

Learning with Nets and Meshes

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Thanks to my collaborators:

Benoit Hudson

Gary Miller

Todd Phillips

Steve Oudot

Point Clouds in low to medium dimensional ambient space.

Rule of thumb: $d!$ or 2^{d^2} is okay but n^d is *not*.

There are many geometric inference problems that could benefit from meshing.

4

Discretize Space

Approximate Functions

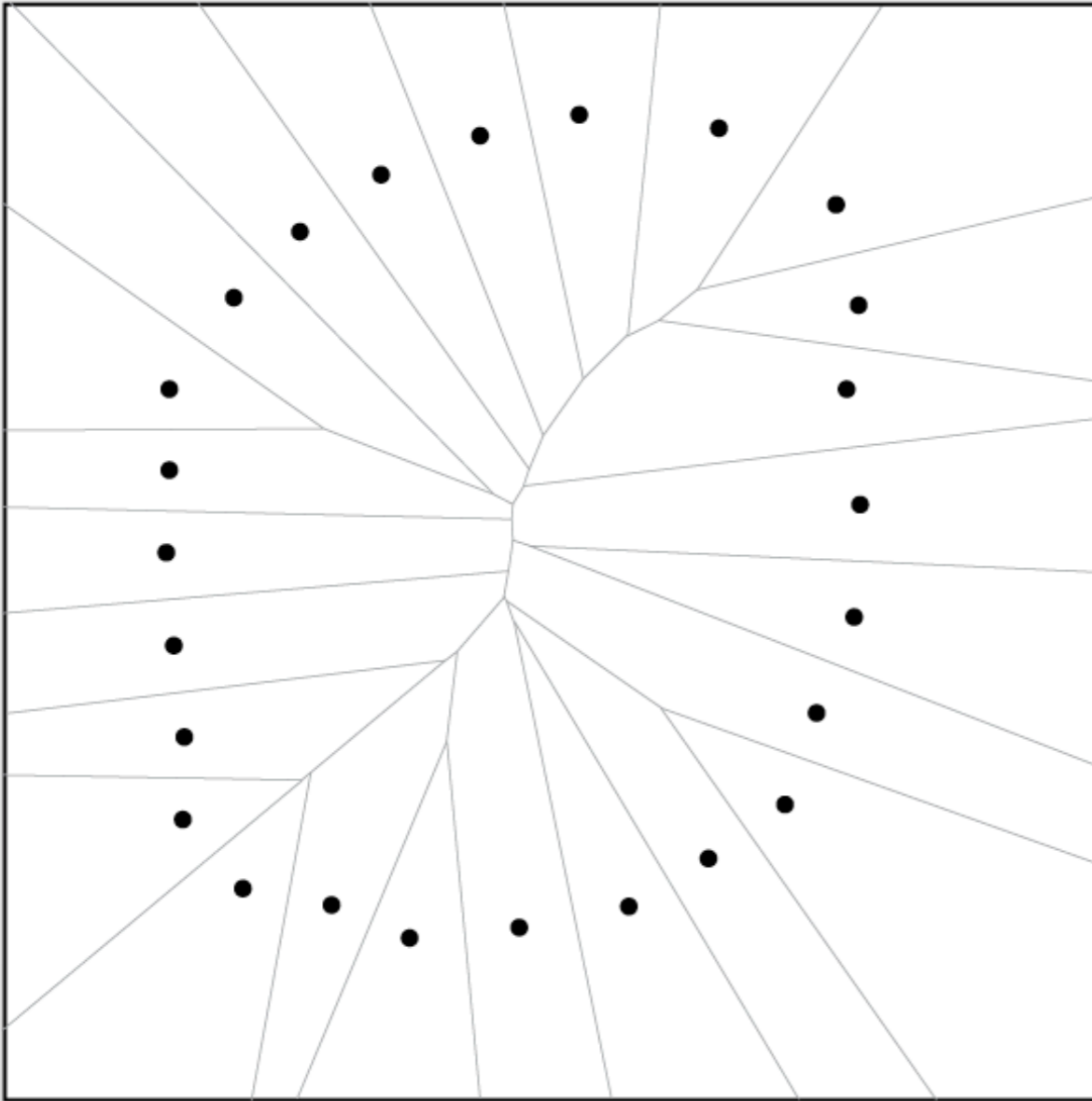
Adapt to density

Describe the space *around* the input.

Meshing

Input: $P \subset \mathbb{R}^d$ $n = |P|$

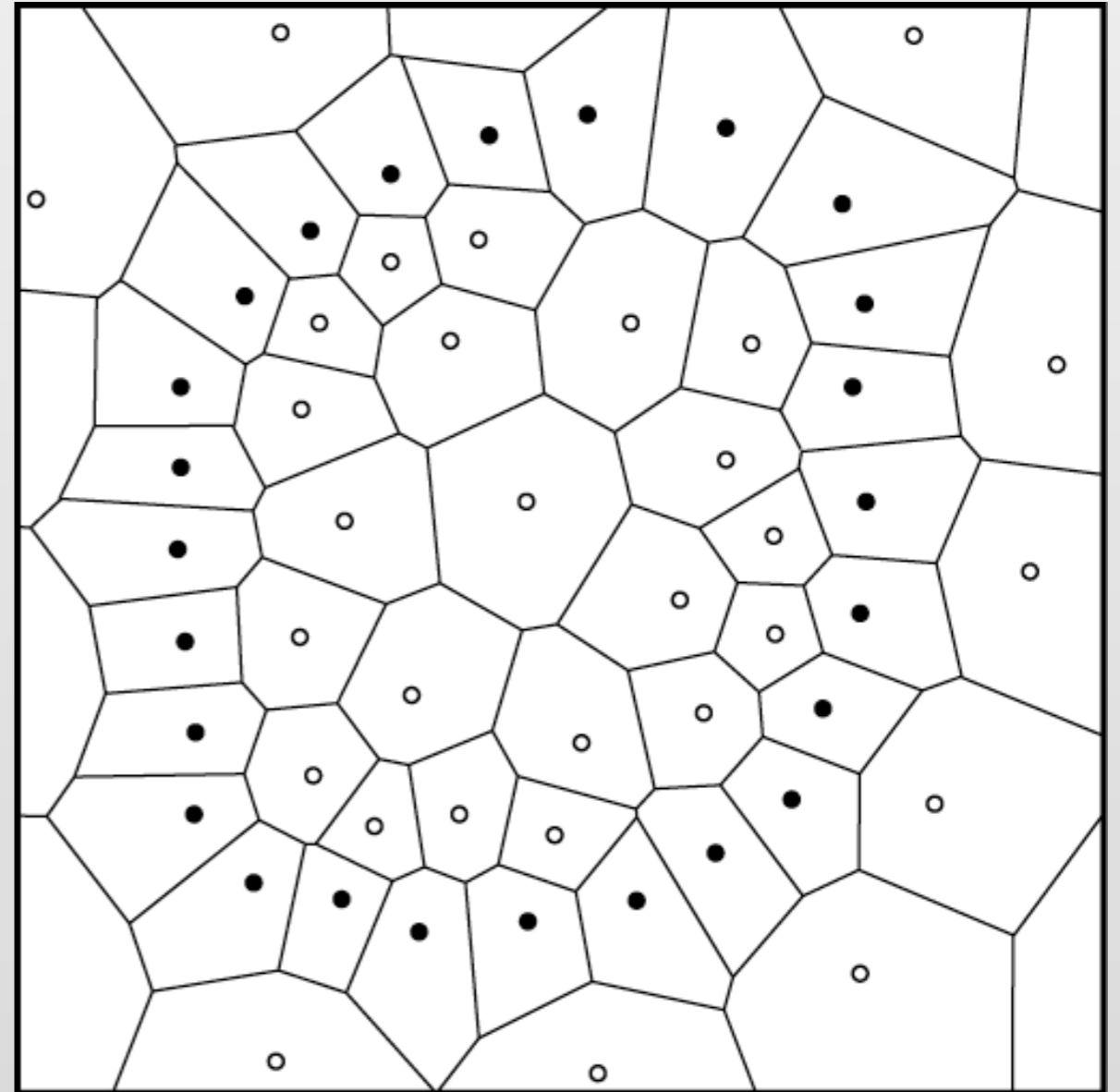
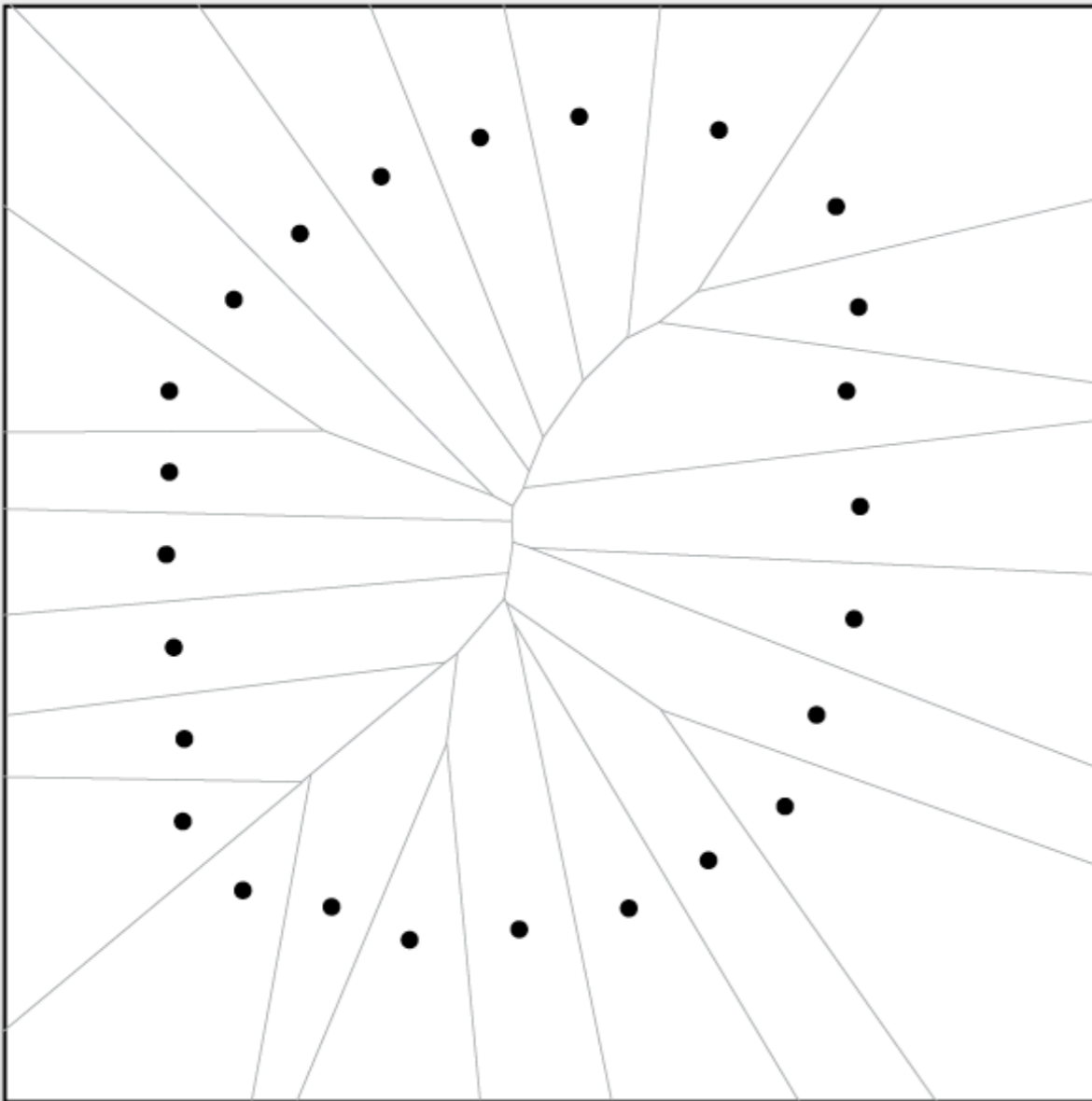
Output: $M \supset P$ with a “nice” Voronoi diagram



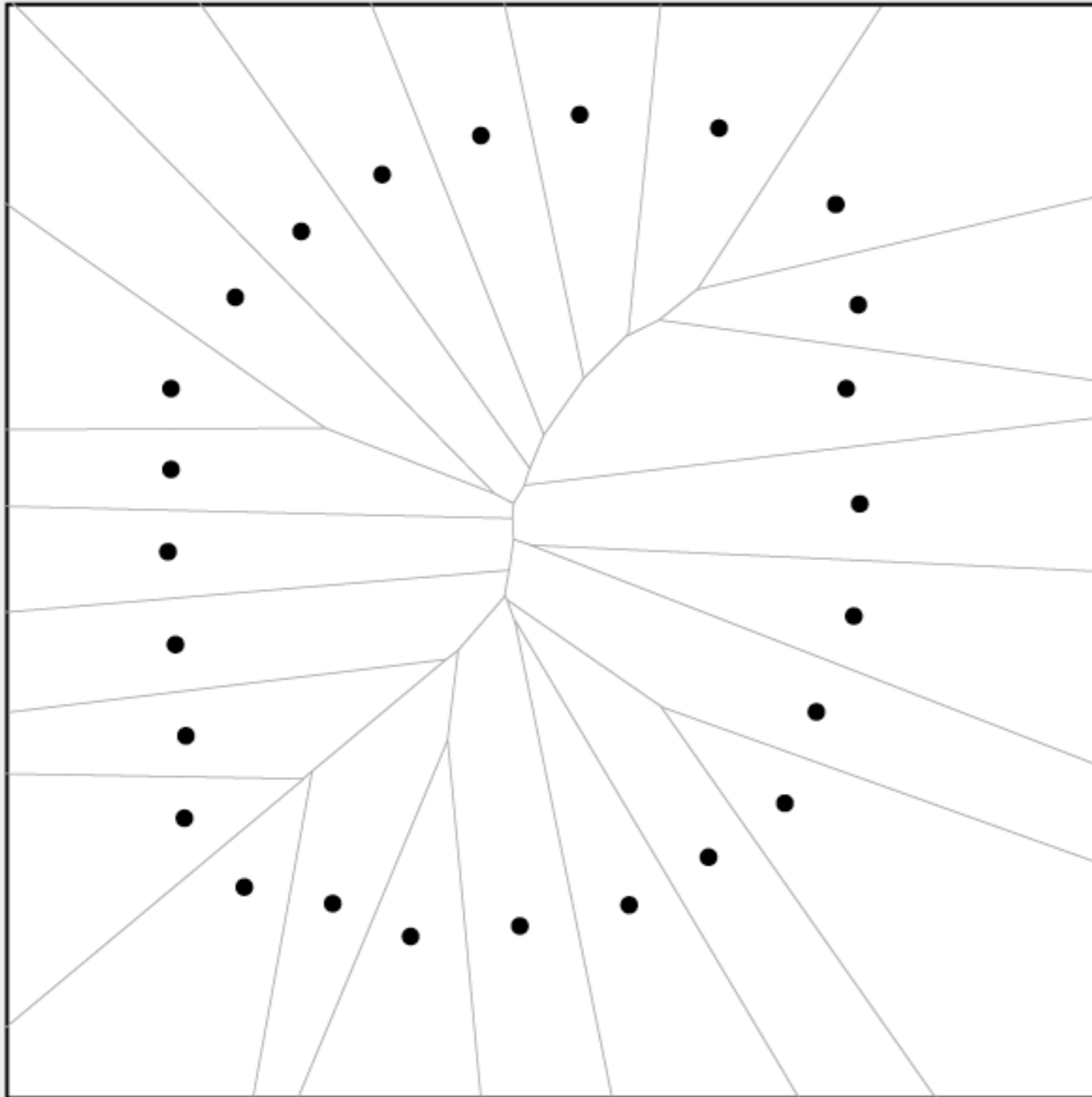
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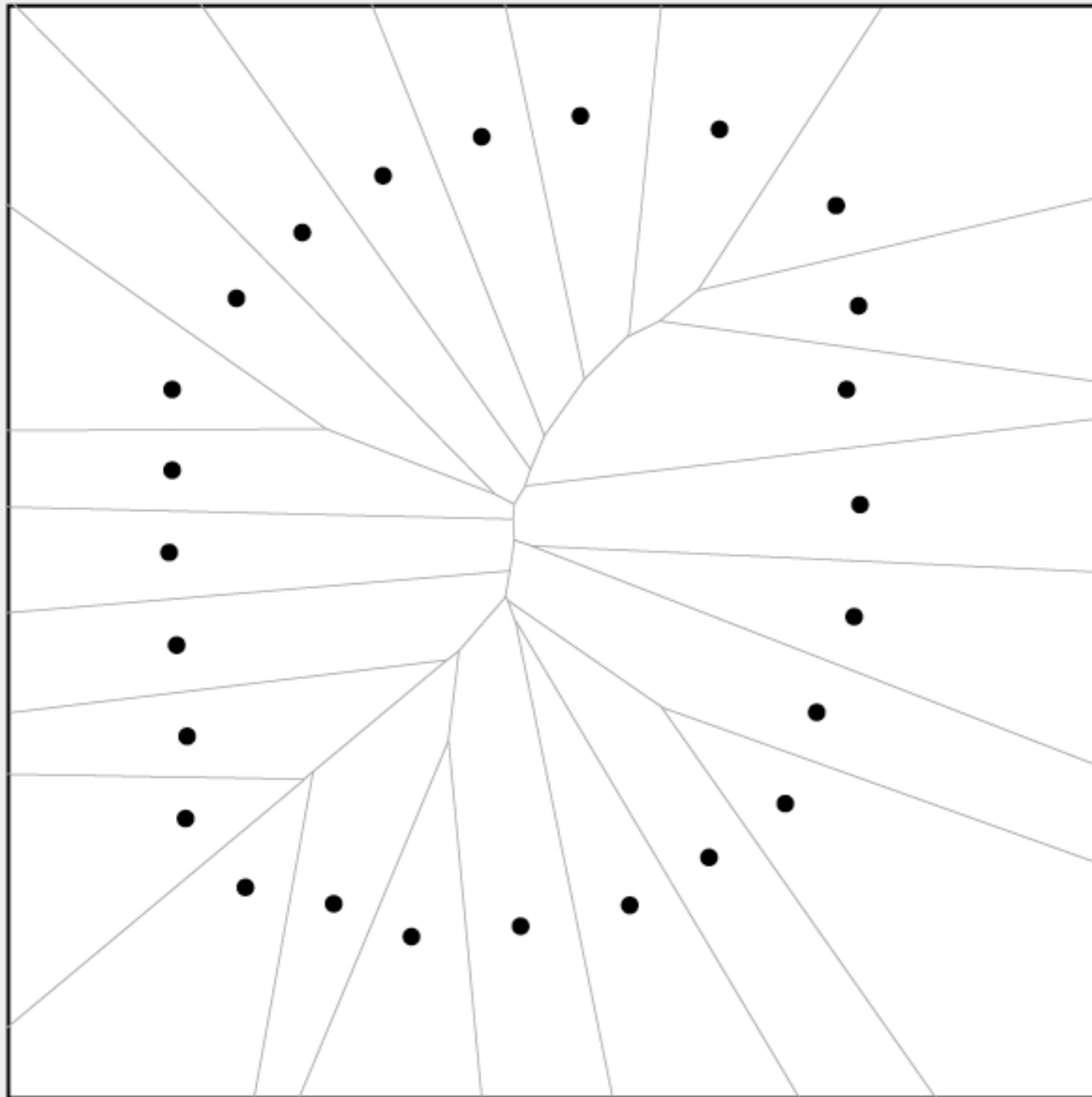
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Points, offsets, homology, and persistence.

