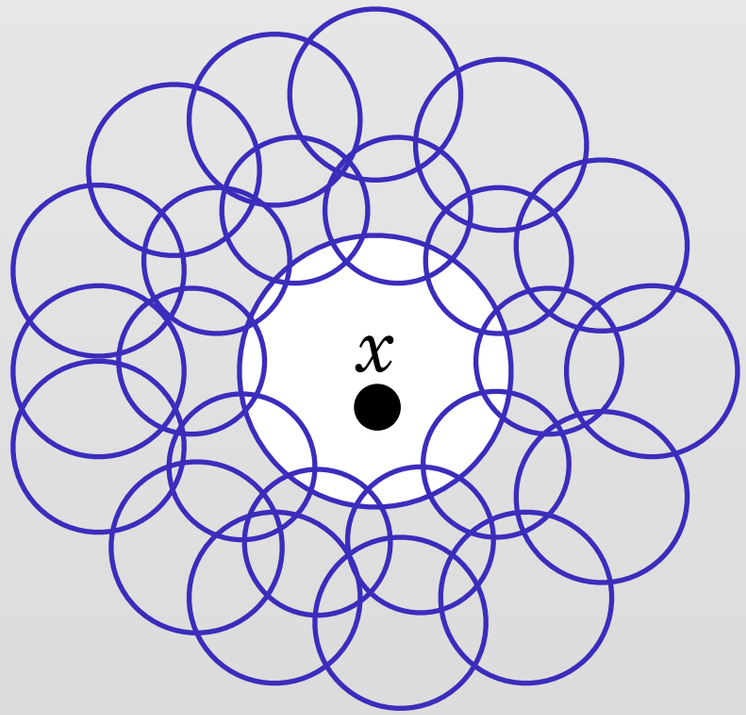

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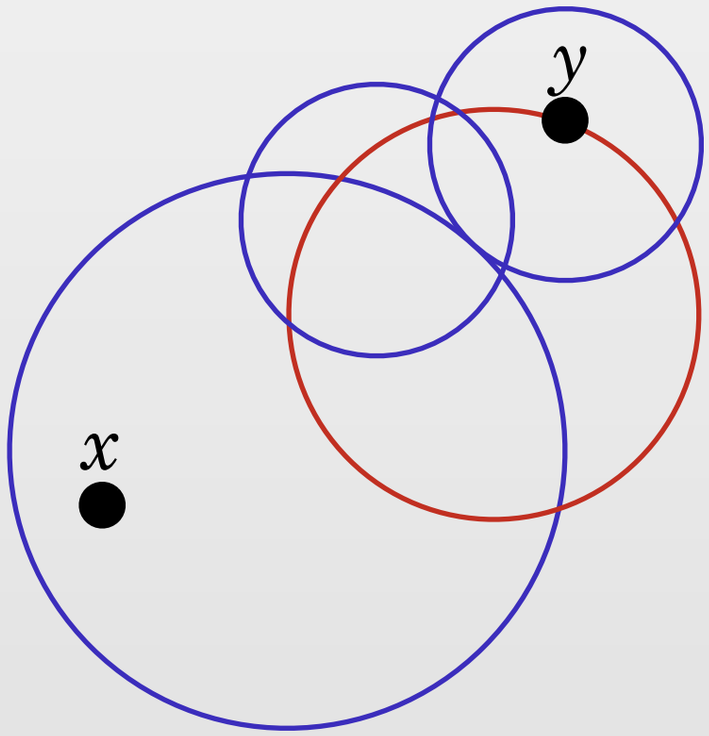

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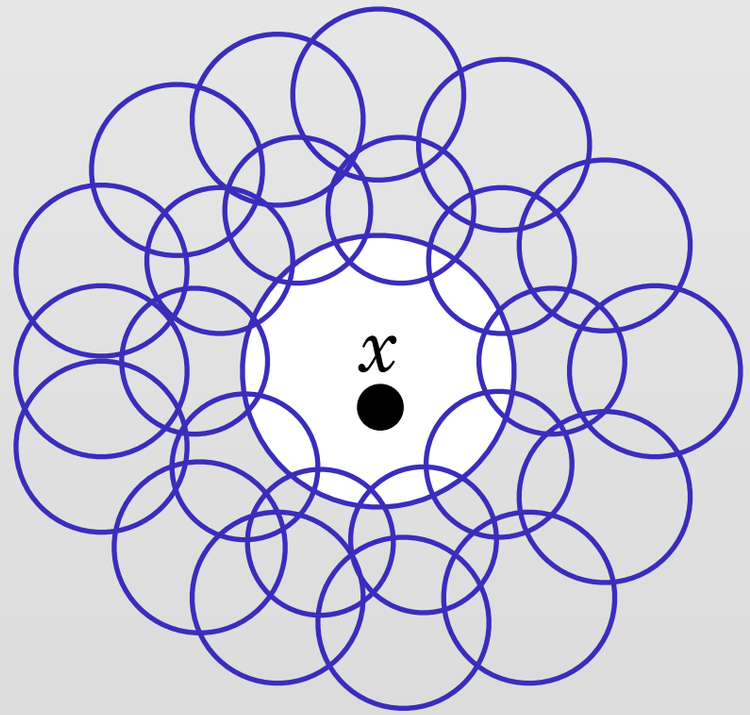

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O(n) total work per round.

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There are many possible new applications, waiting to be discovered (i.e. any time you would have used a Voronoi diagram).

Thanks.