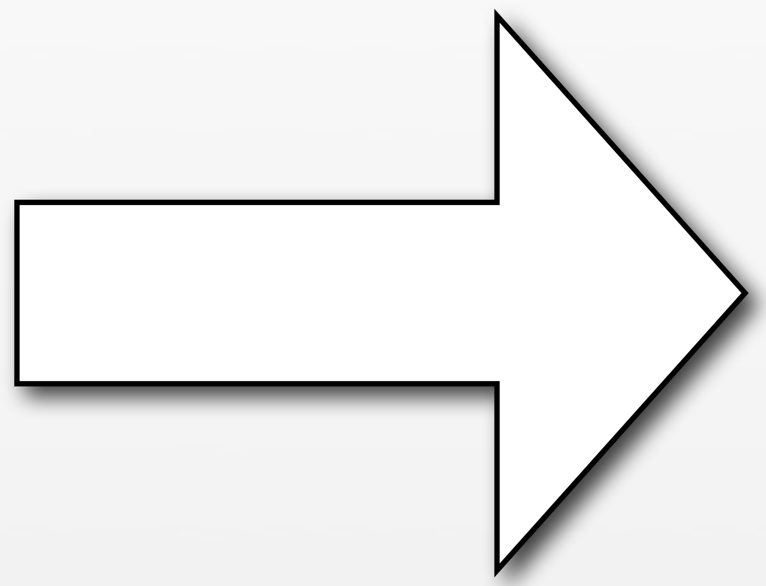
Searching for the center.

Don Sheehy
CMU Theory Lunch
October 8, 2008


THIS IS
Spinal Tap


It’s a fine line between stupid and clever.

The Divide and Conquer Game



How to win

The Divide and Conquer Game

How to win

The Divide and Conquer Game

Pick a center point.

How to win

The Divide and Conquer Game

Pick a center point.

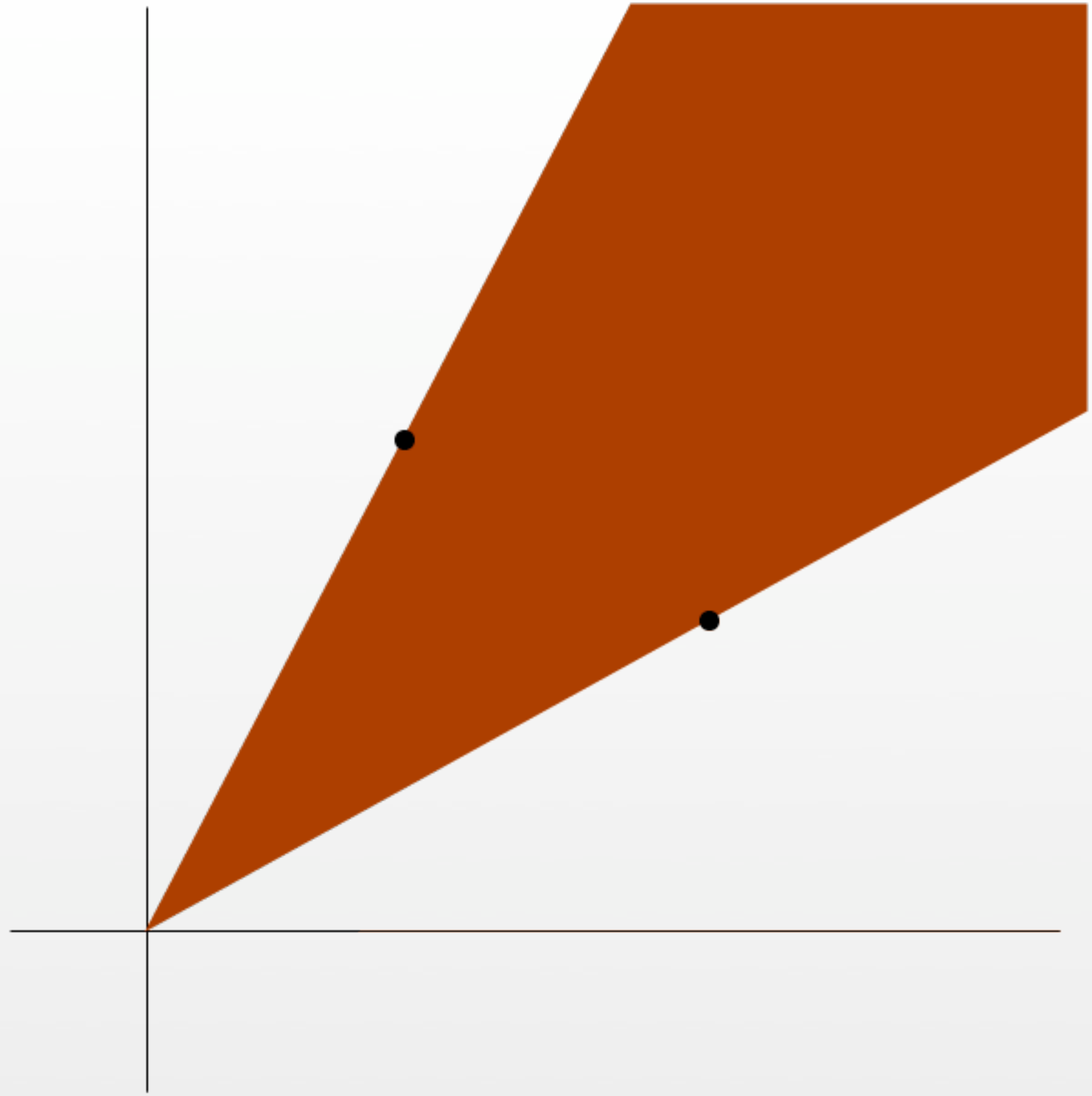
Given a set $S \subset \mathbb{R}^d$, a *center point* p is a point such that every closed halfspace with p on its boundary contains at least $\frac{n}{d+1}$ points of S.