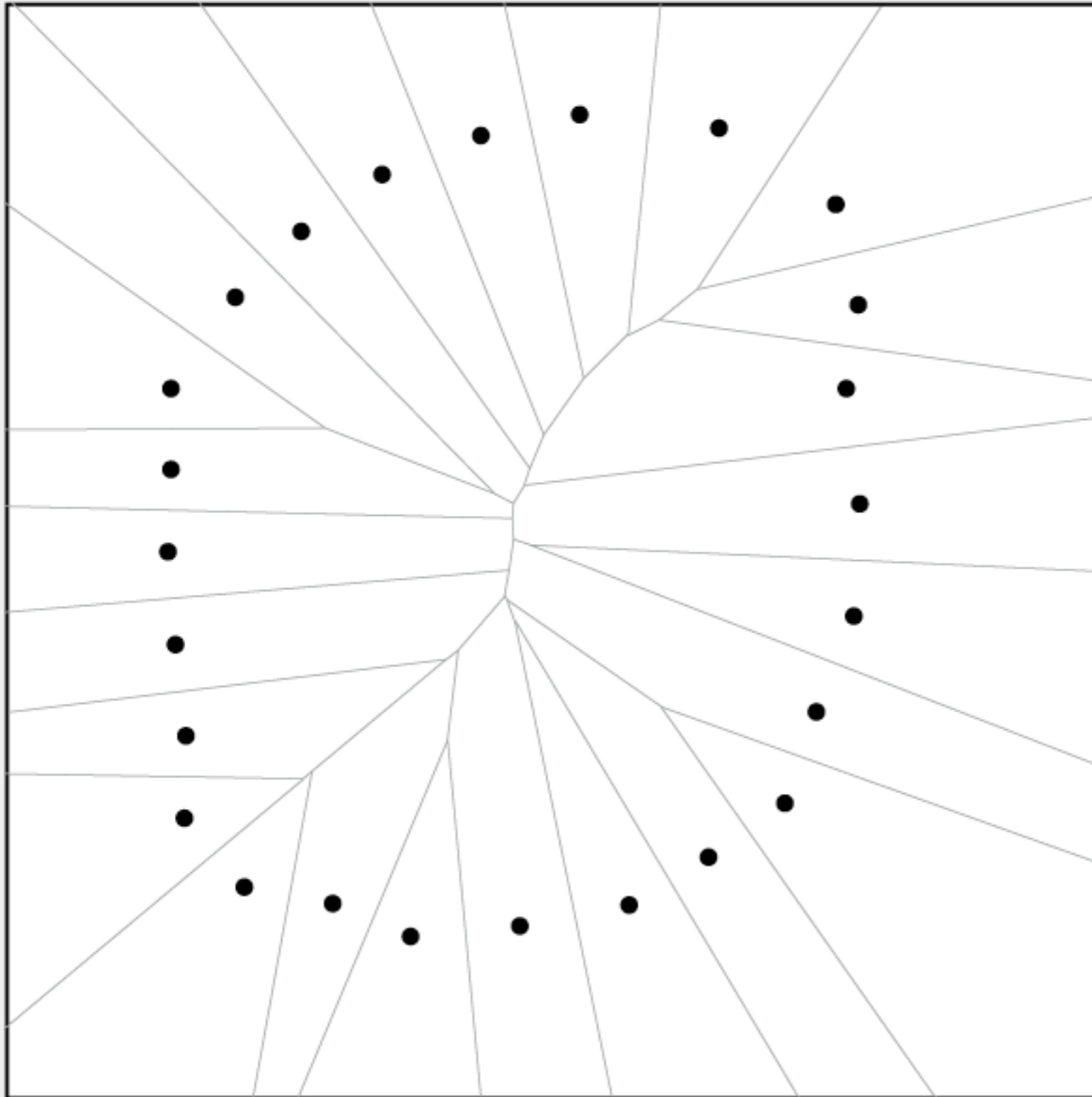


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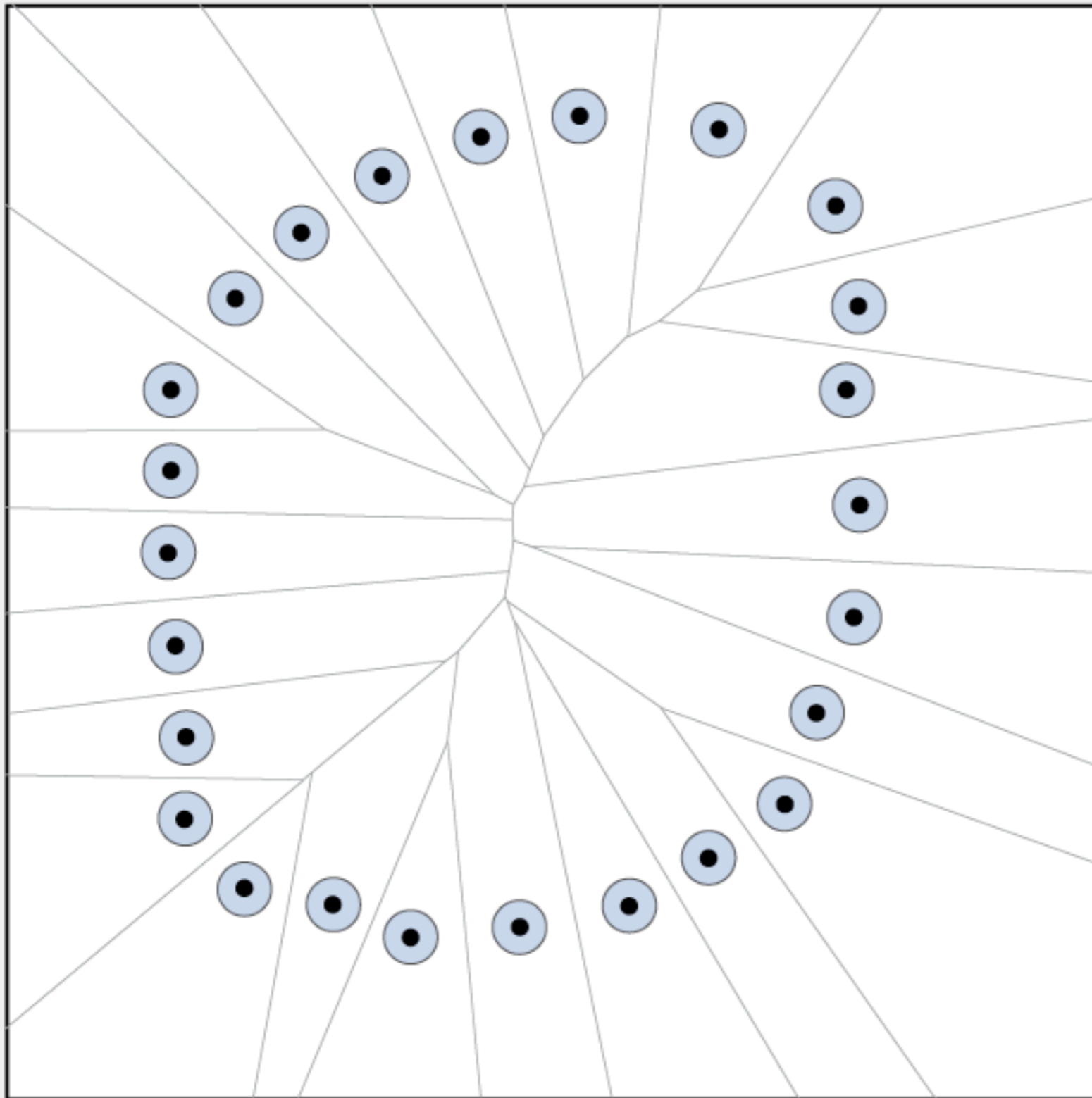
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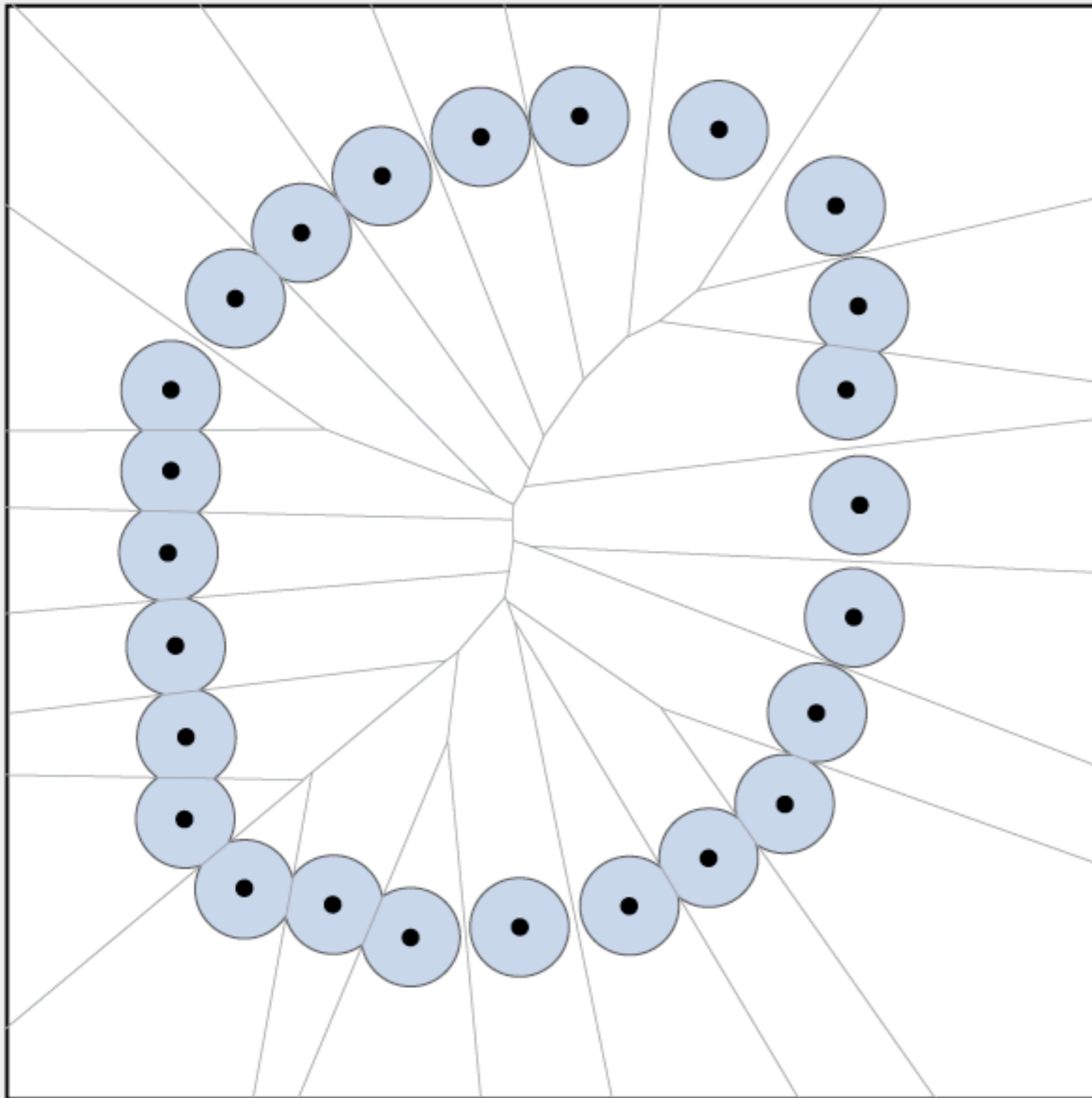
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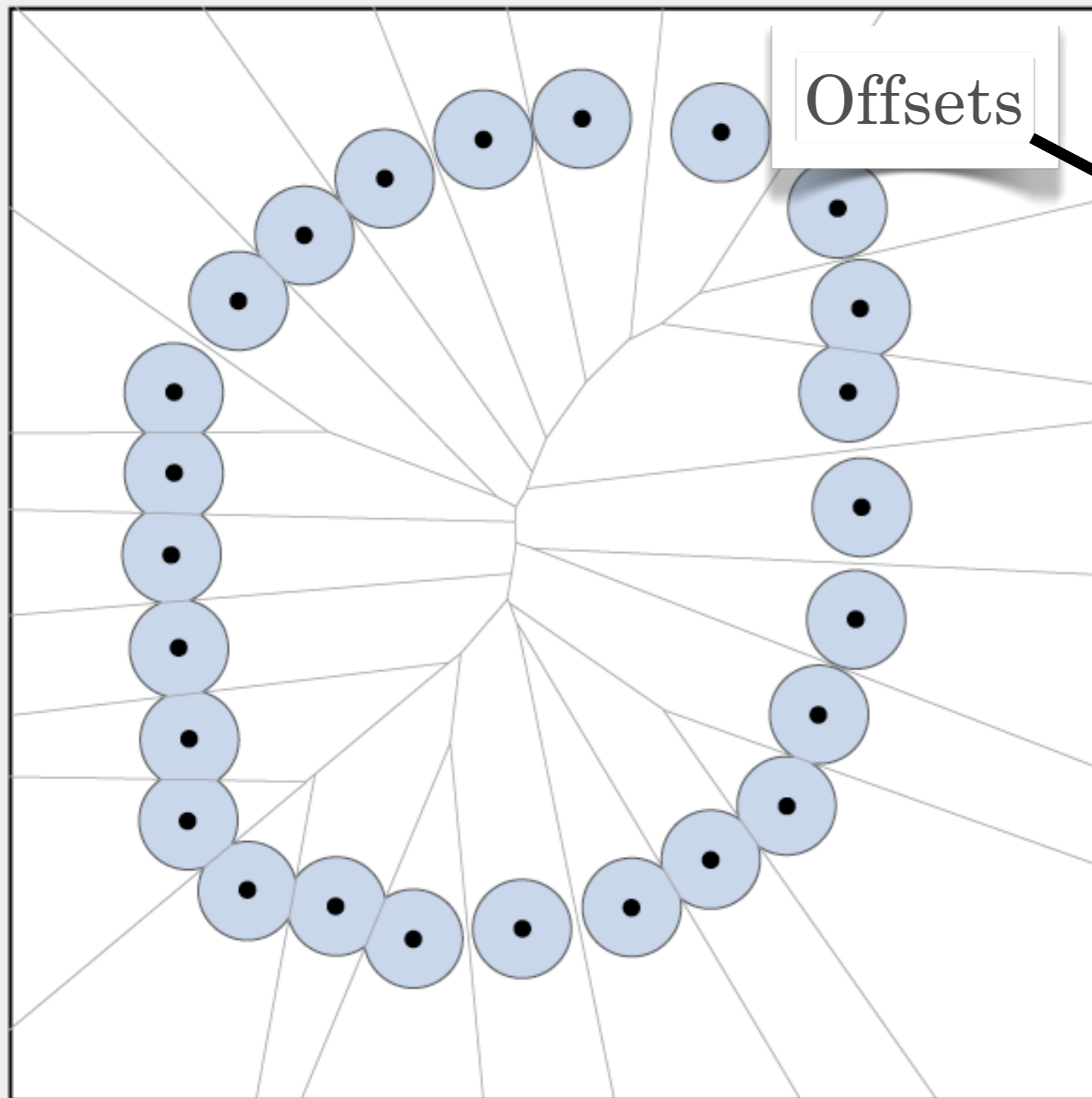
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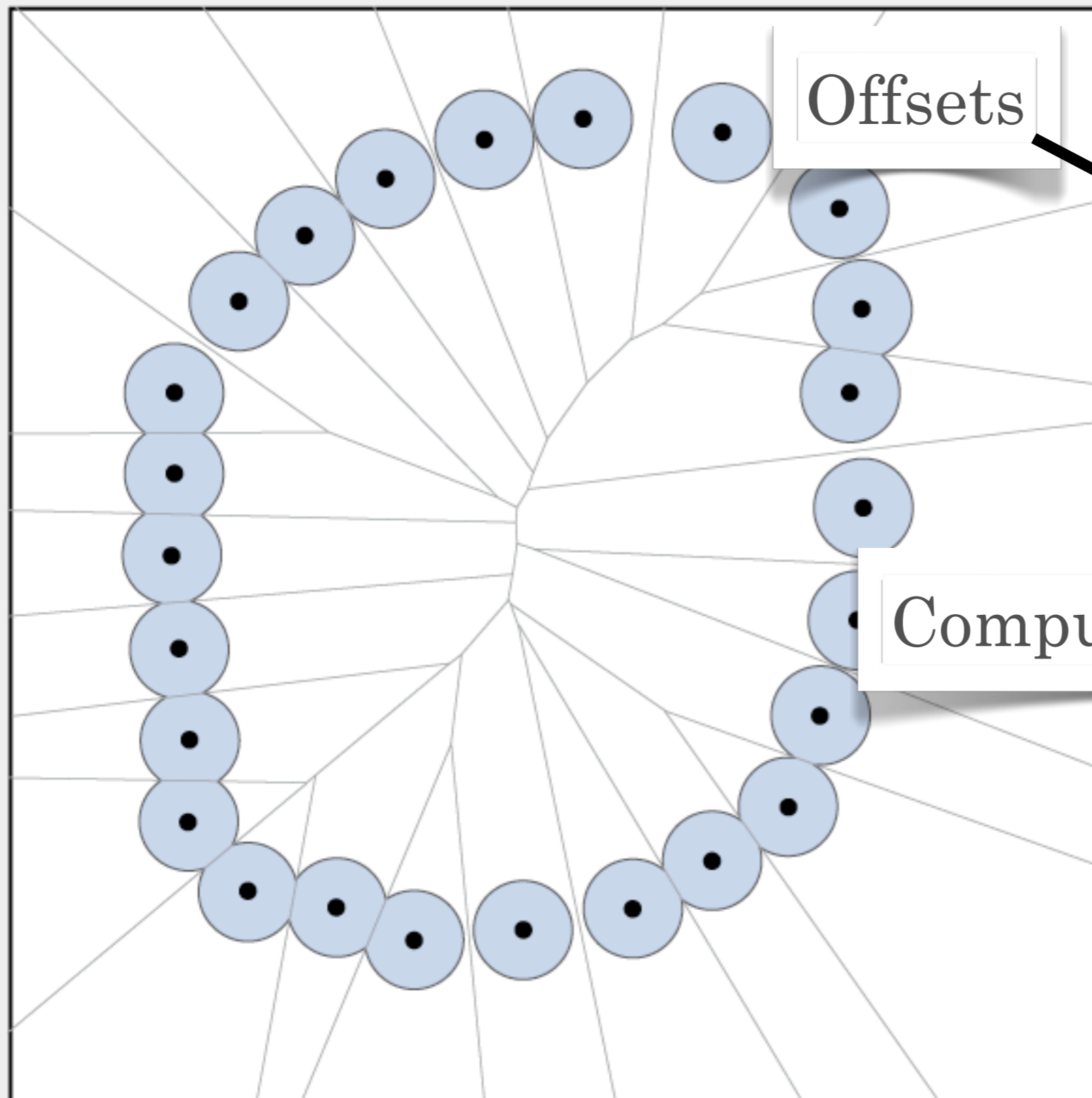
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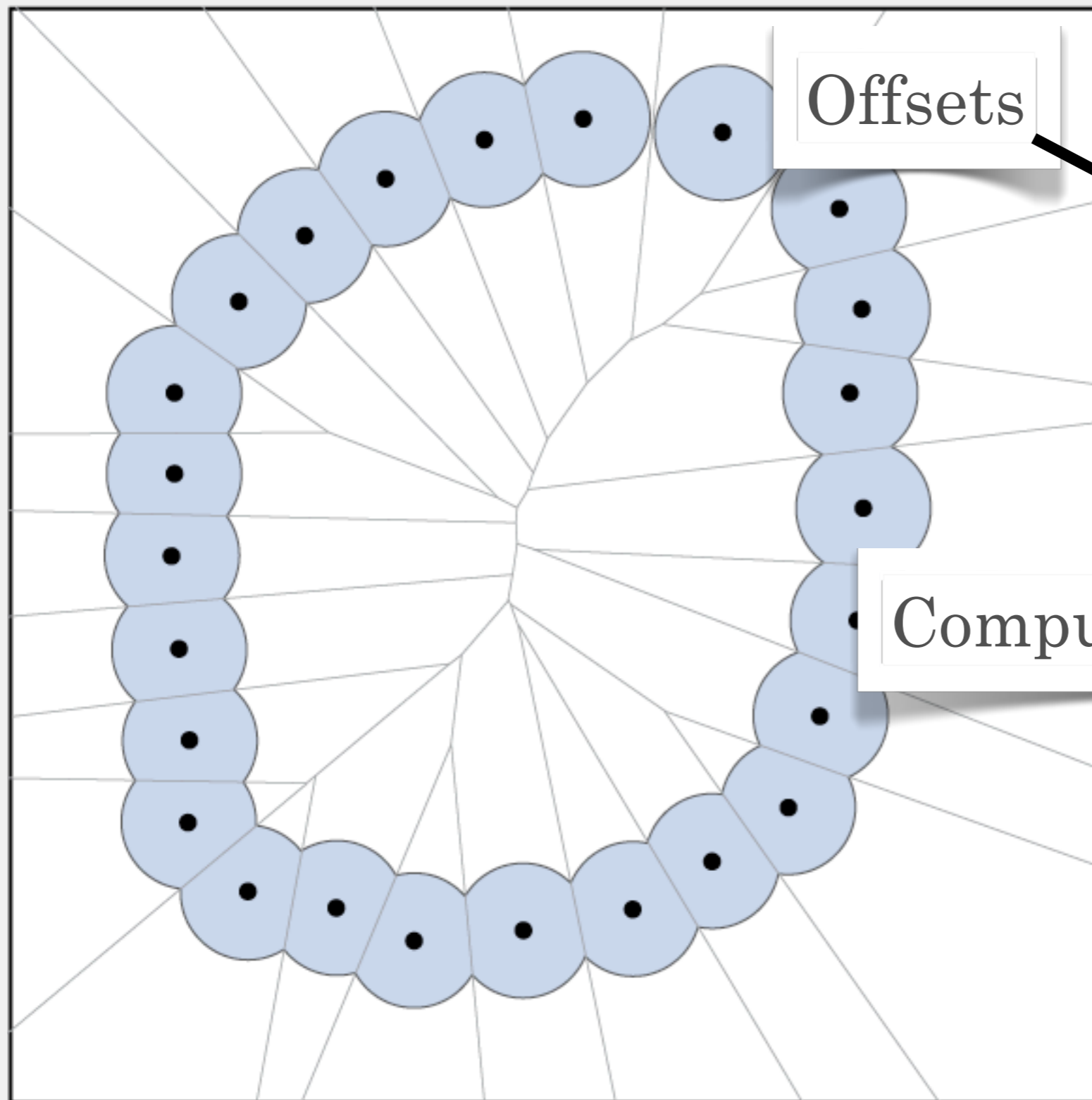


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Points, offsets, homology, and persistence.



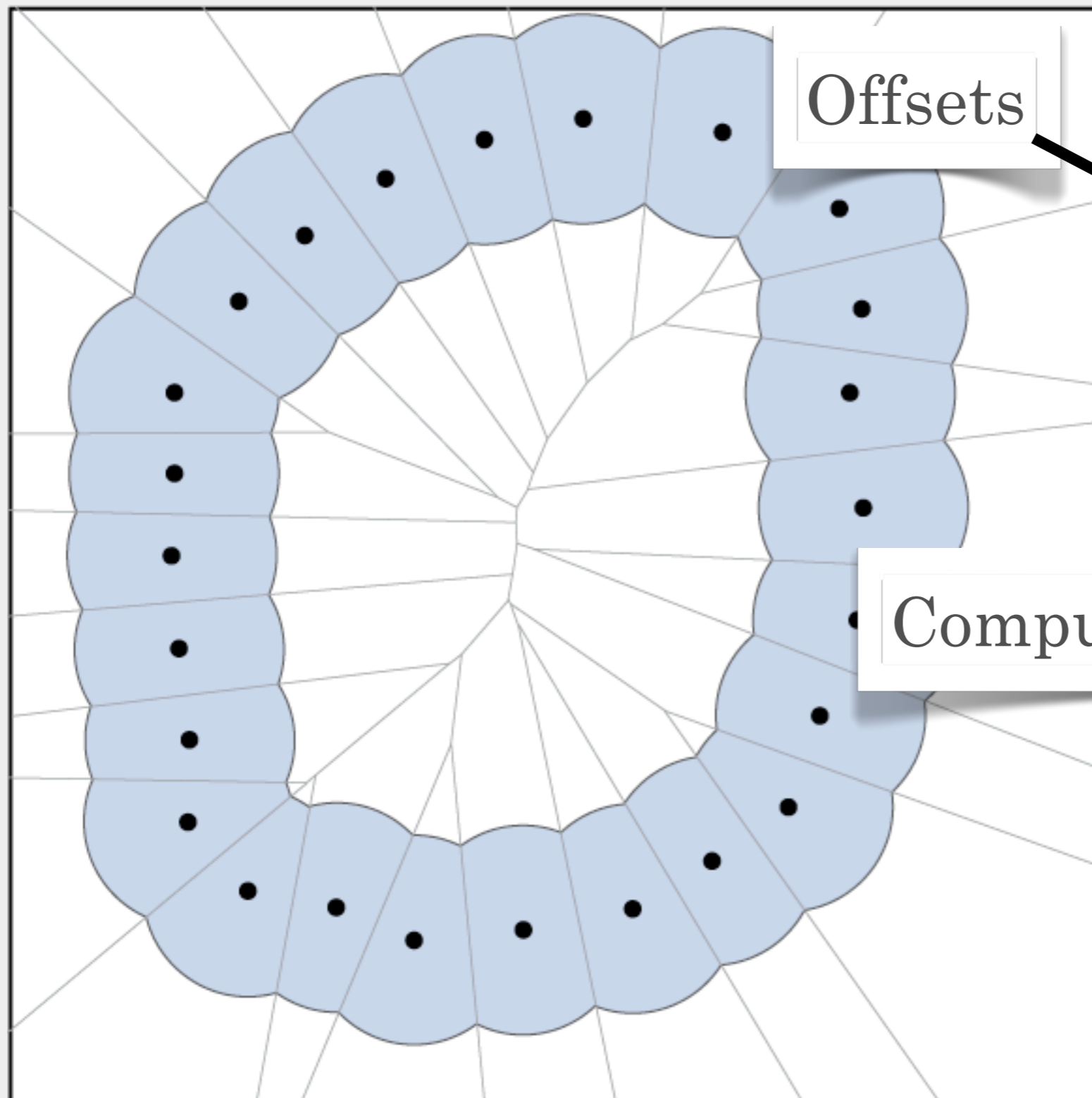
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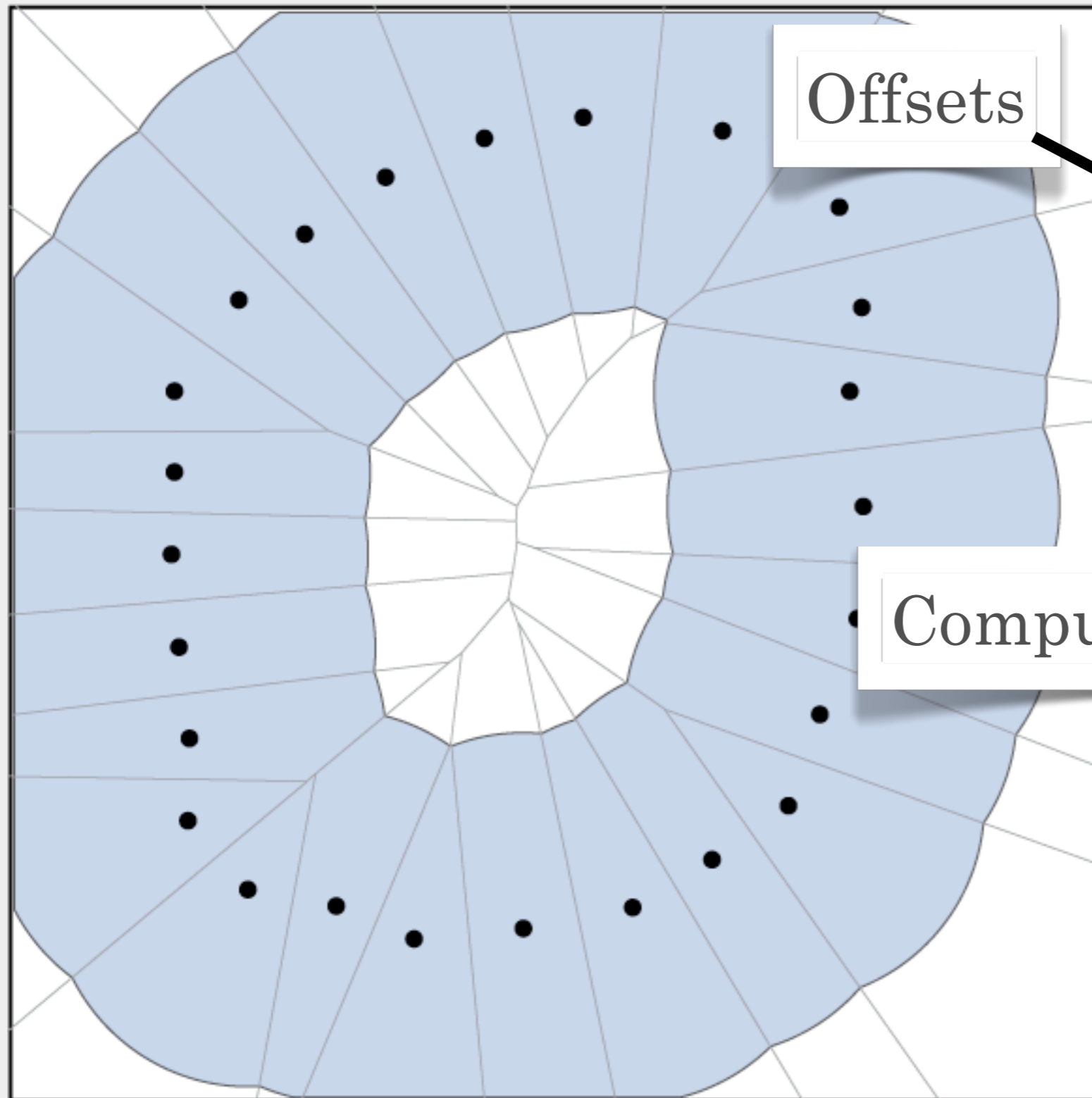
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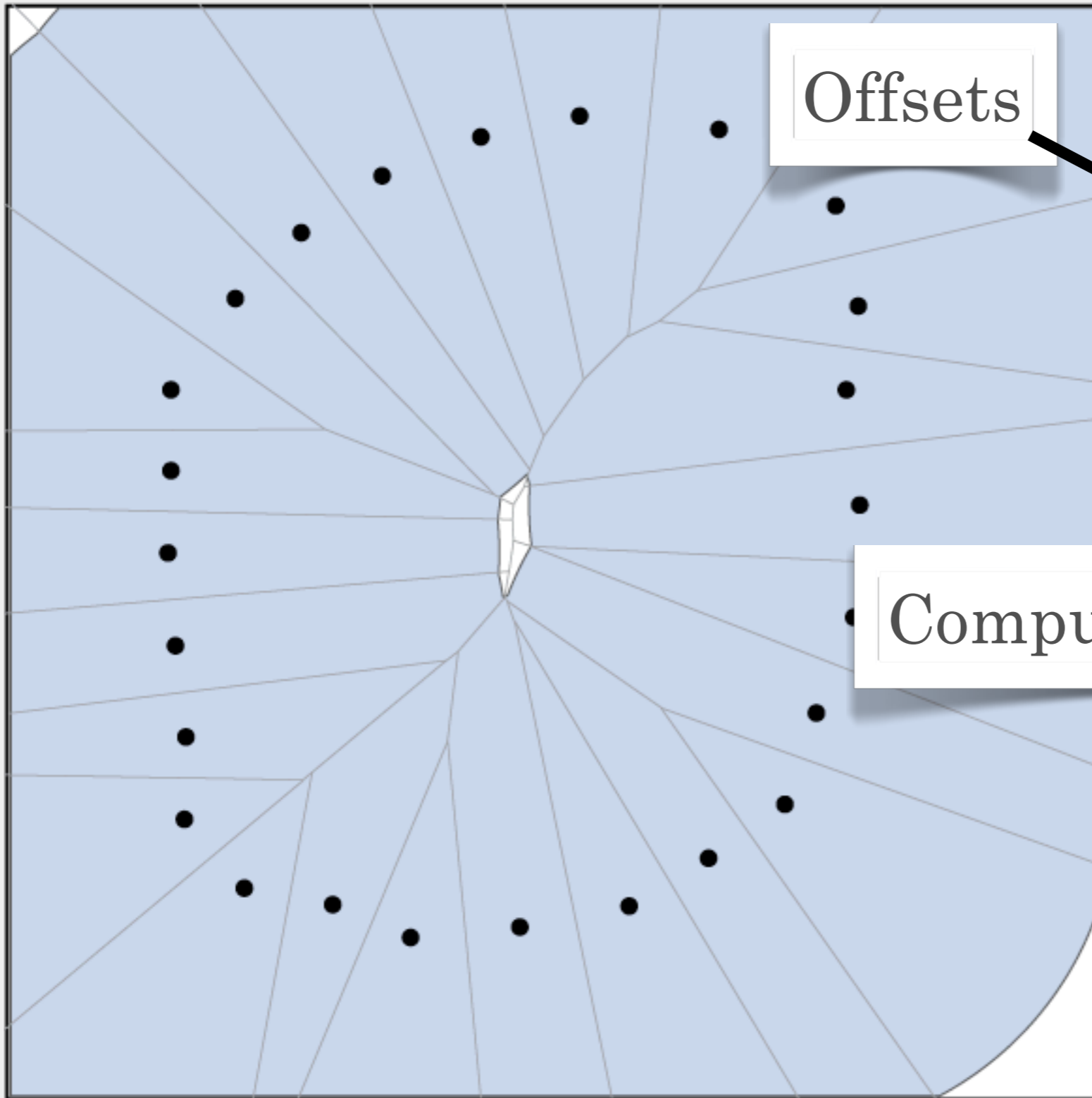
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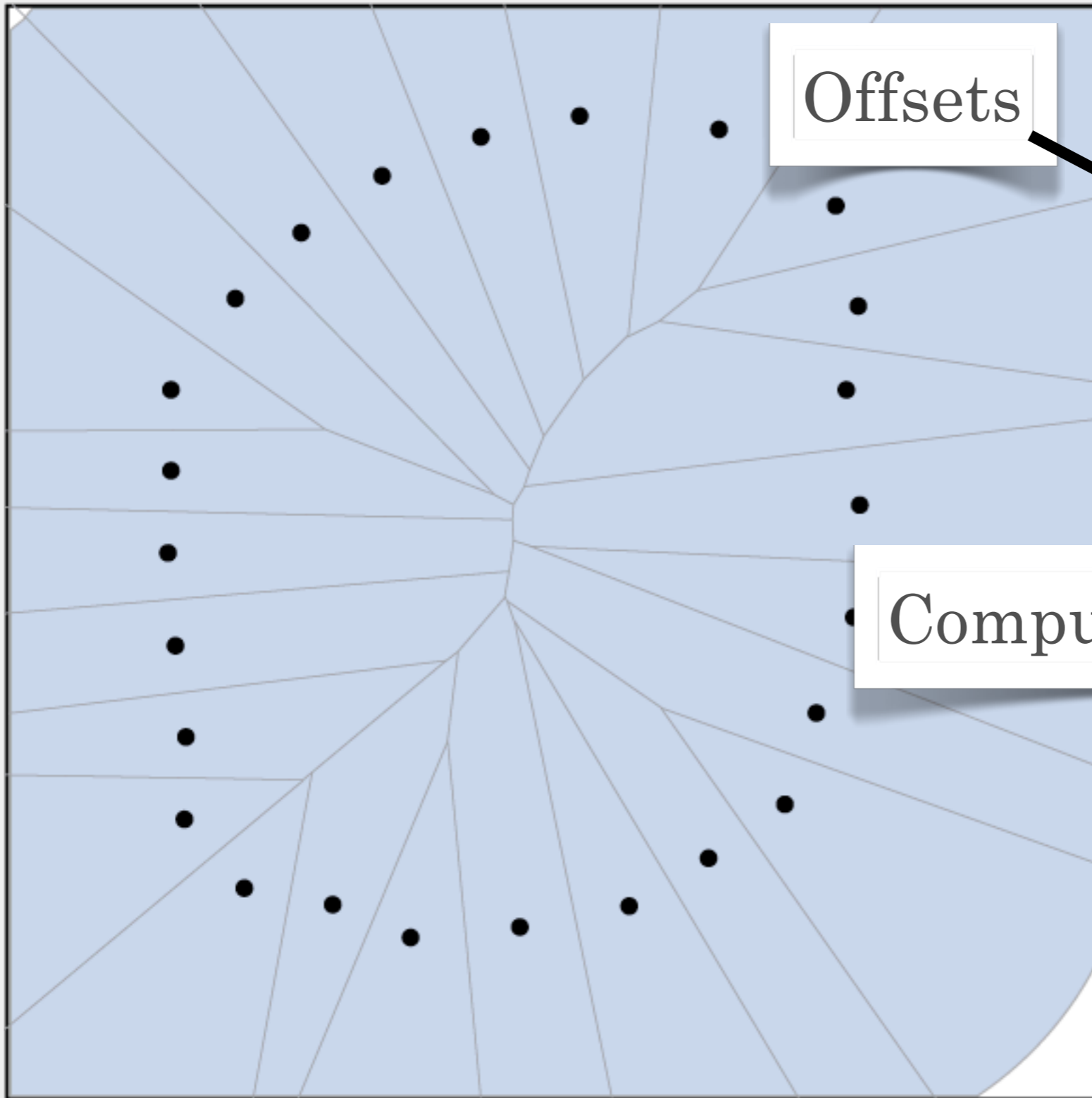
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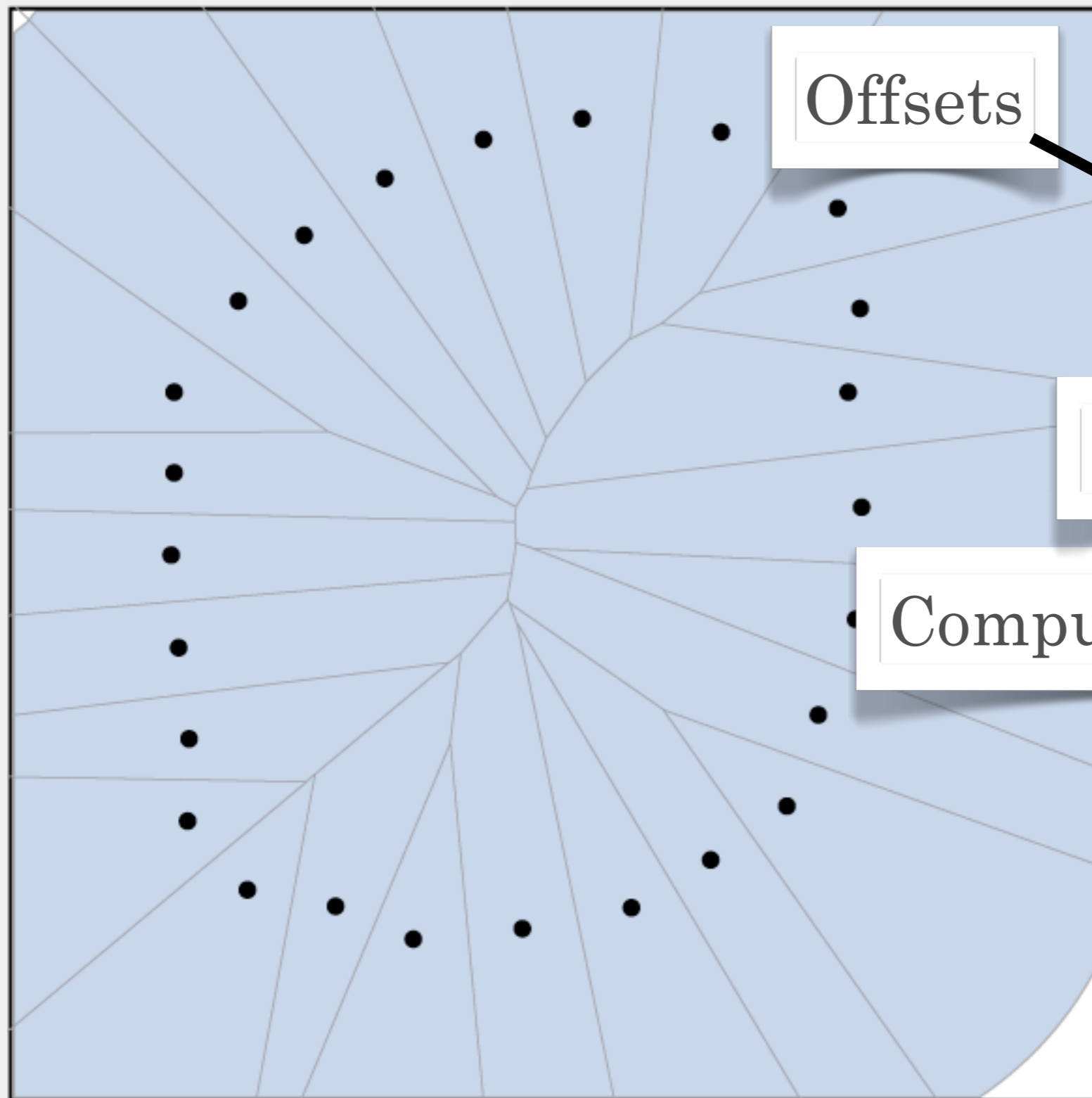
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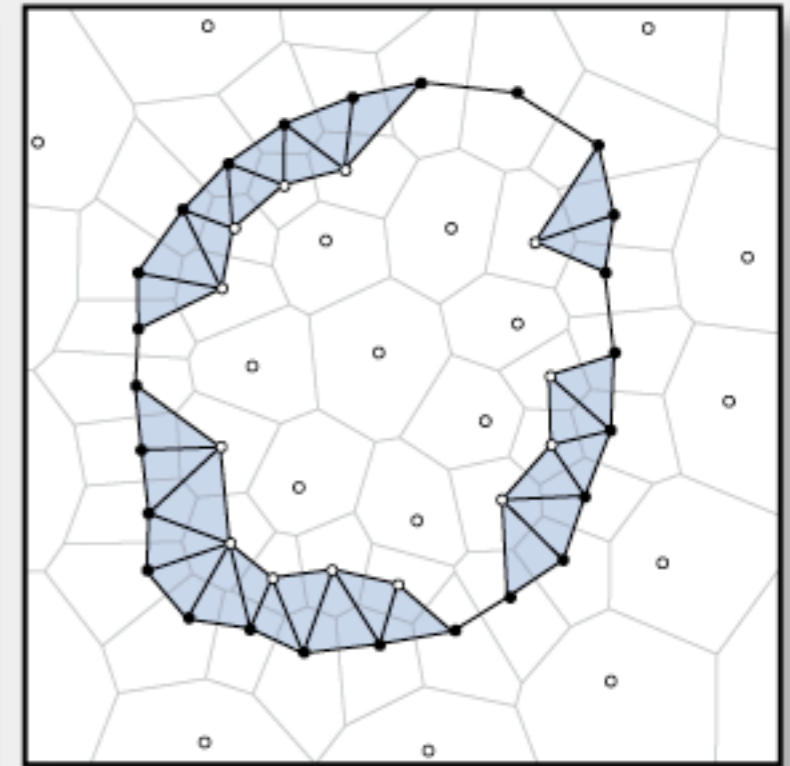
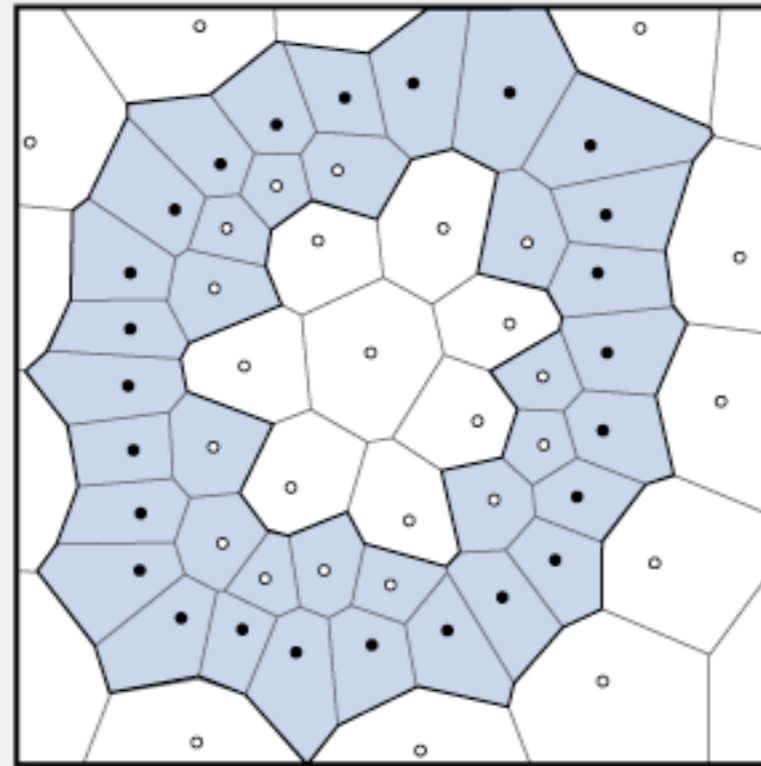
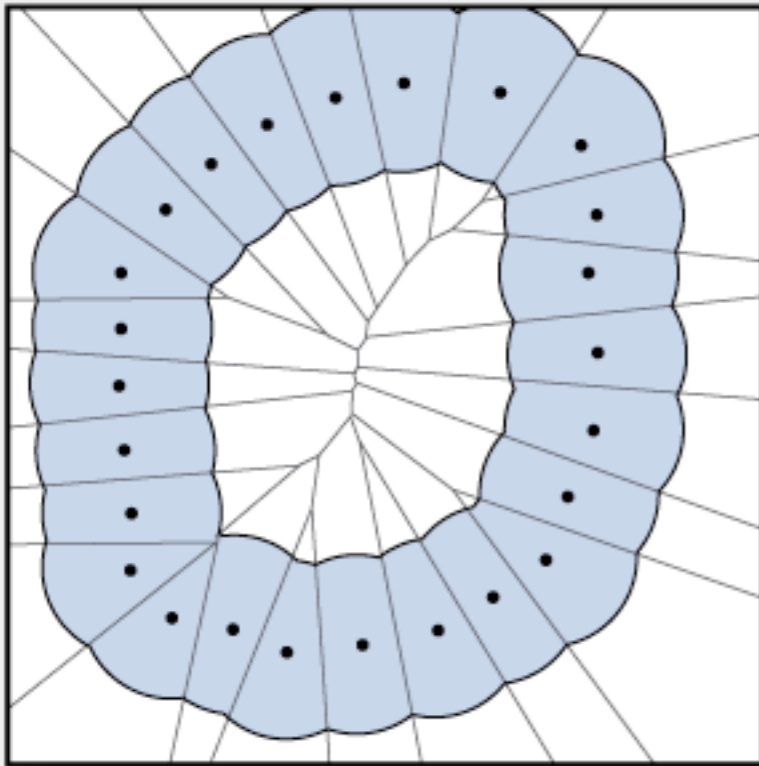
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Persistent

Compute the **Homology**

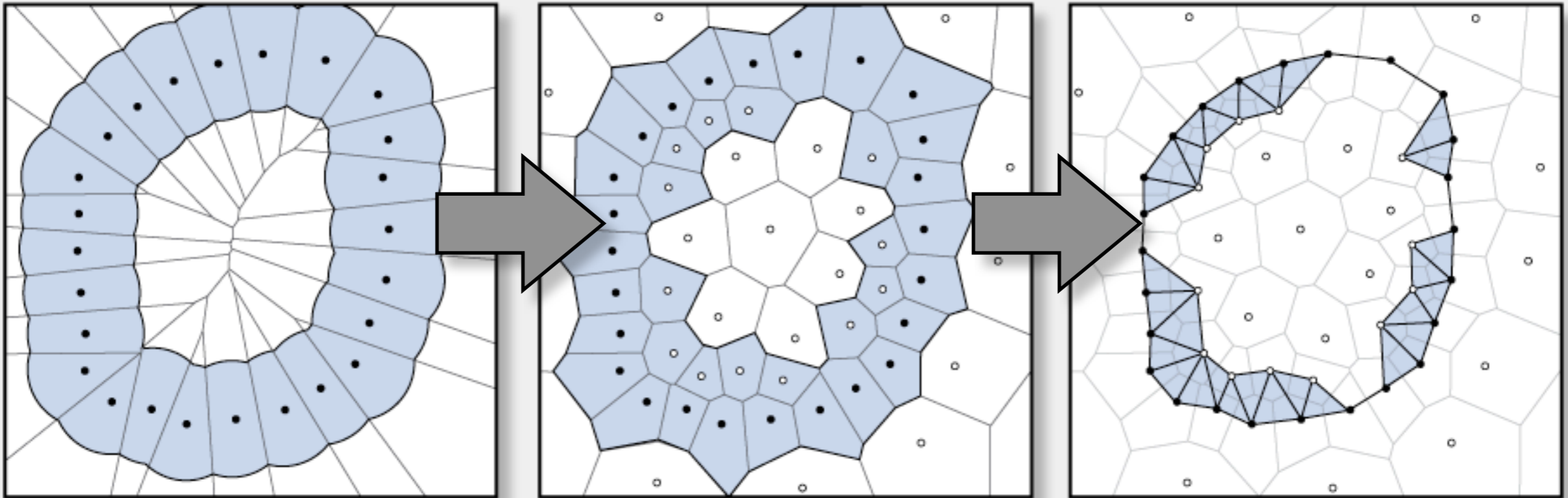
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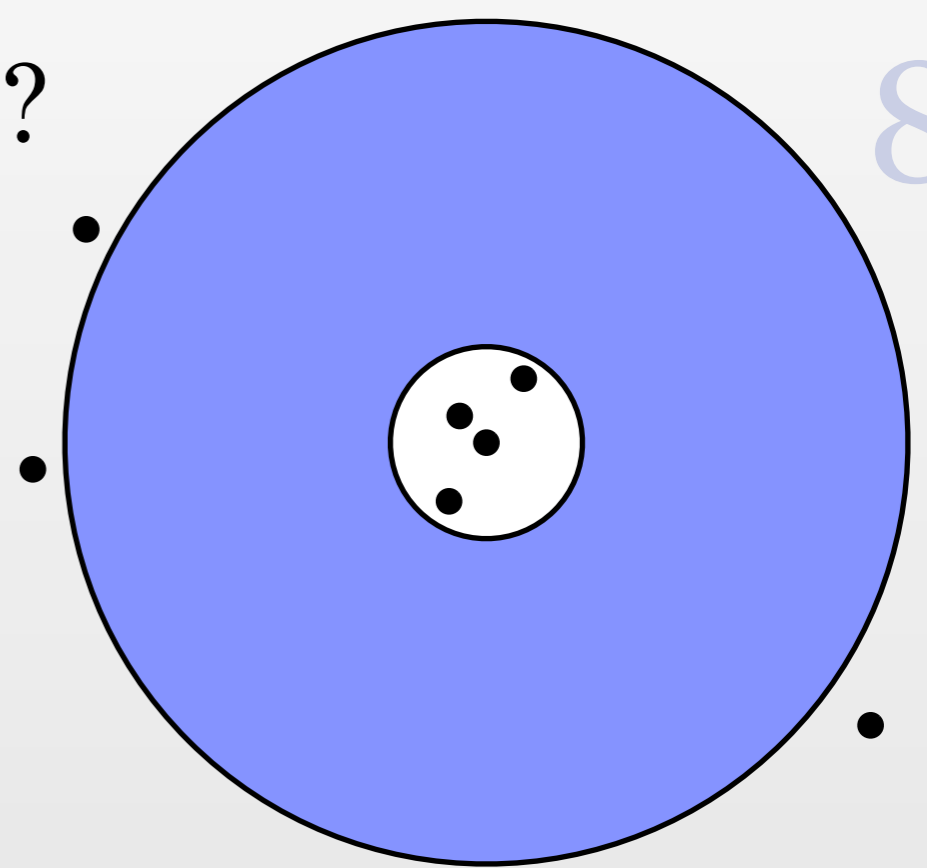
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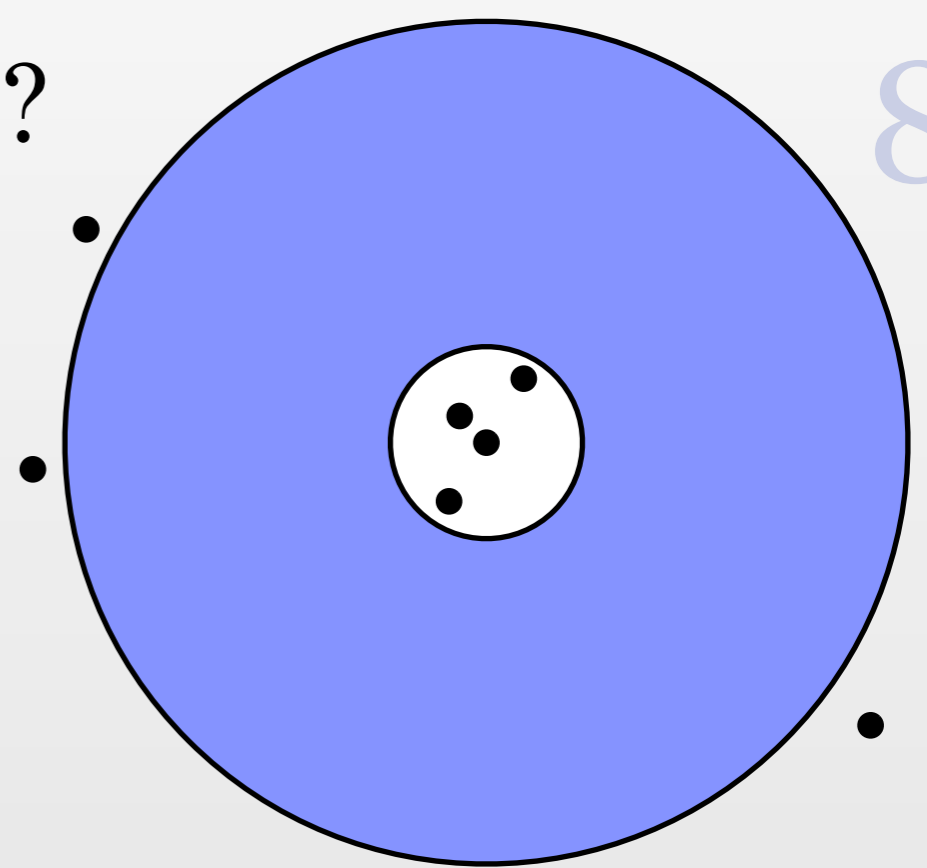
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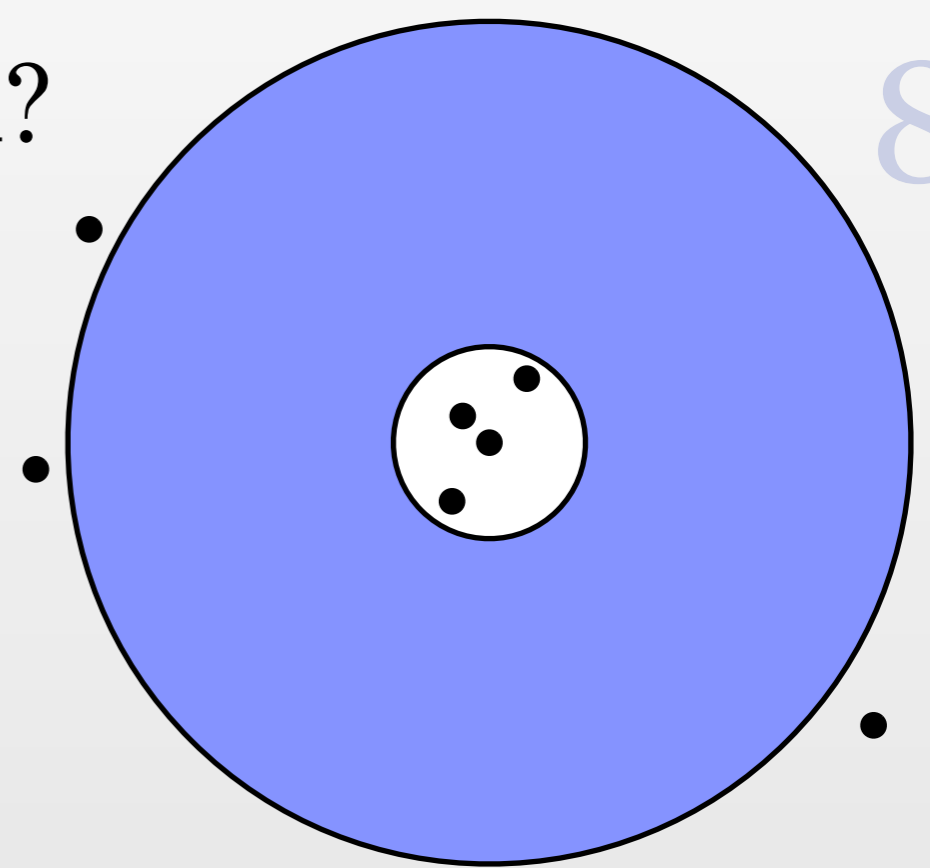


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How many simplices?

Only $O(|M|)$ simplices.

Compare to $|M|^{\lceil d/2 \rceil}$ for general Delaunay triangulations.

Constants depend on aspect ratio.

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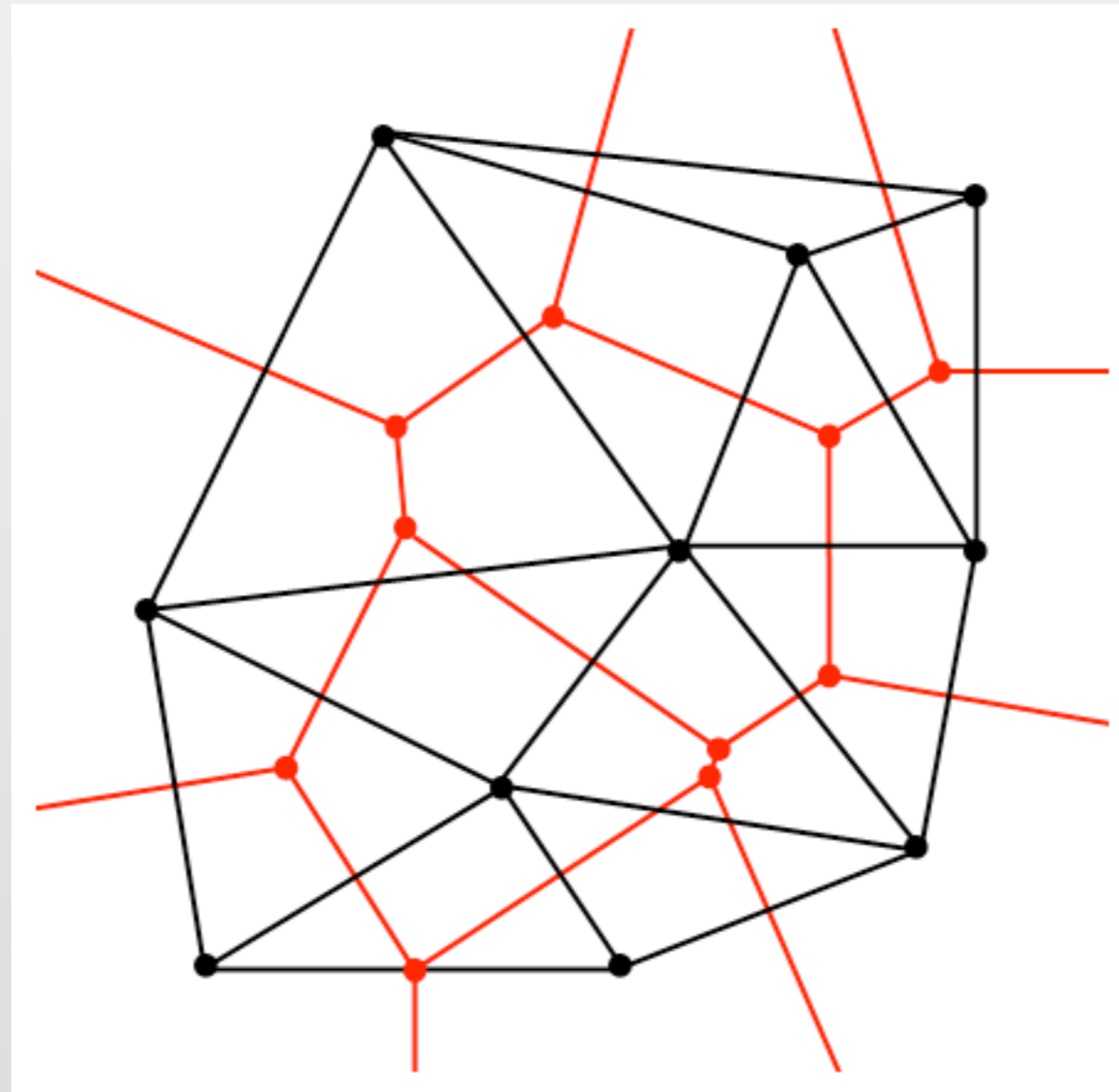
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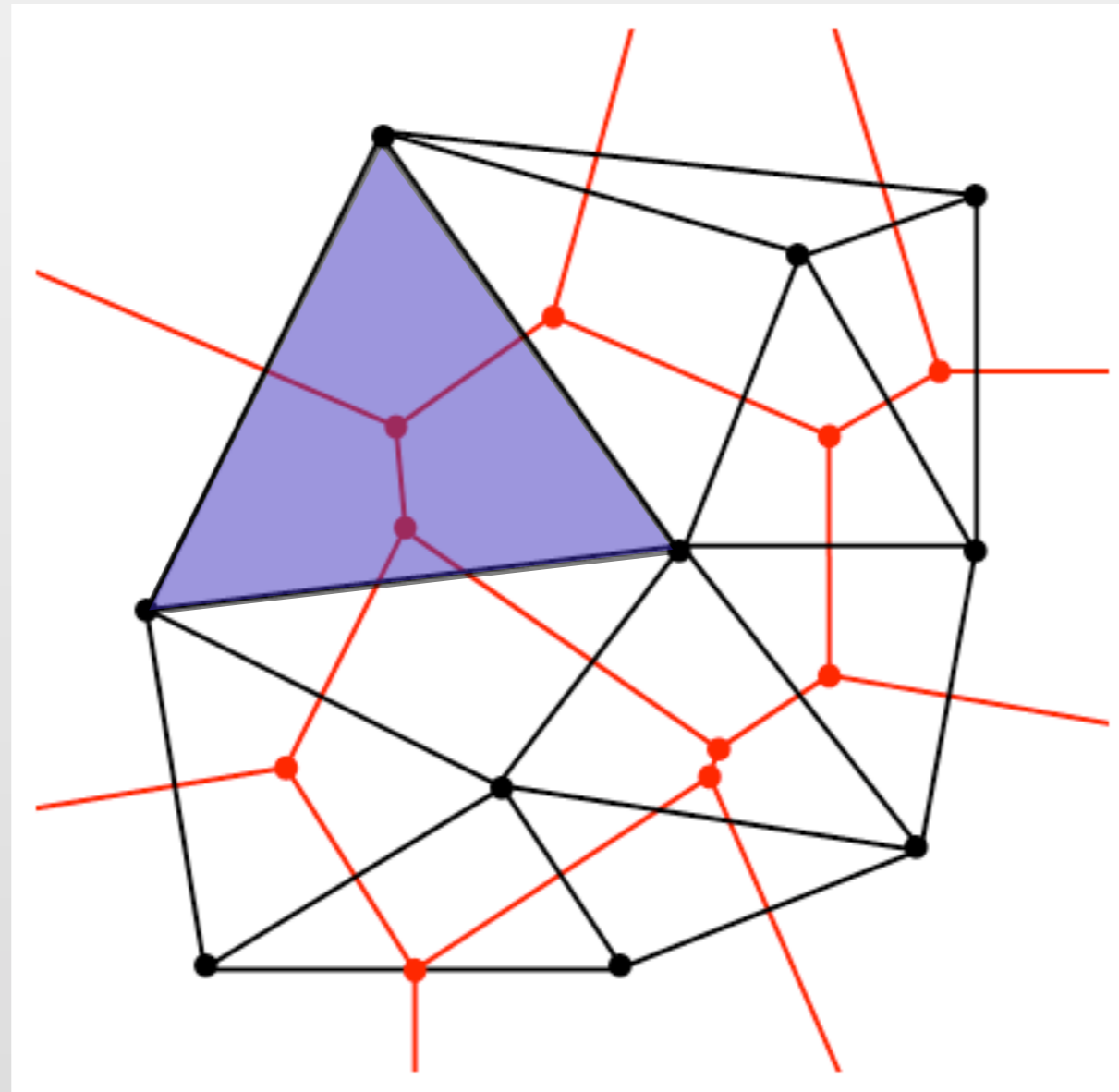
Compute a *hierarchical quality* mesh.

Finishing post-process.

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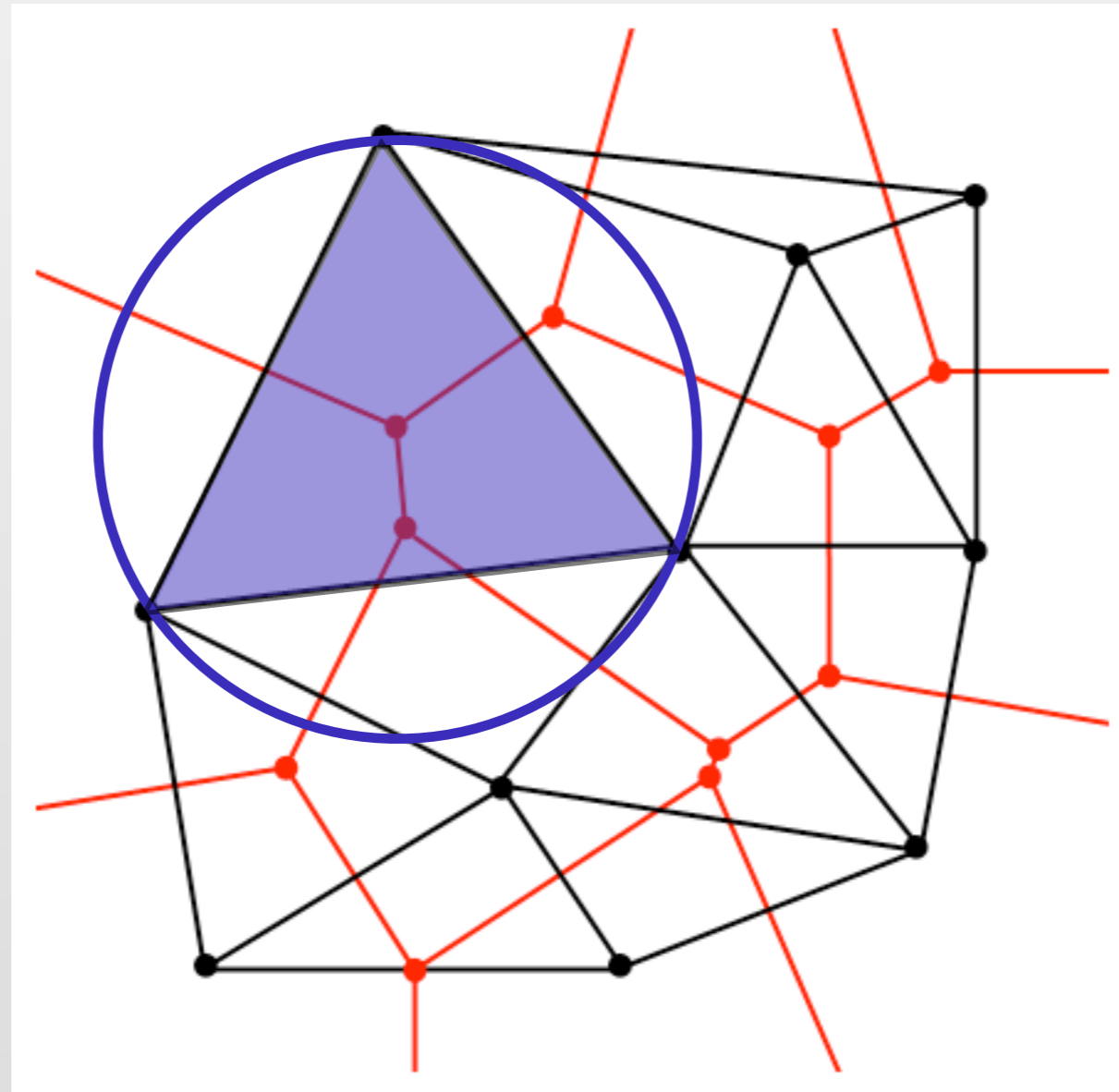


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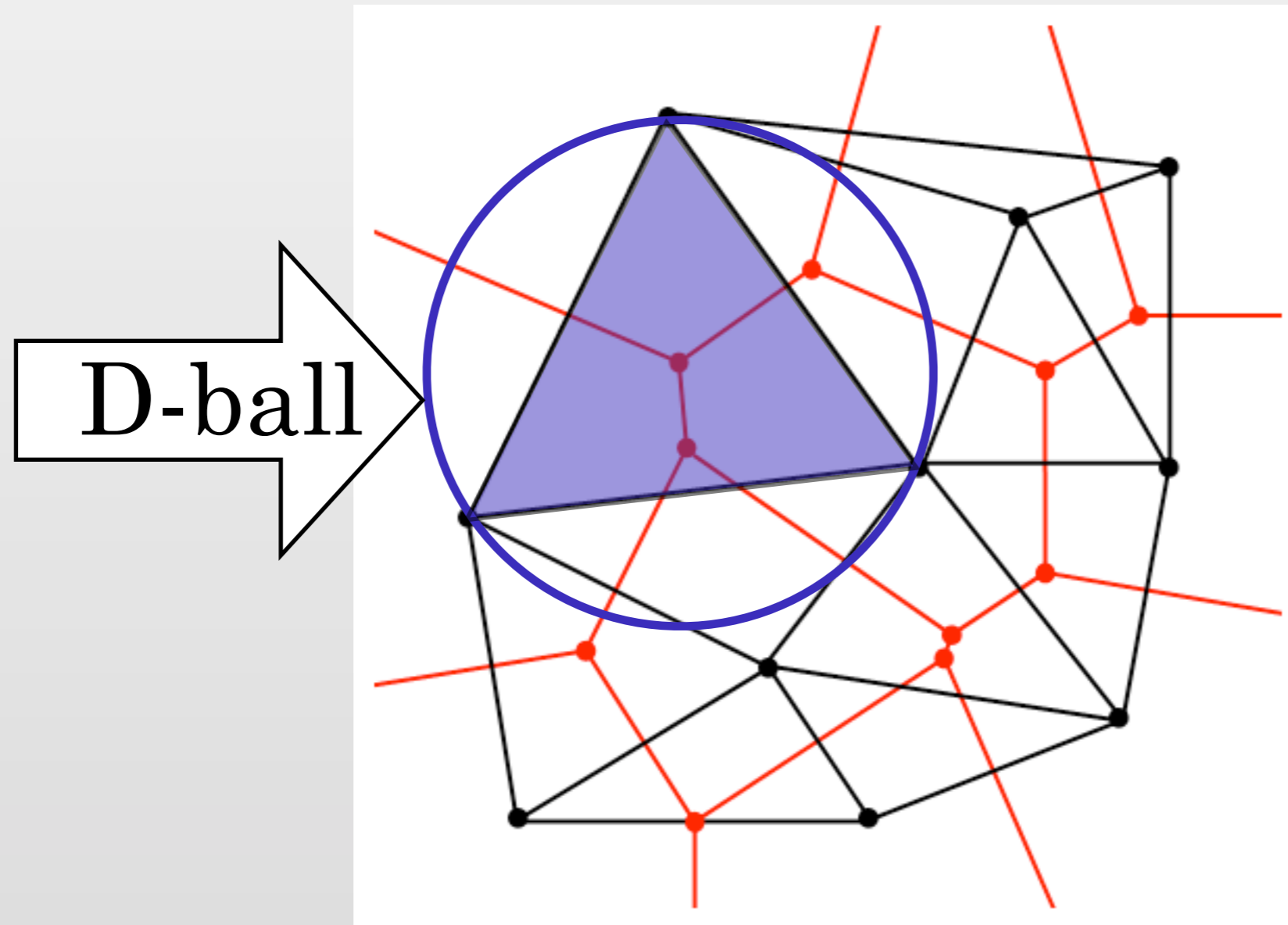


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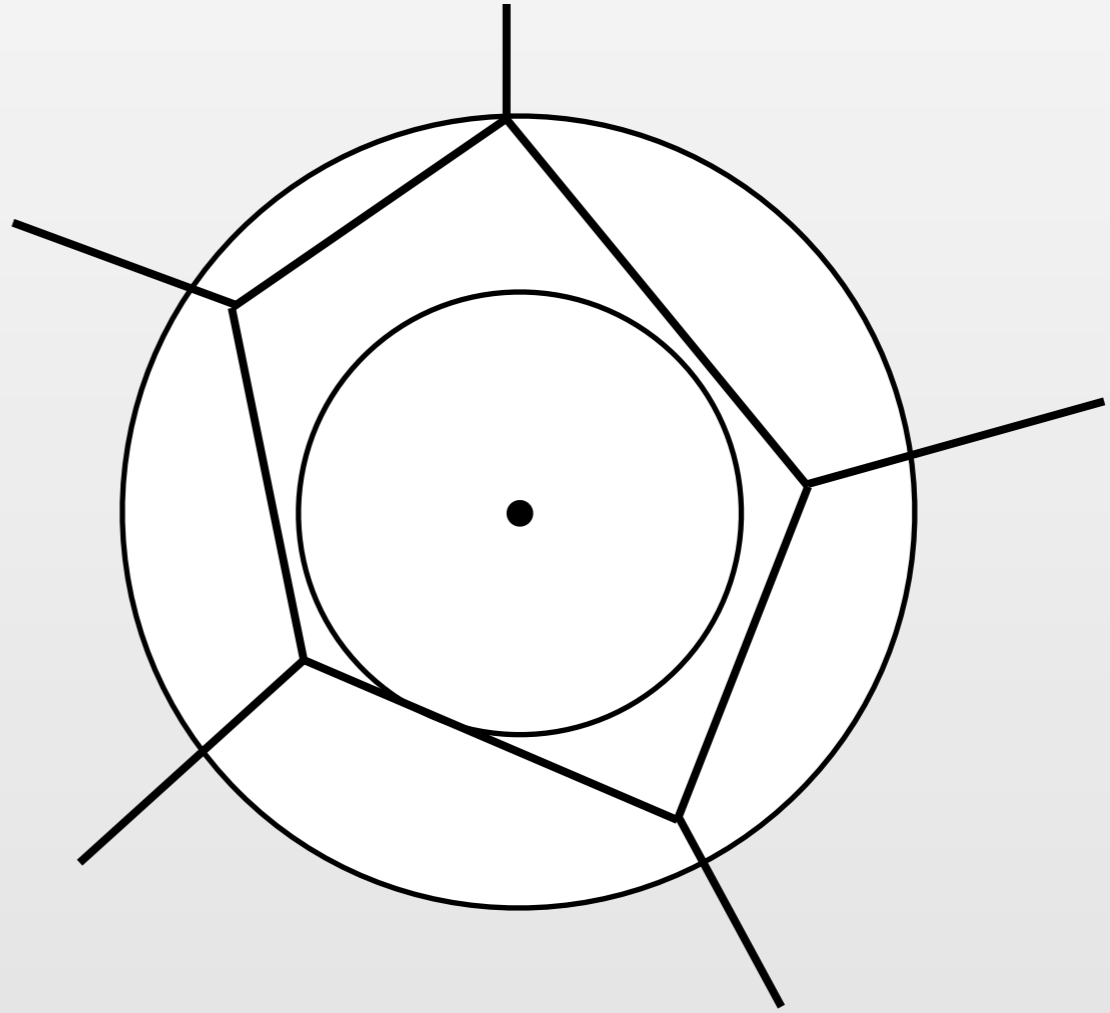
10

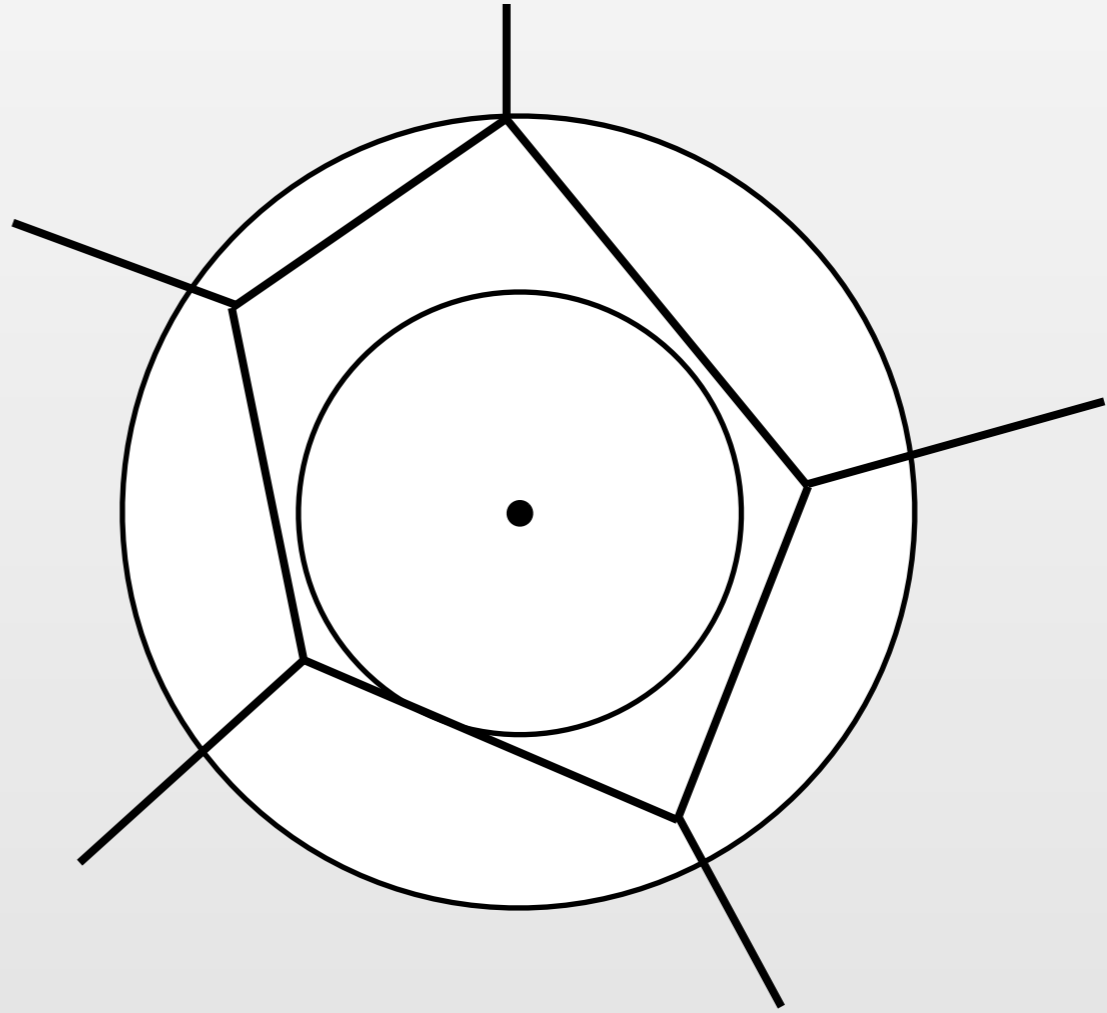


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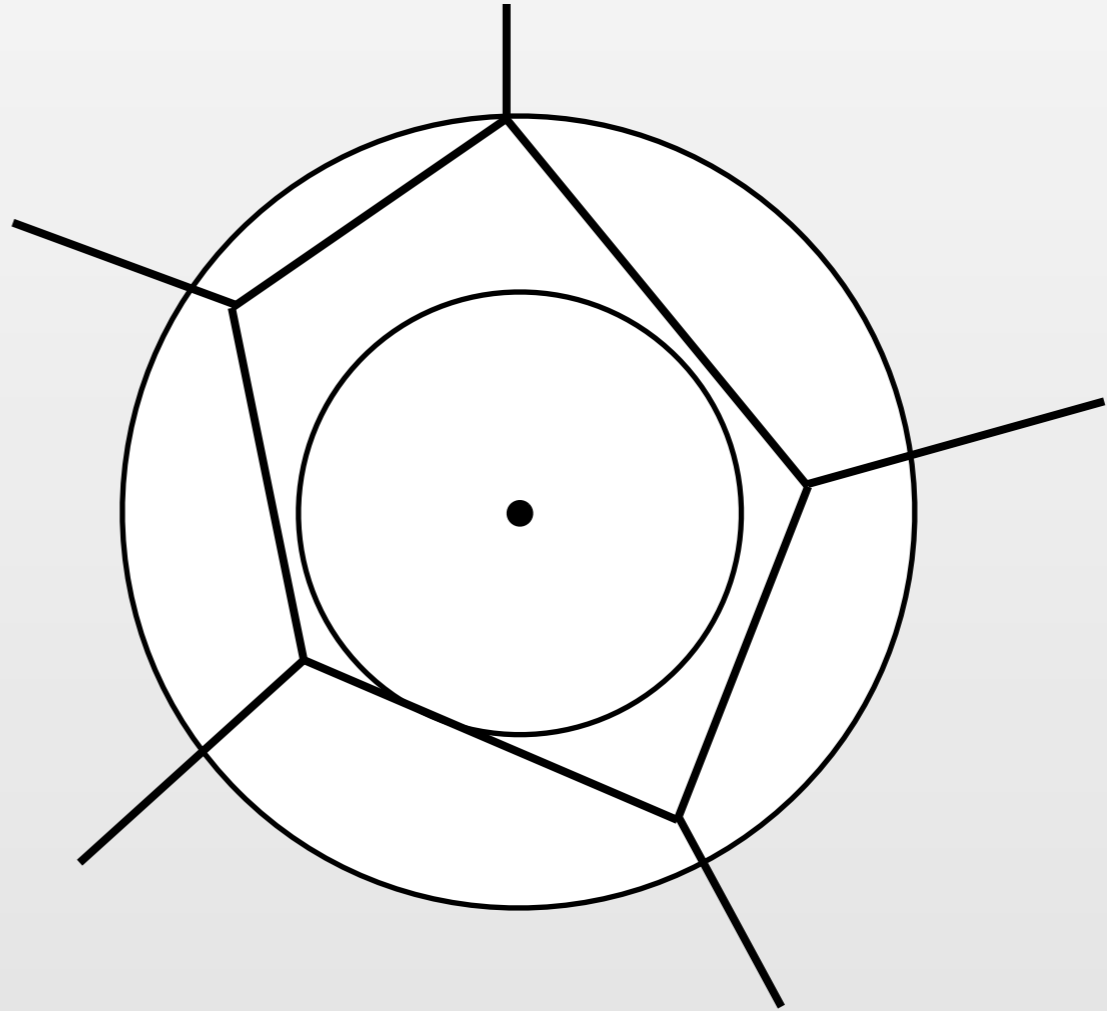


Quality Meshes



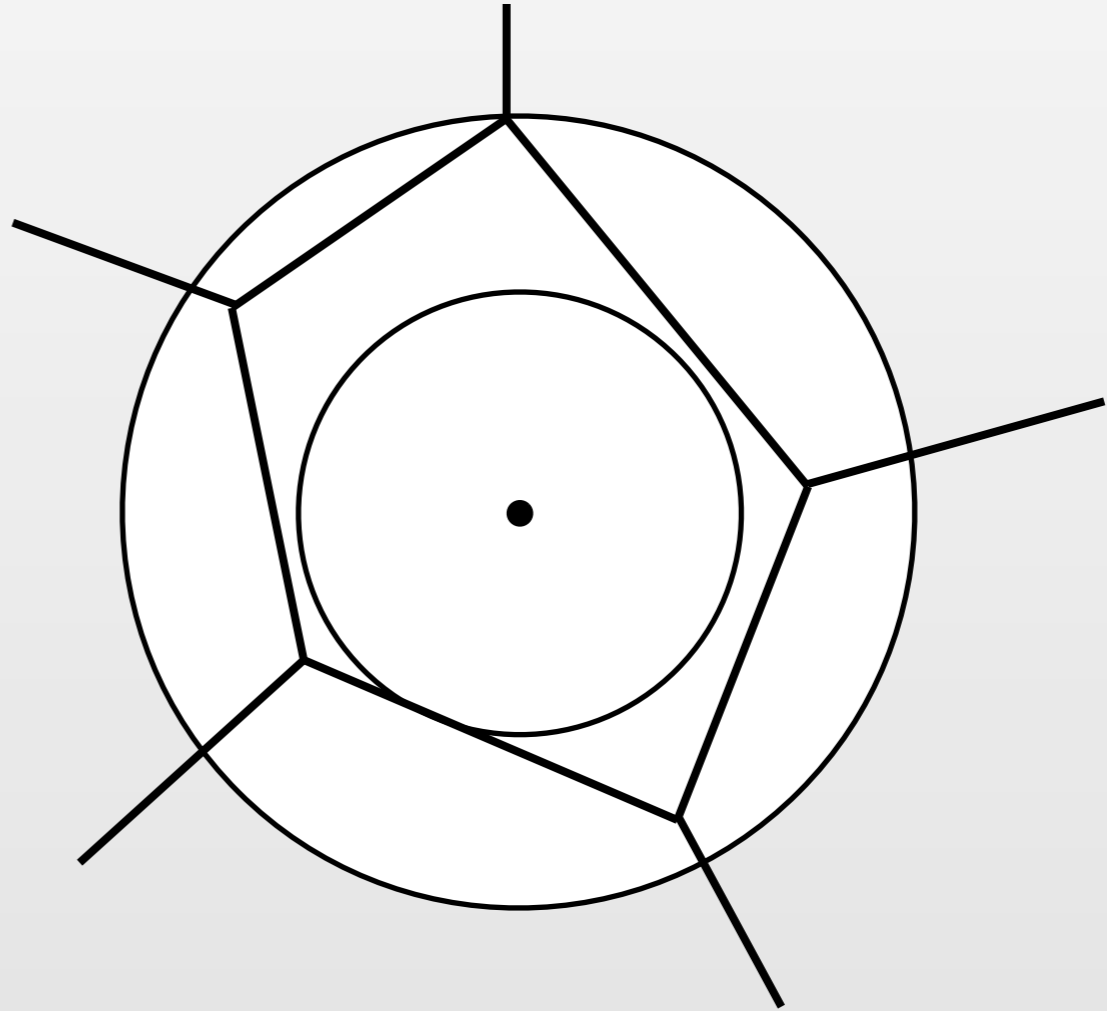


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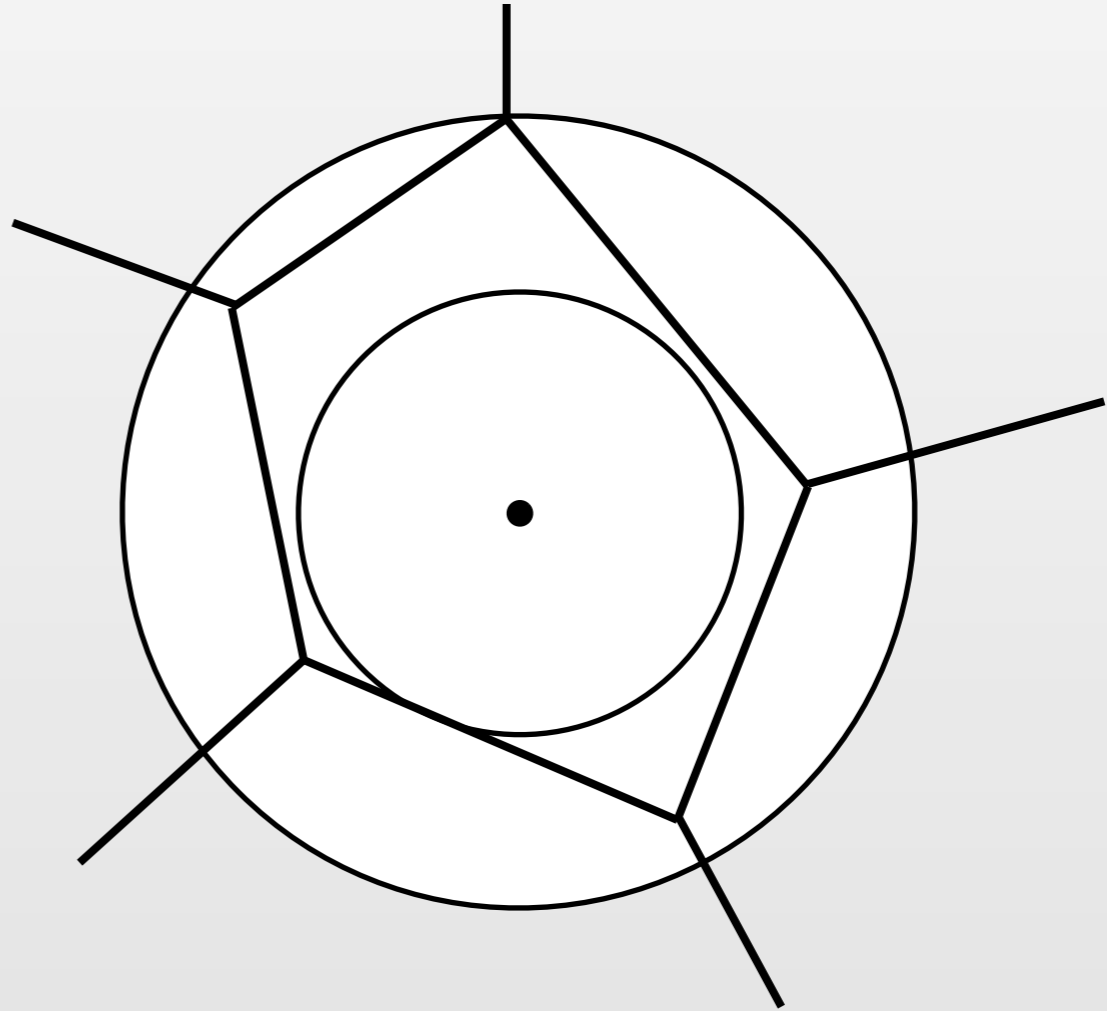
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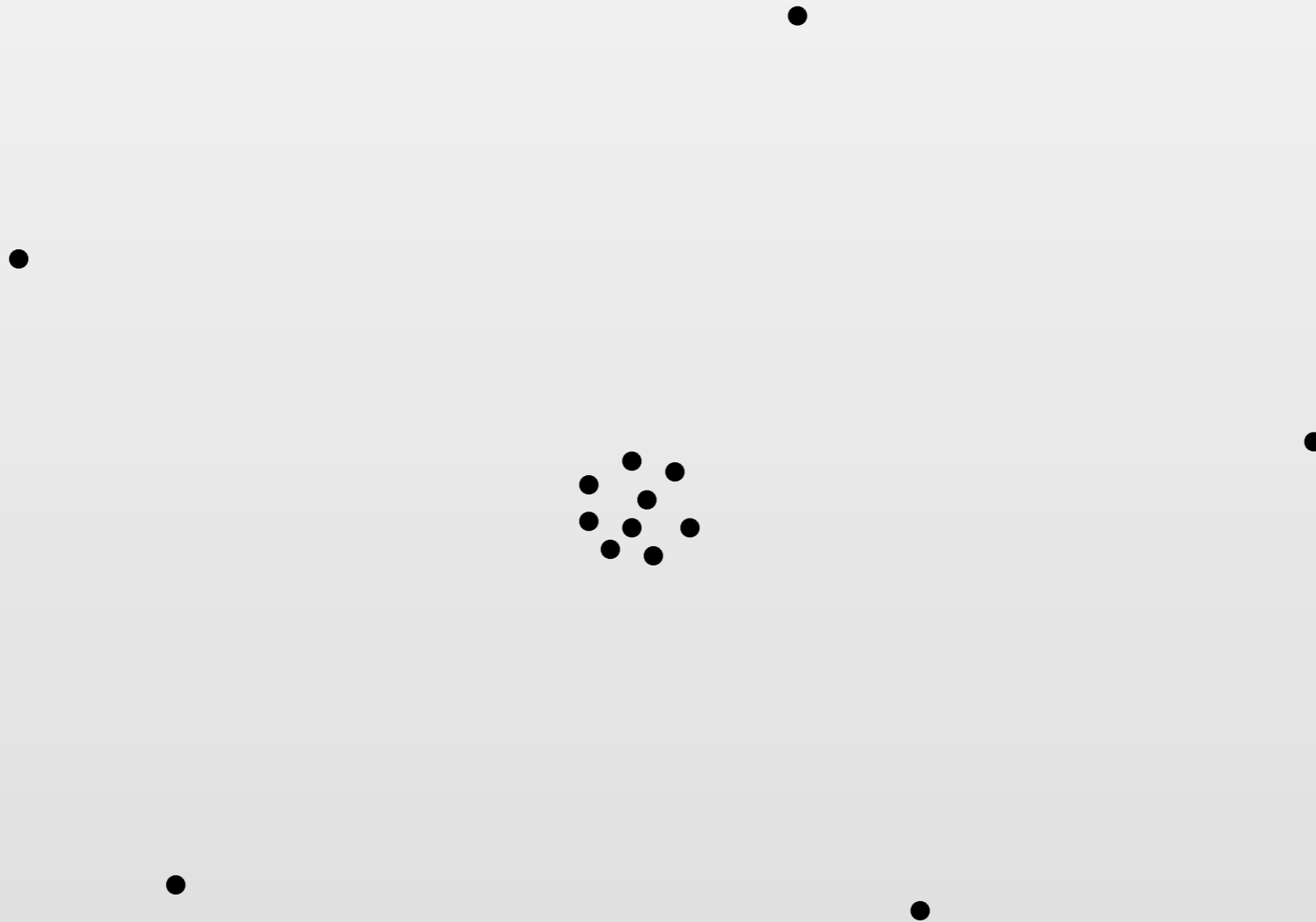
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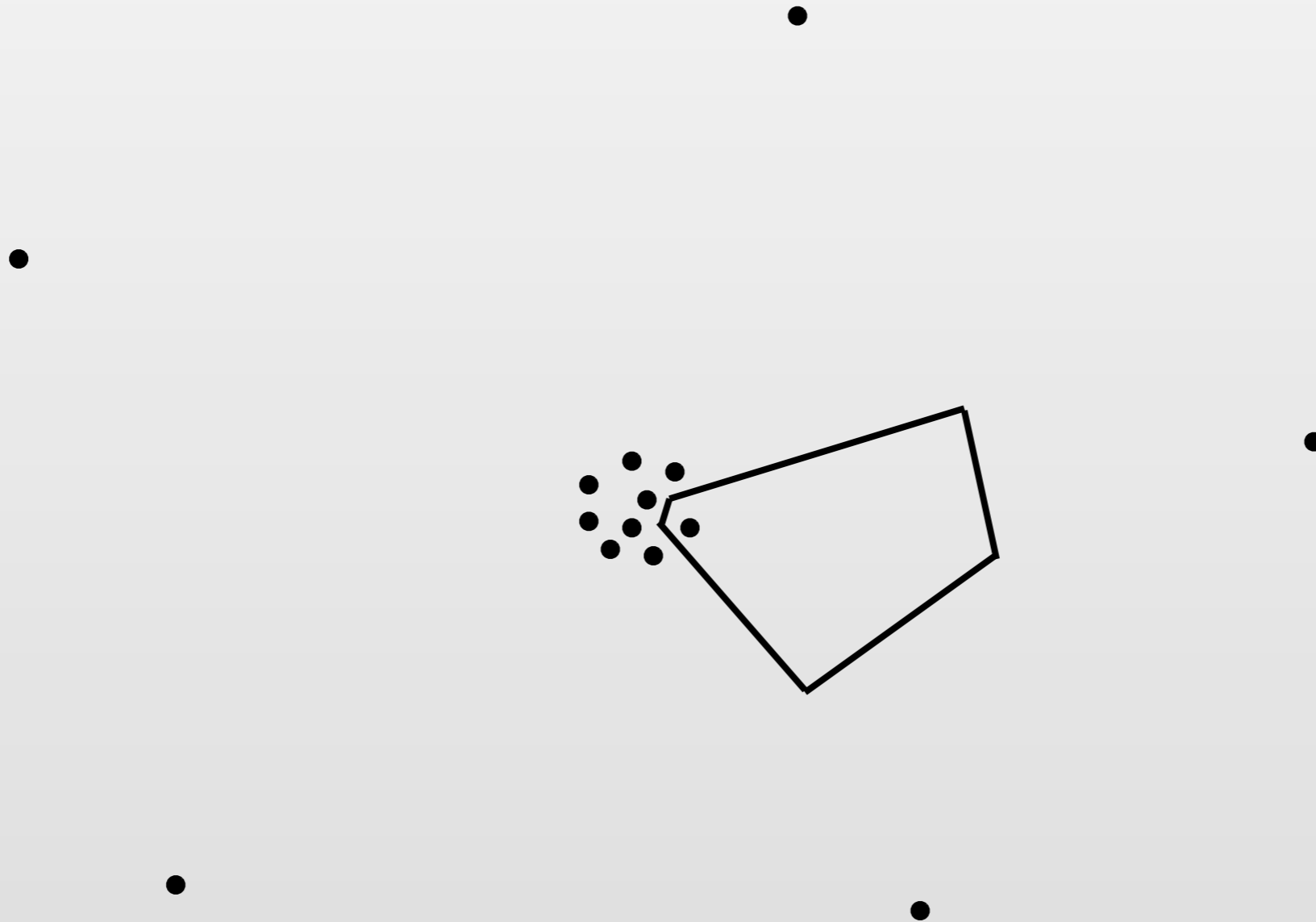
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Refine poor-quality cells by adding a Steiner point at its farthest vertex.

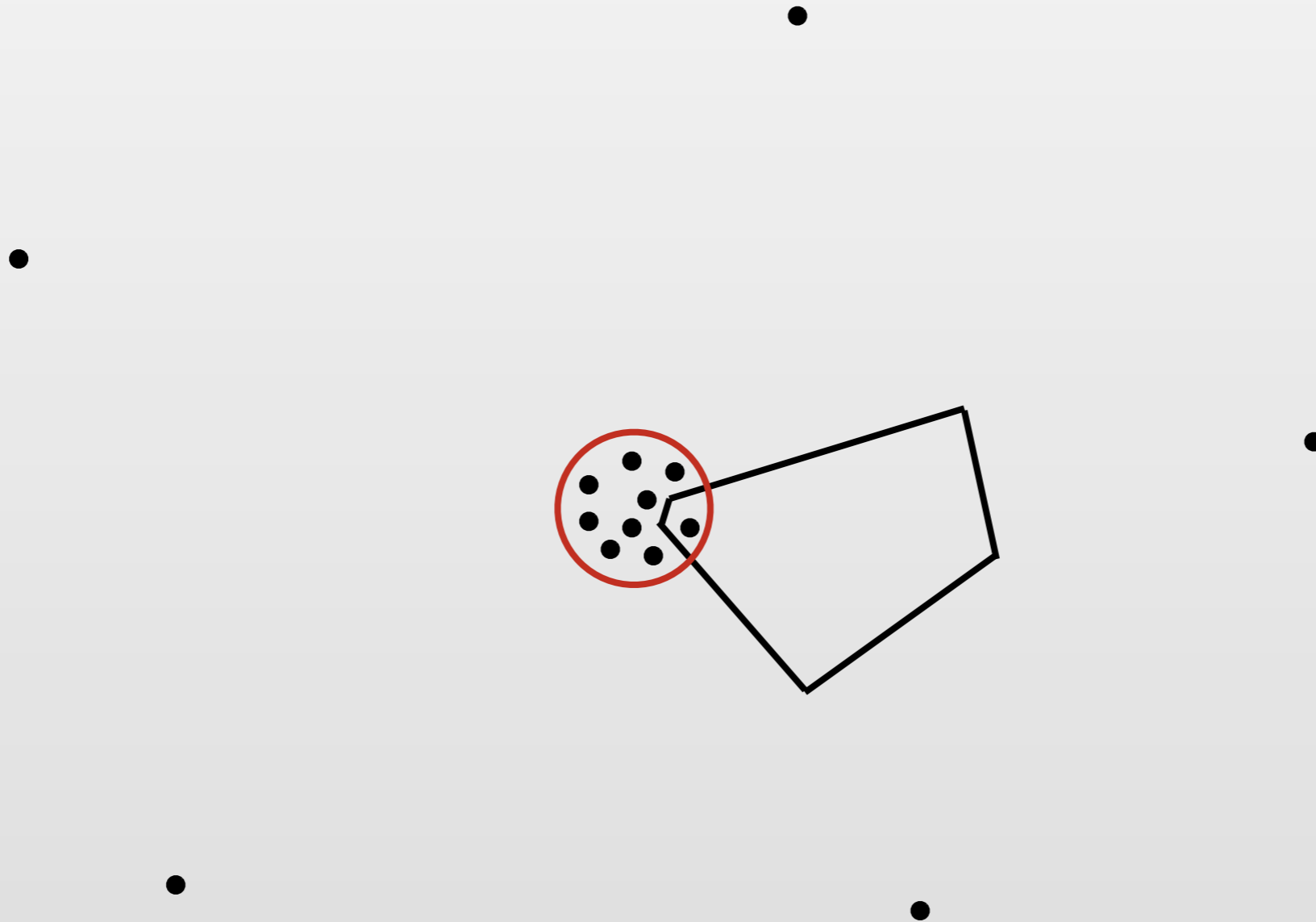
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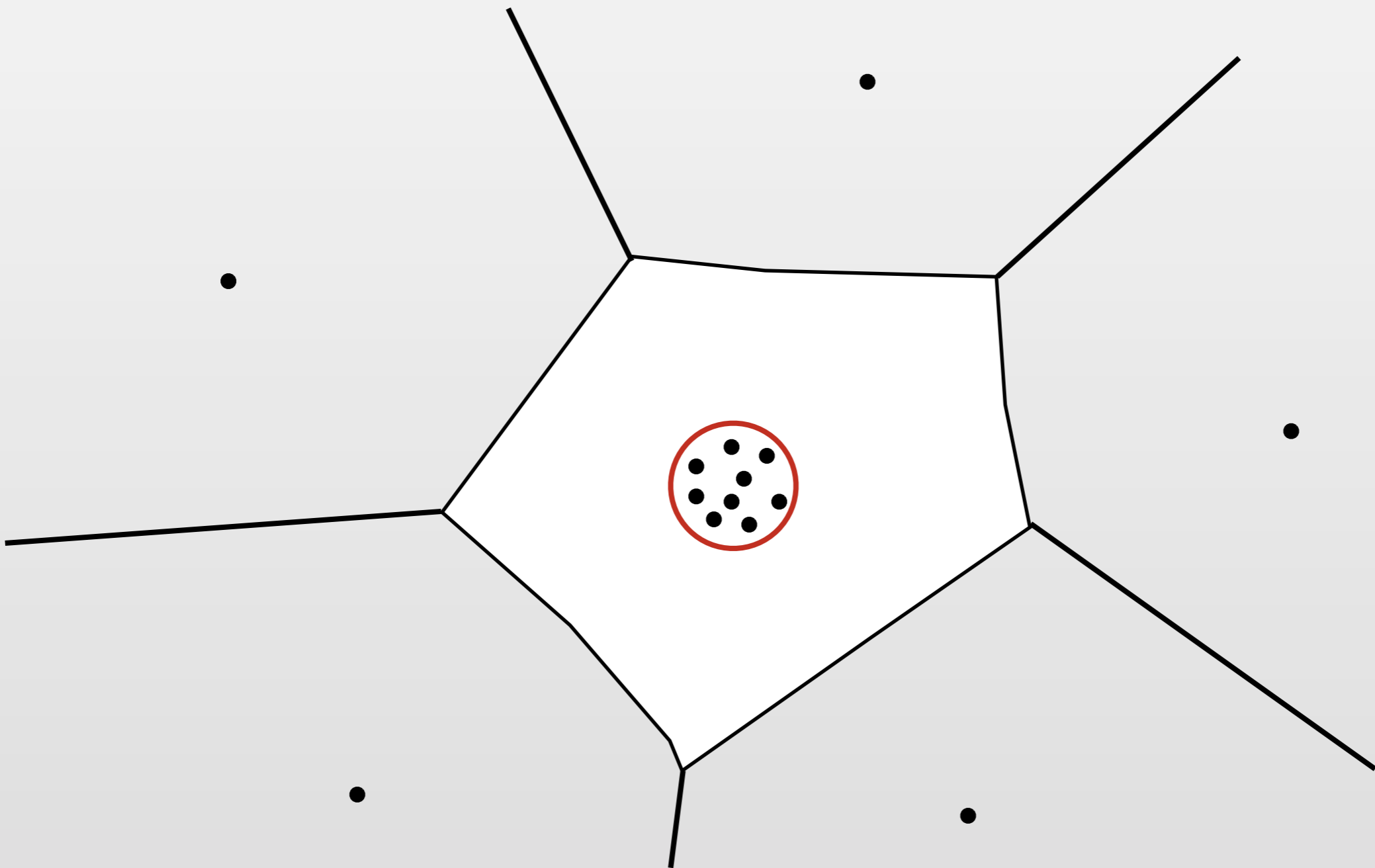
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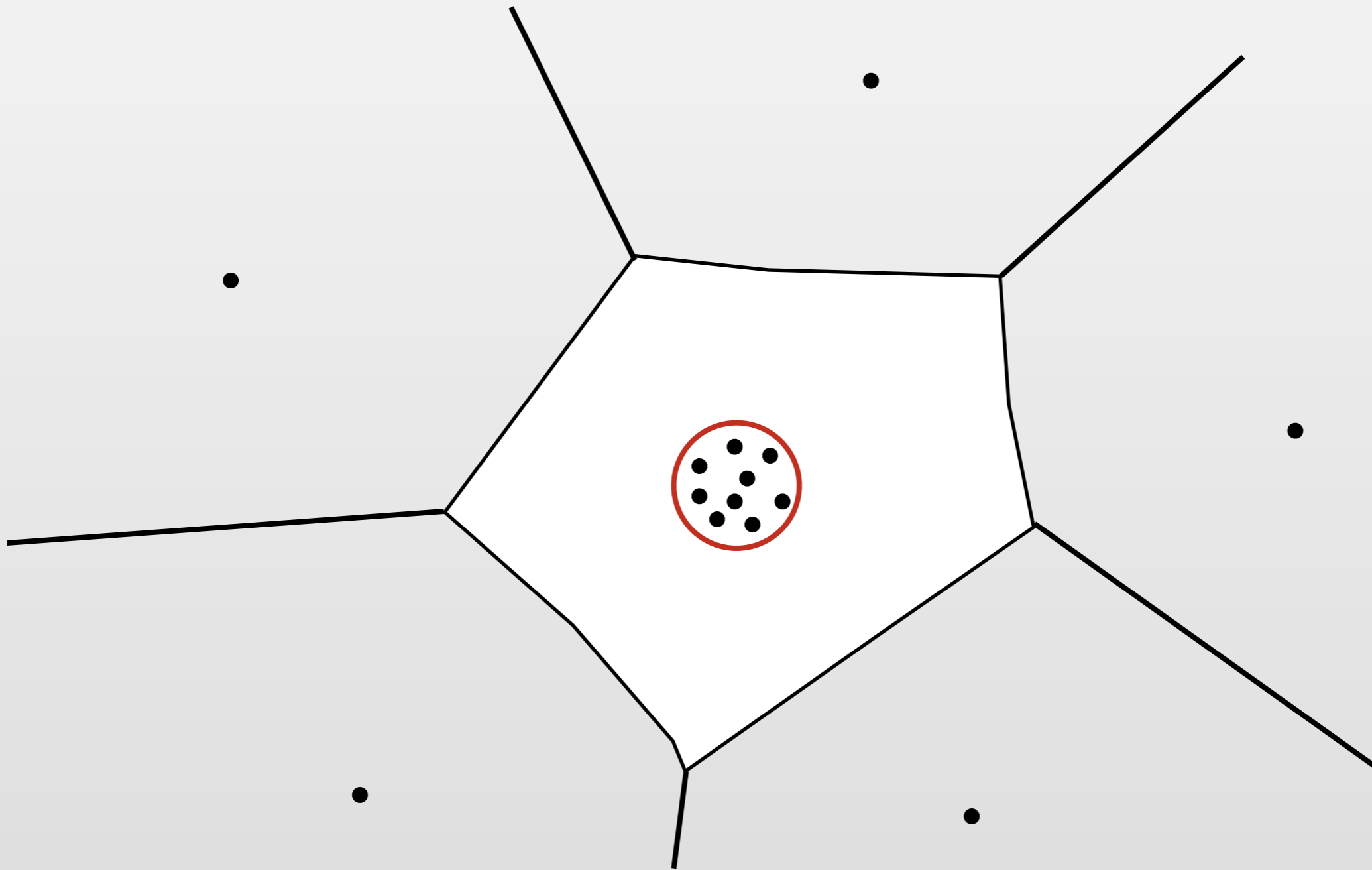


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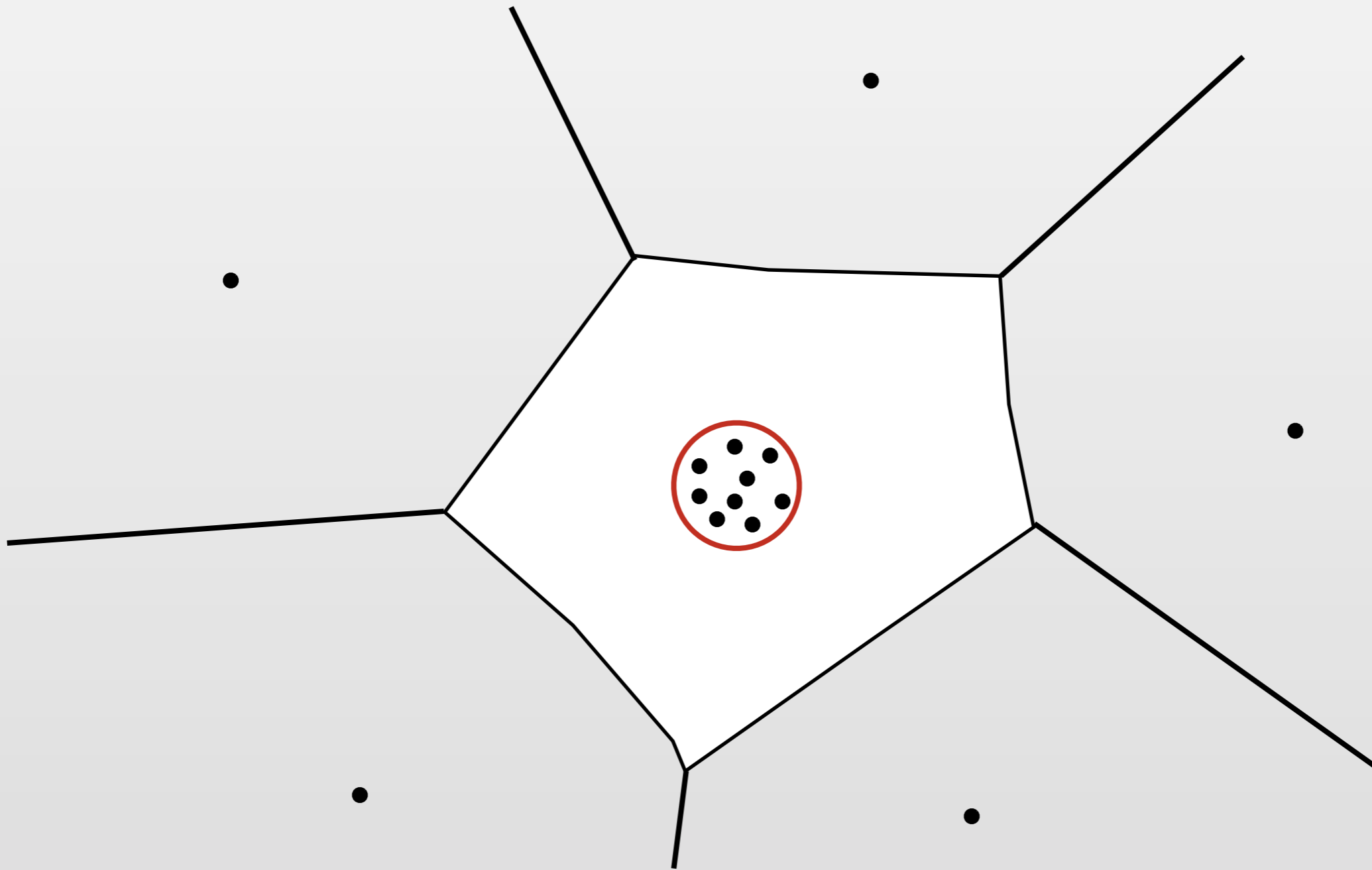
12



Inside the cage: Old definition of quality.

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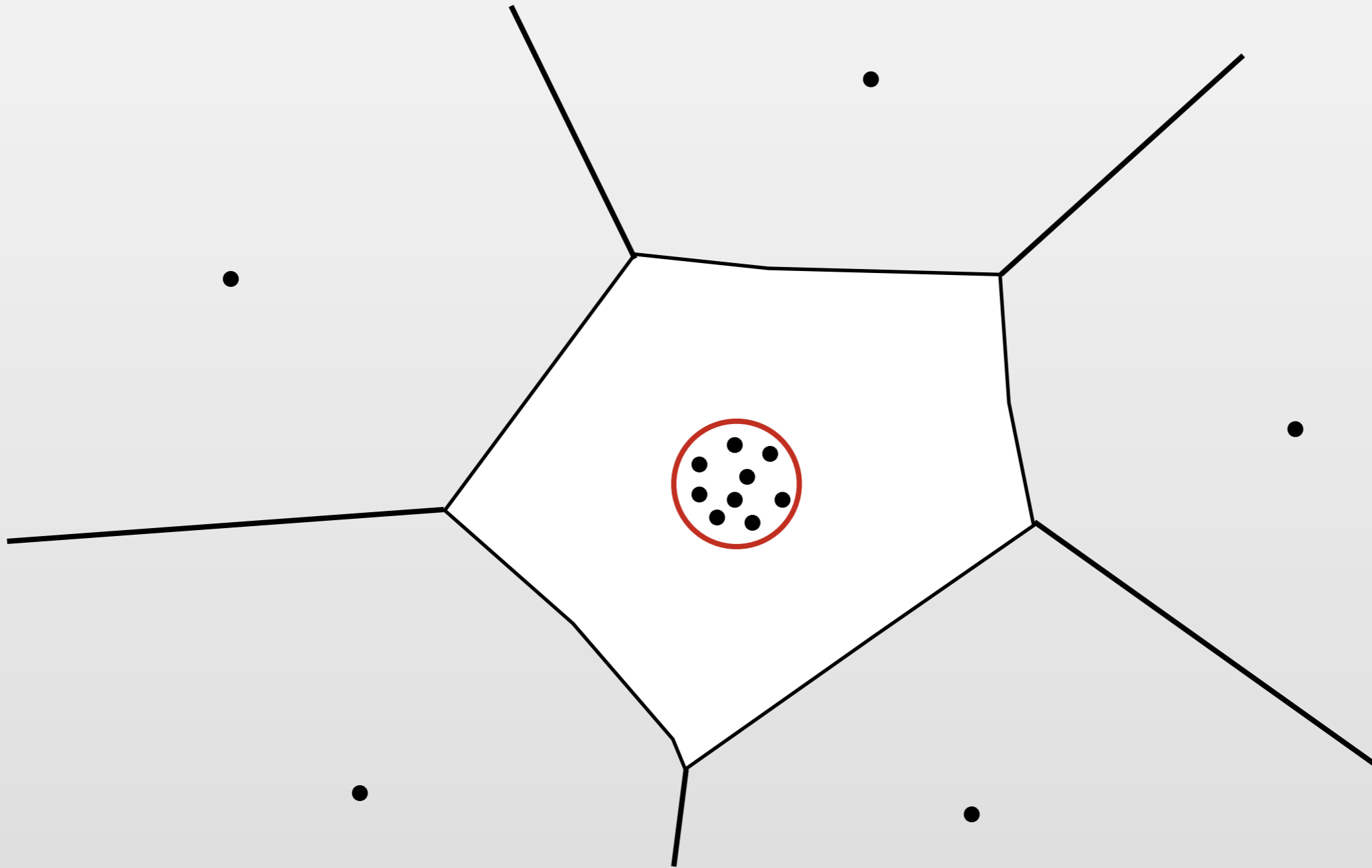


Inside the cage: Old definition of quality.

Outside: Treat the whole cage as a single object.

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Inside the cage: Old definition of quality.

Outside: Treat the whole cage as a single object.

All the same properties as quality meshes: ply, degree, etc.

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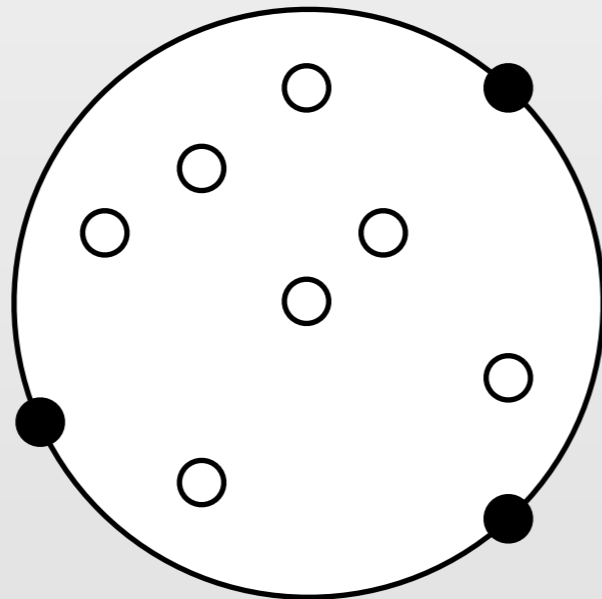
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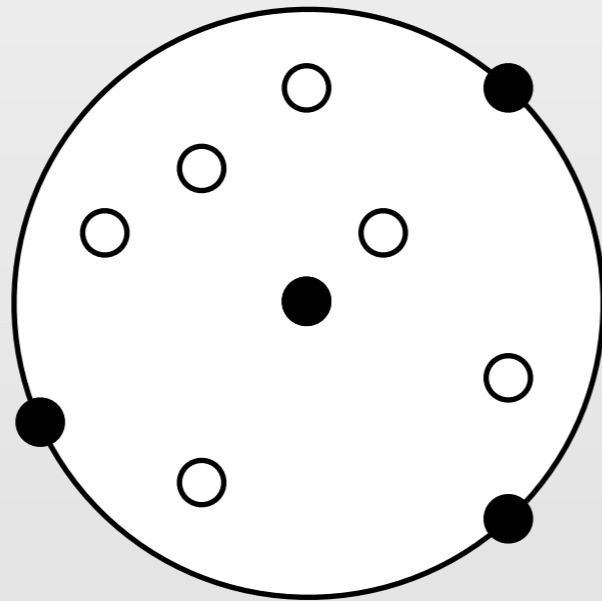
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This is how we avoid the worst-case Voronoi bounds.

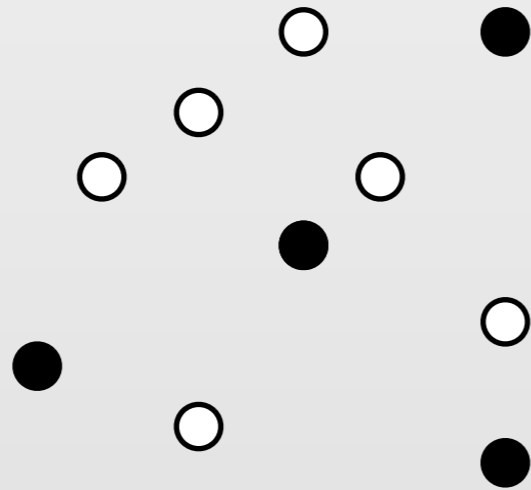
Idea: Store the uninserted points in the D-balls.
When the balls change, make local updates.



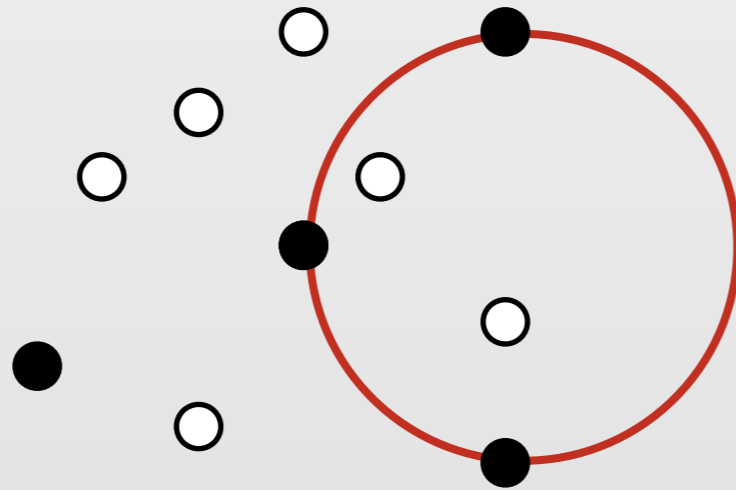
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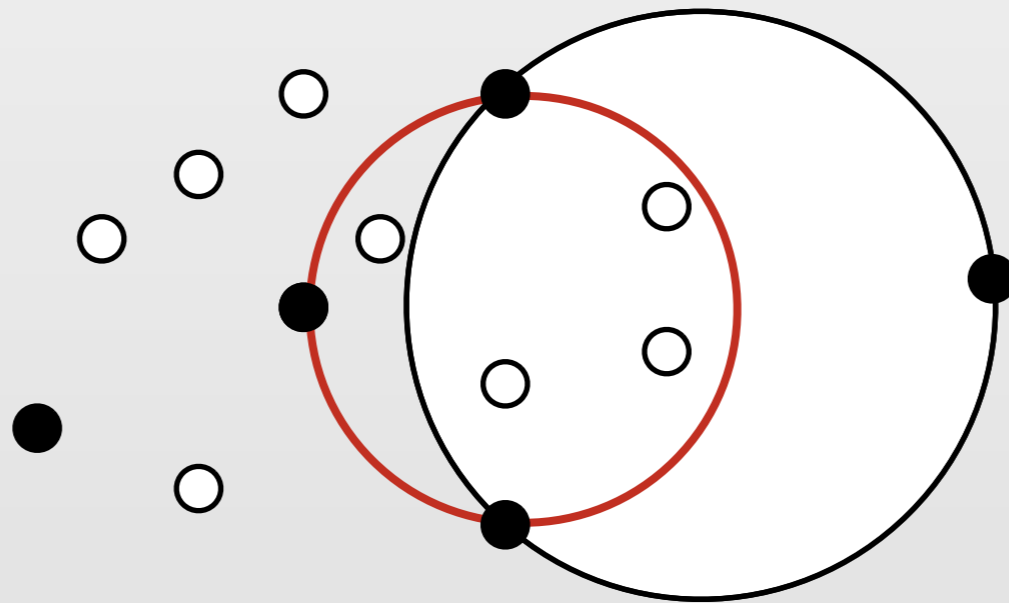
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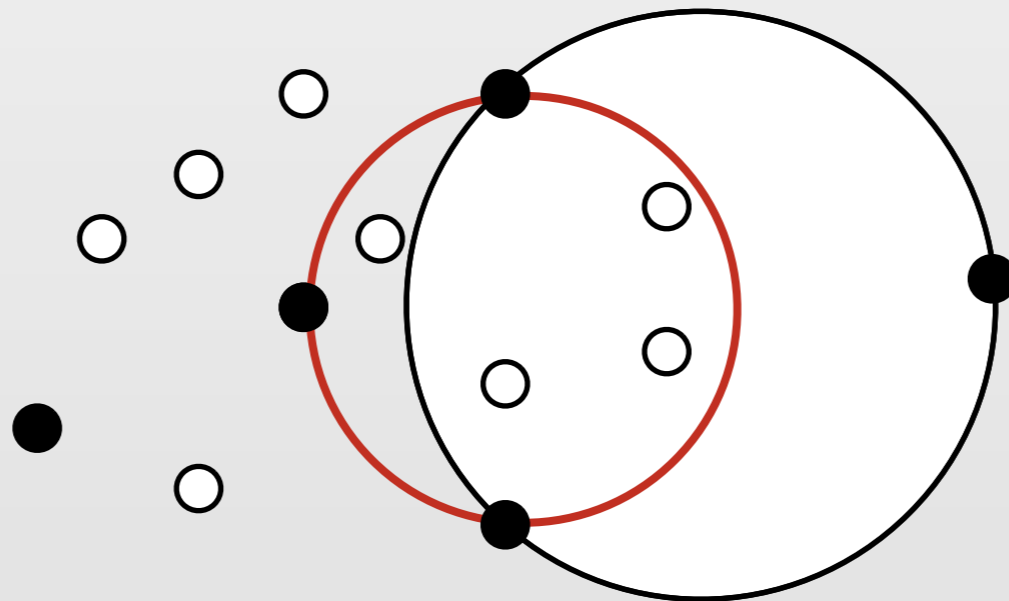
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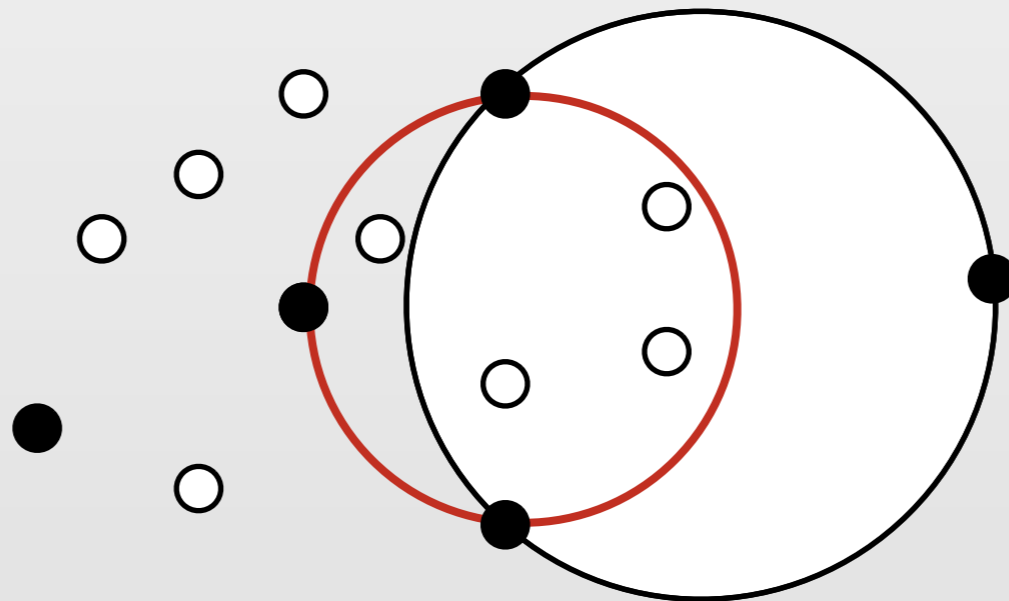


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Geometric Divide and Conquer: $O(n \log \Delta)$

With hierarchical meshes, we can add inputs
in any order!

15



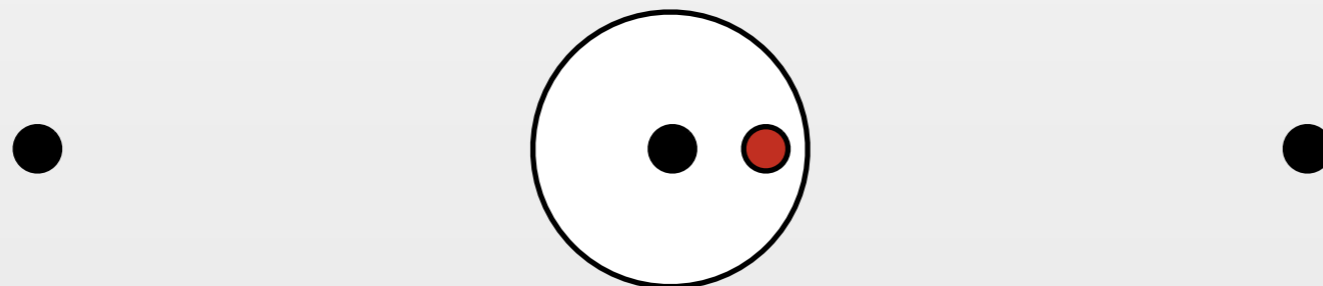
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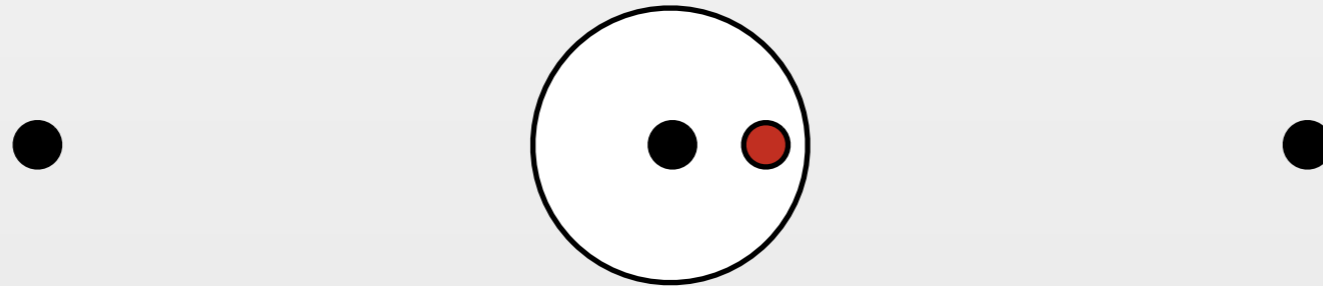
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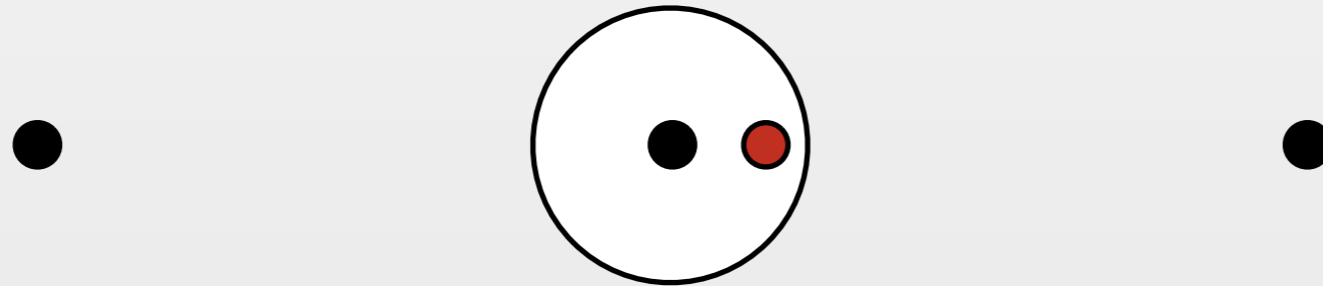
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Goal: Combinatorial Divide and Conquer

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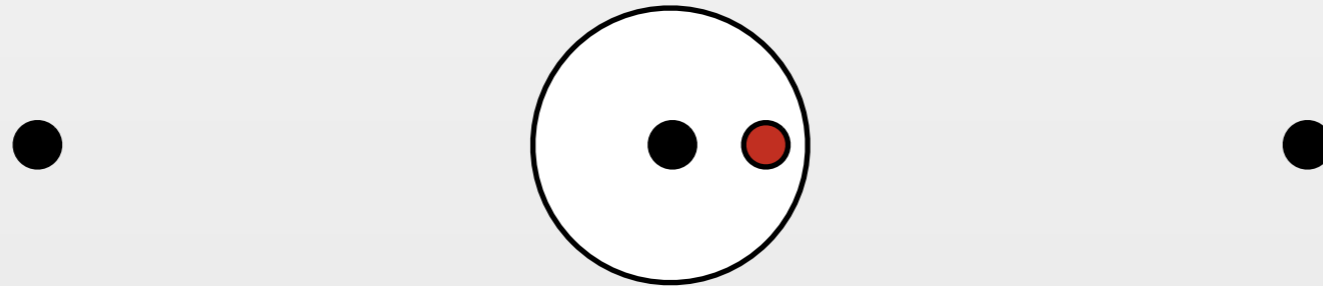


Goal: Combinatorial Divide and Conquer

Old: Progress was decreasing the radius by a factor of 2.

With hierarchical meshes, we can add inputs in any order!

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Goal: Combinatorial Divide and Conquer

Old: Progress was decreasing the radius by a factor of 2.

New: Decrease number of points in a ball by a factor of 2.

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Definition. *A range space is a pair (X, R) , where X is a set (the vertices) and R is a collection of subsets (the ranges).*

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Theorem: [Chazelle & Matousek 96] For ε, d fixed constants, ε -nets of size $O(1)$ can be computed in $O(n)$ deterministic time.

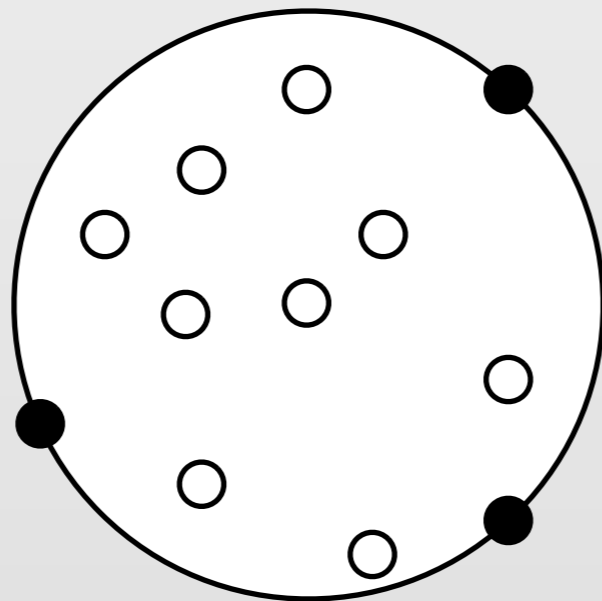
Ordering the inputs

For each D-Ball, select a $1/2d$ -net of the points it contains.

Take the union of these nets and call it a round.

Insert these.

Repeat.



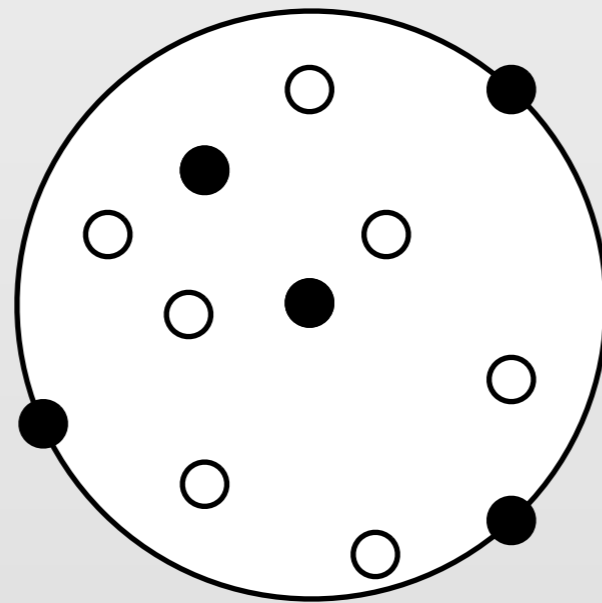
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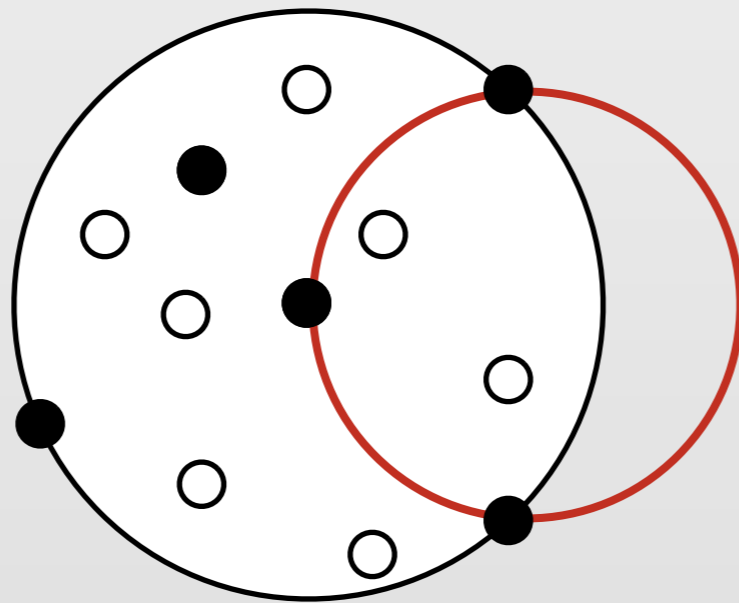
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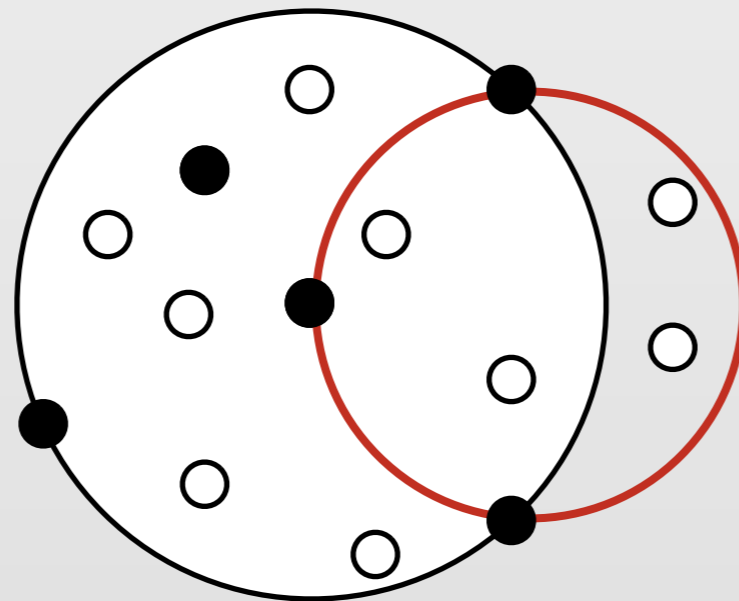


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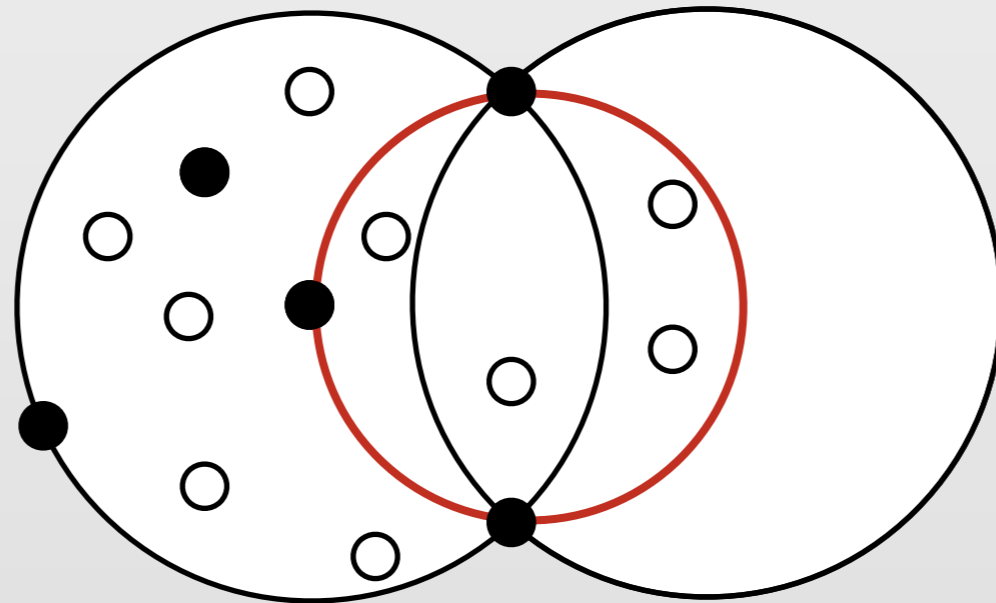


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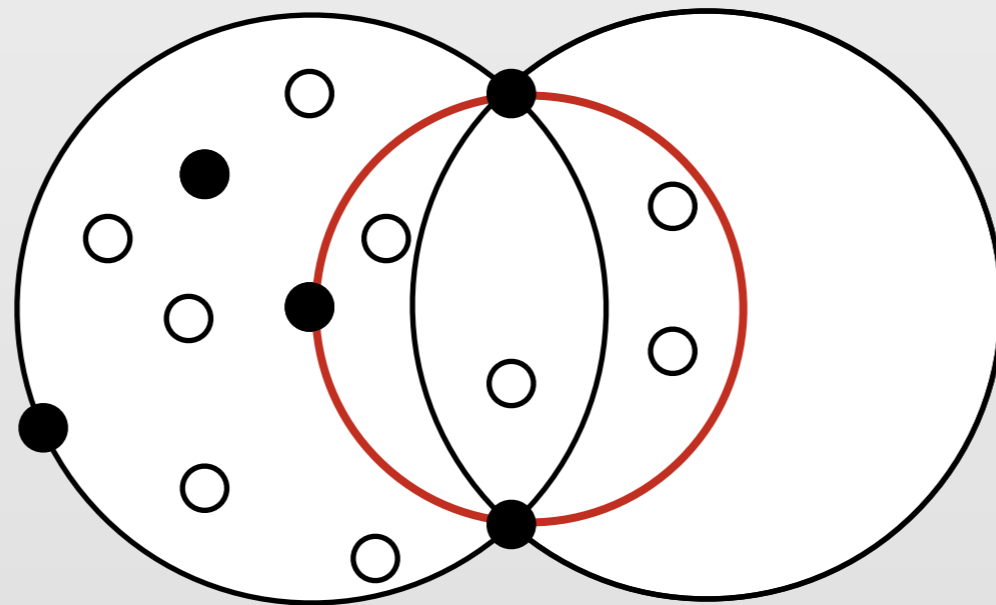


Ordering the inputs

For each D -Ball, select a $1/2d$ -net of the points it contains.
Take the union of these nets and call it a round.

Insert these.

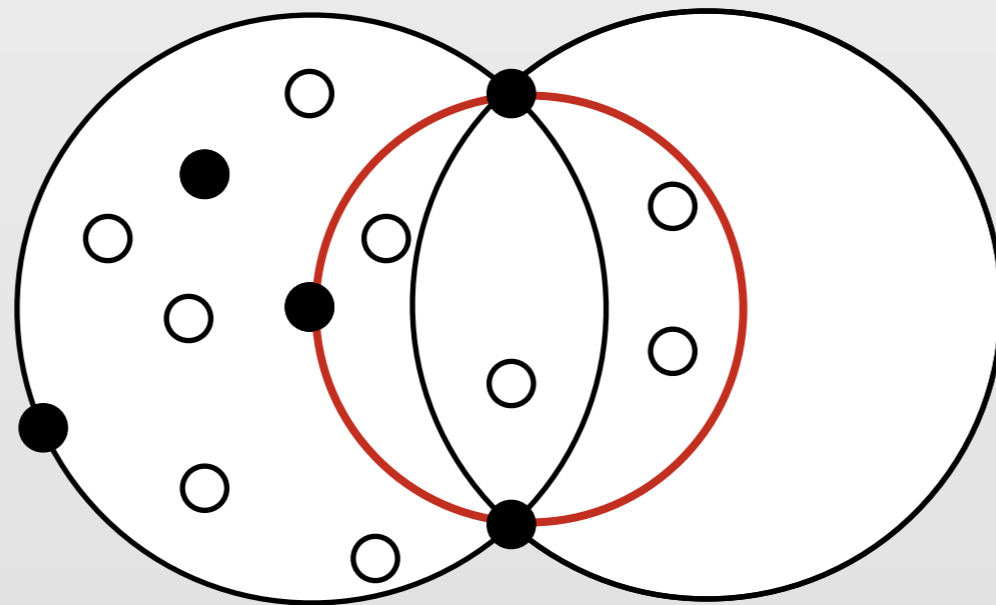
Repeat.



Lemma. *Let M be a set of vertices. If an open ball B contains no points of M , then B is contained in the union of d D -balls of M .*

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$\log(n)$ Rounds

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Amortized Cost of a Round is $O(n)$

Watch an uninserted point x .

Claim: x only gets touched $O(1)$ times per round.

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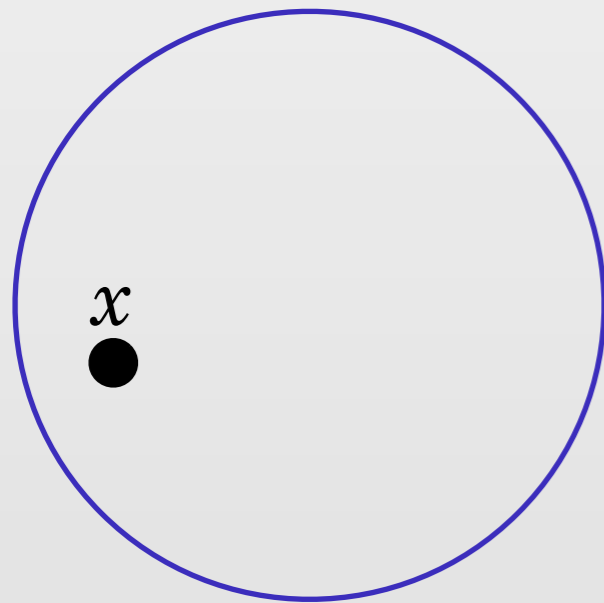
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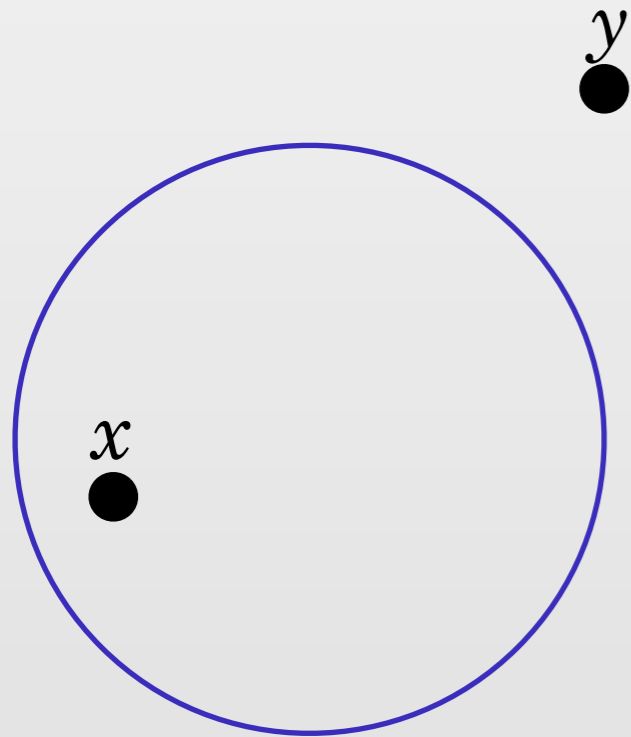
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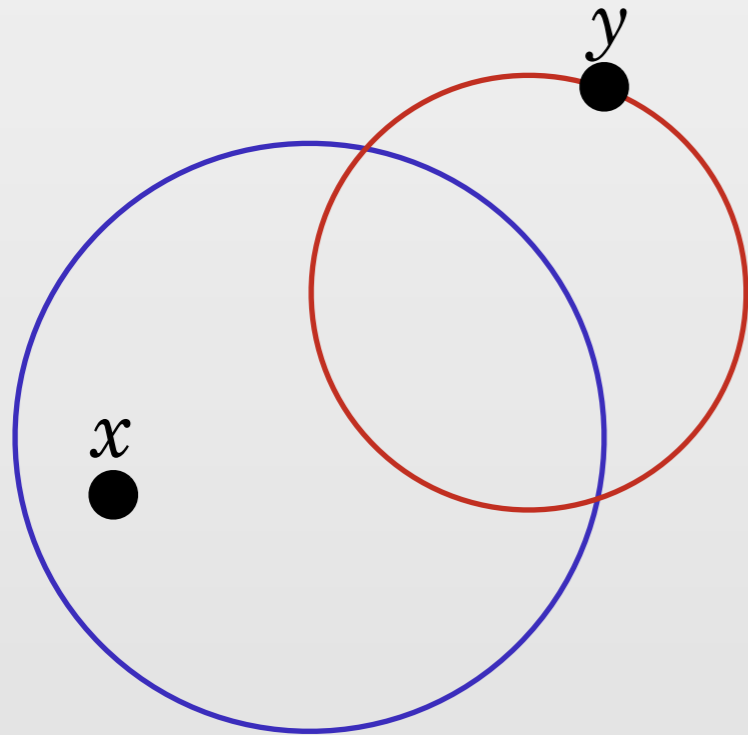
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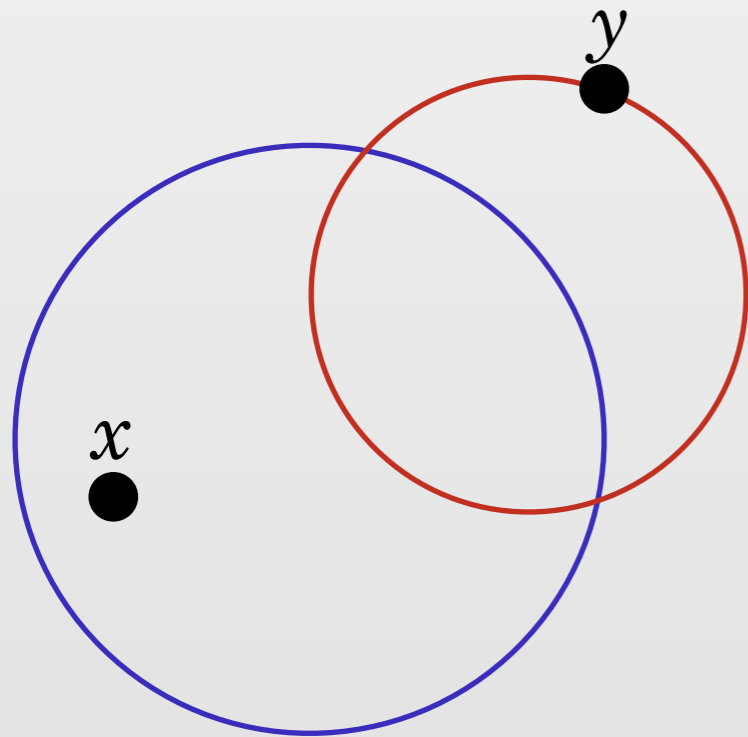
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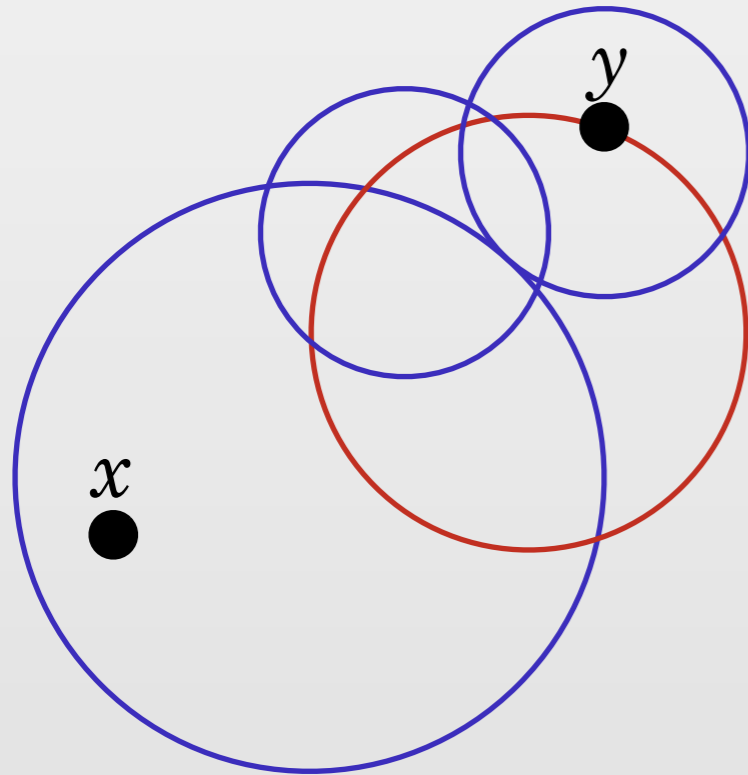


y touches $x \Rightarrow y$ is “close” to x
close = 2 hops among D-balls

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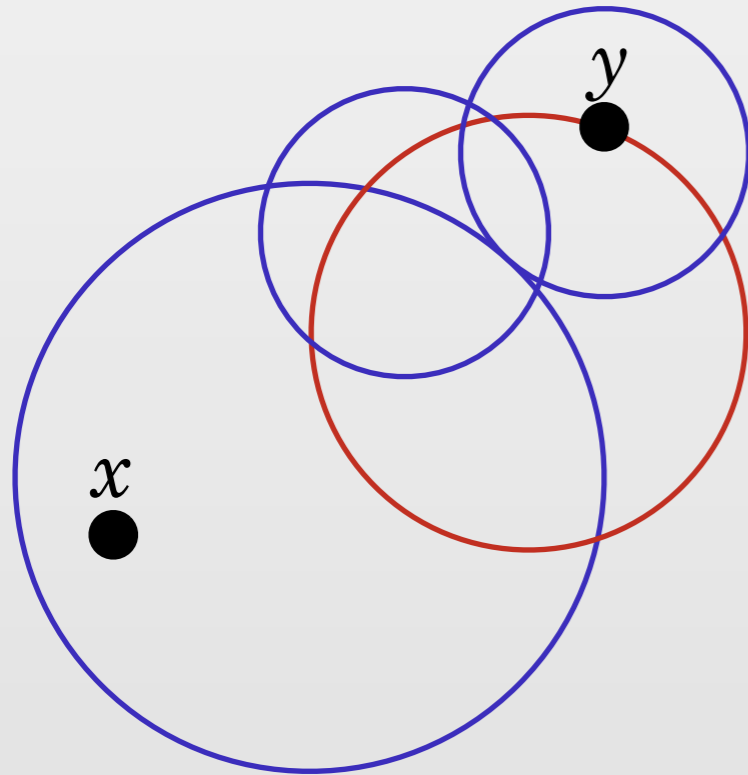


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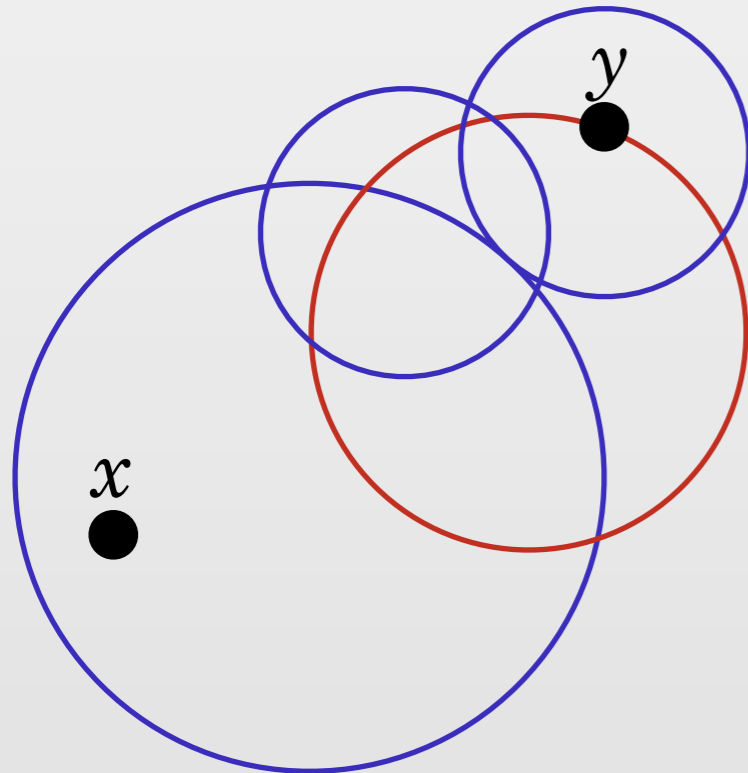
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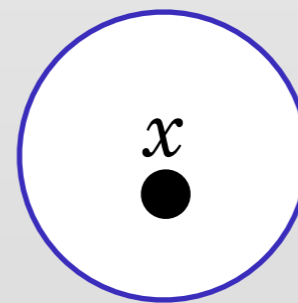
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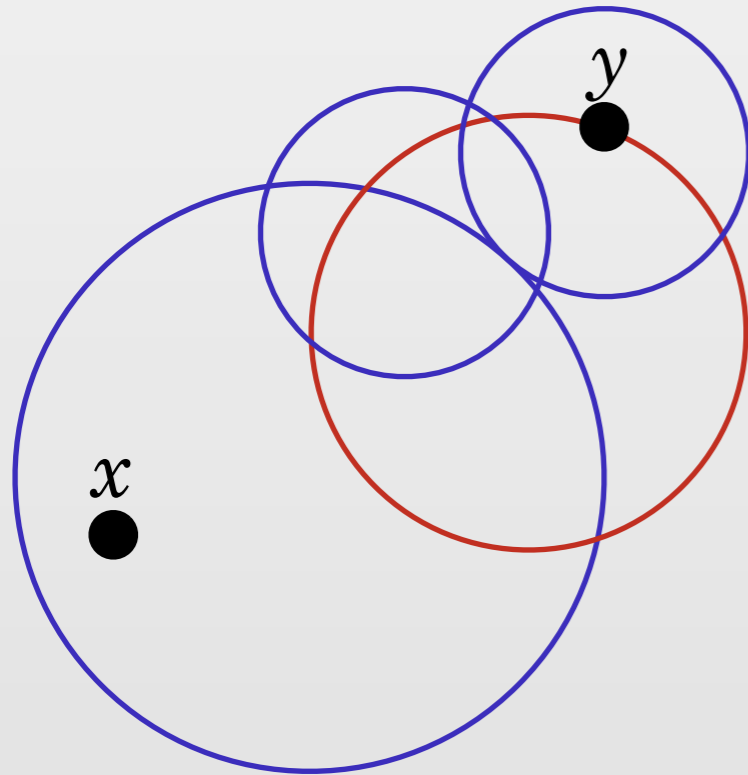
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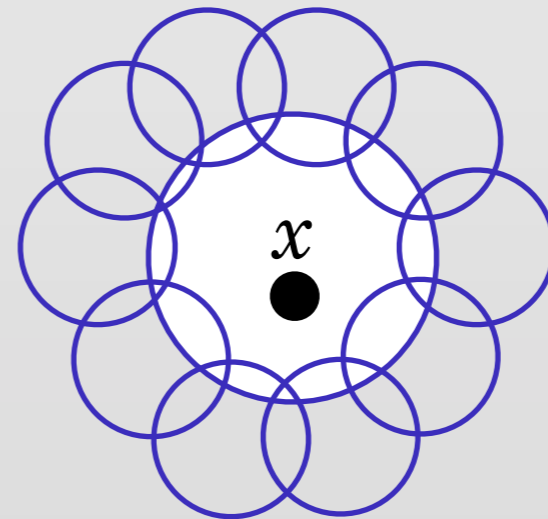
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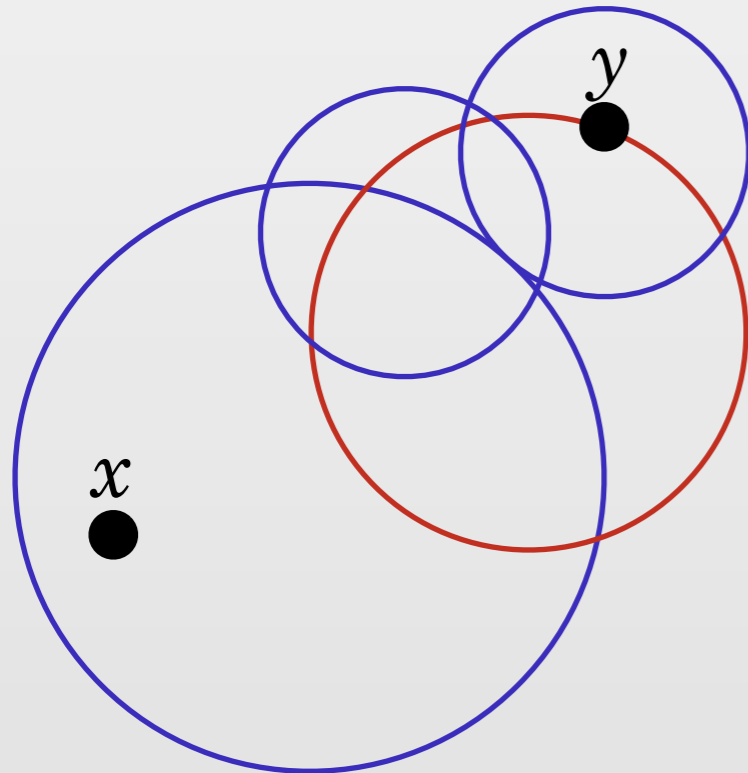
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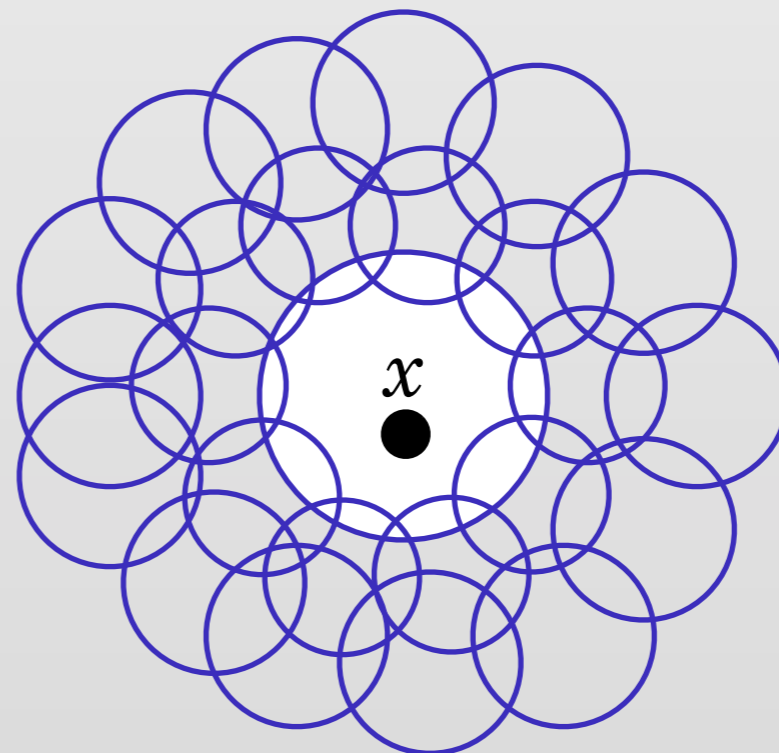
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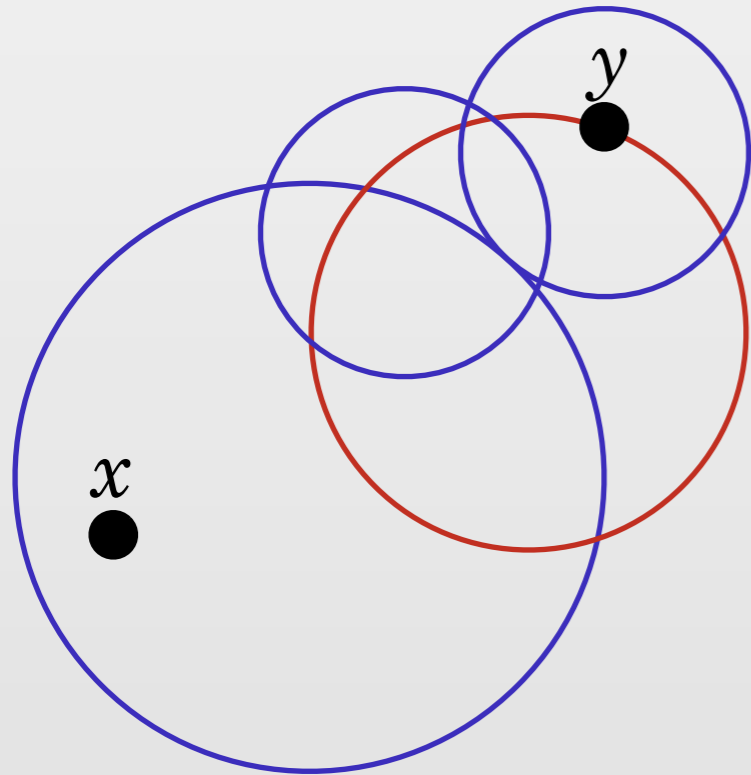
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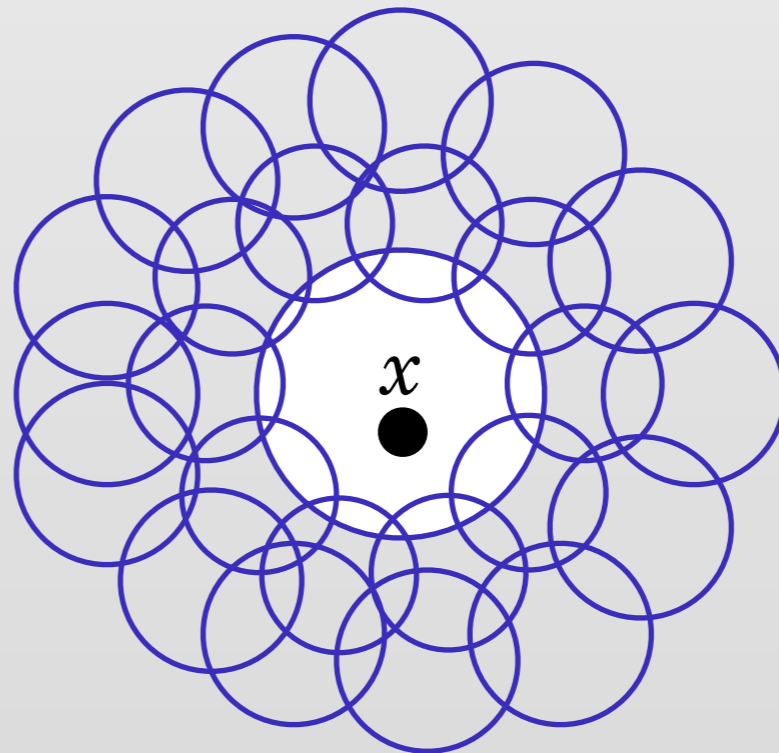
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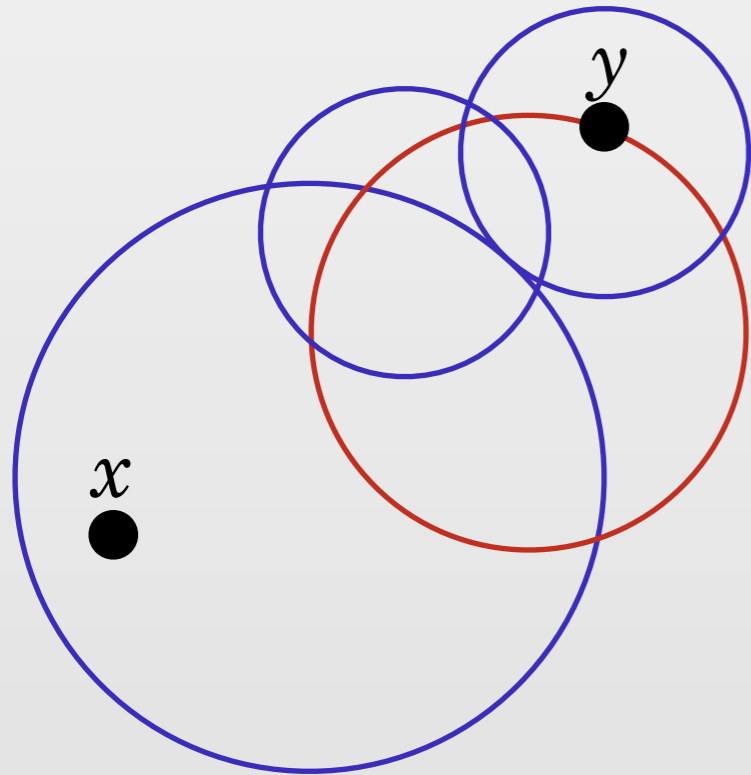
Only $O(1)$ points are added to any D-ball in a round.



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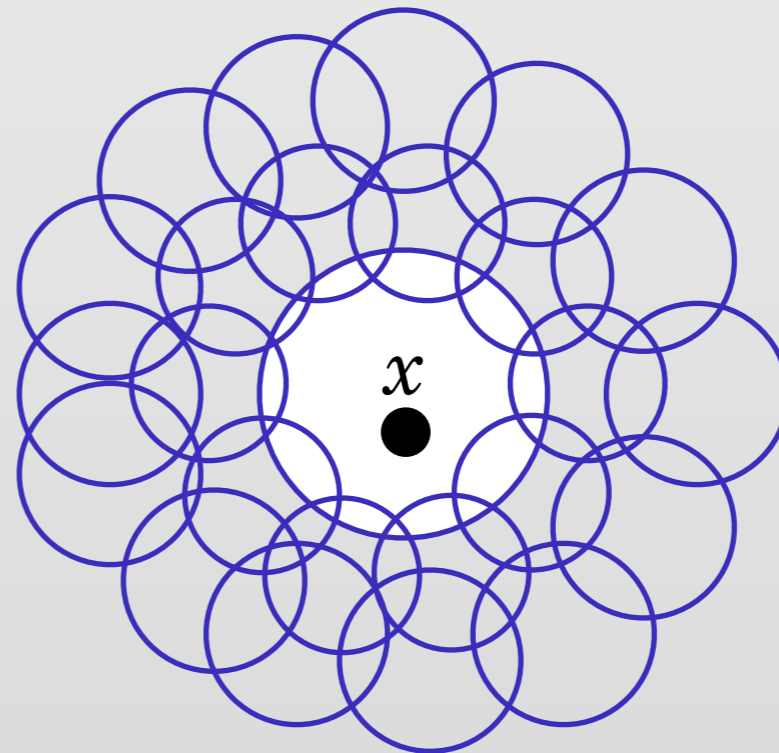


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Only $O(1)$ D-balls are within 2 hops.

Only $O(1)$ points are added to any D-ball in a round.

$O(n)$ total work per round.



Some Takeaways

Voronoi refinement meshes give structure to points, especially with respect to the ambient space.

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There are many possible new applications, waiting to be discovered (i.e. any time you would have used a Voronoi diagram).

Thanks.