Some definitions you probably already know.

Linear: $$\sum_{p_i \in P} c_i p_i$$

Linear: $$\sum_{p_i \in P} c_i p_i$$

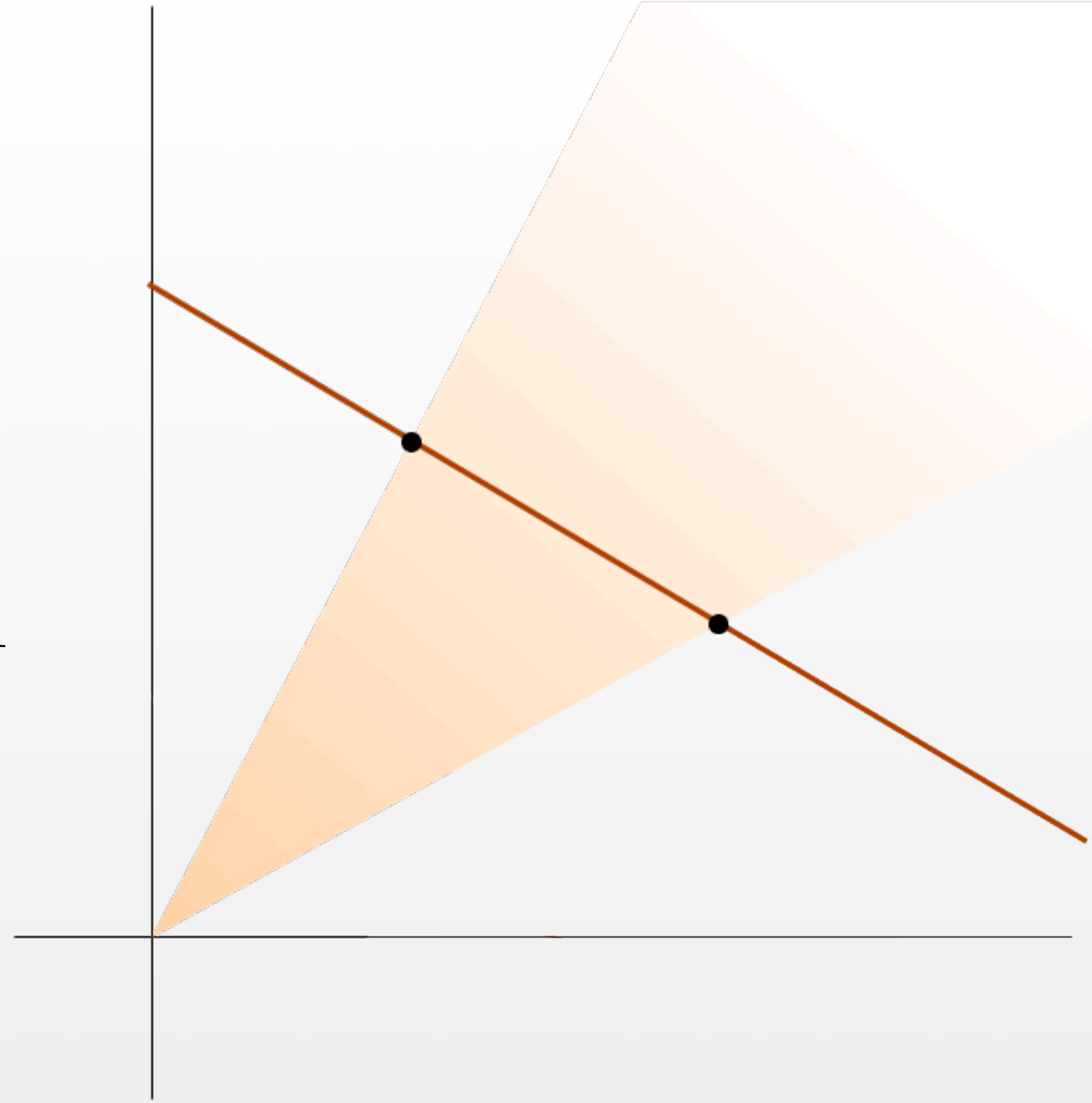
Nonnegative: $c_i \geq 0$



Linear: $\sum_{p_i \in P} c_i p_i$

Nonnegative: $c_i \geq 0$

Affine: $\sum c_i = 1$

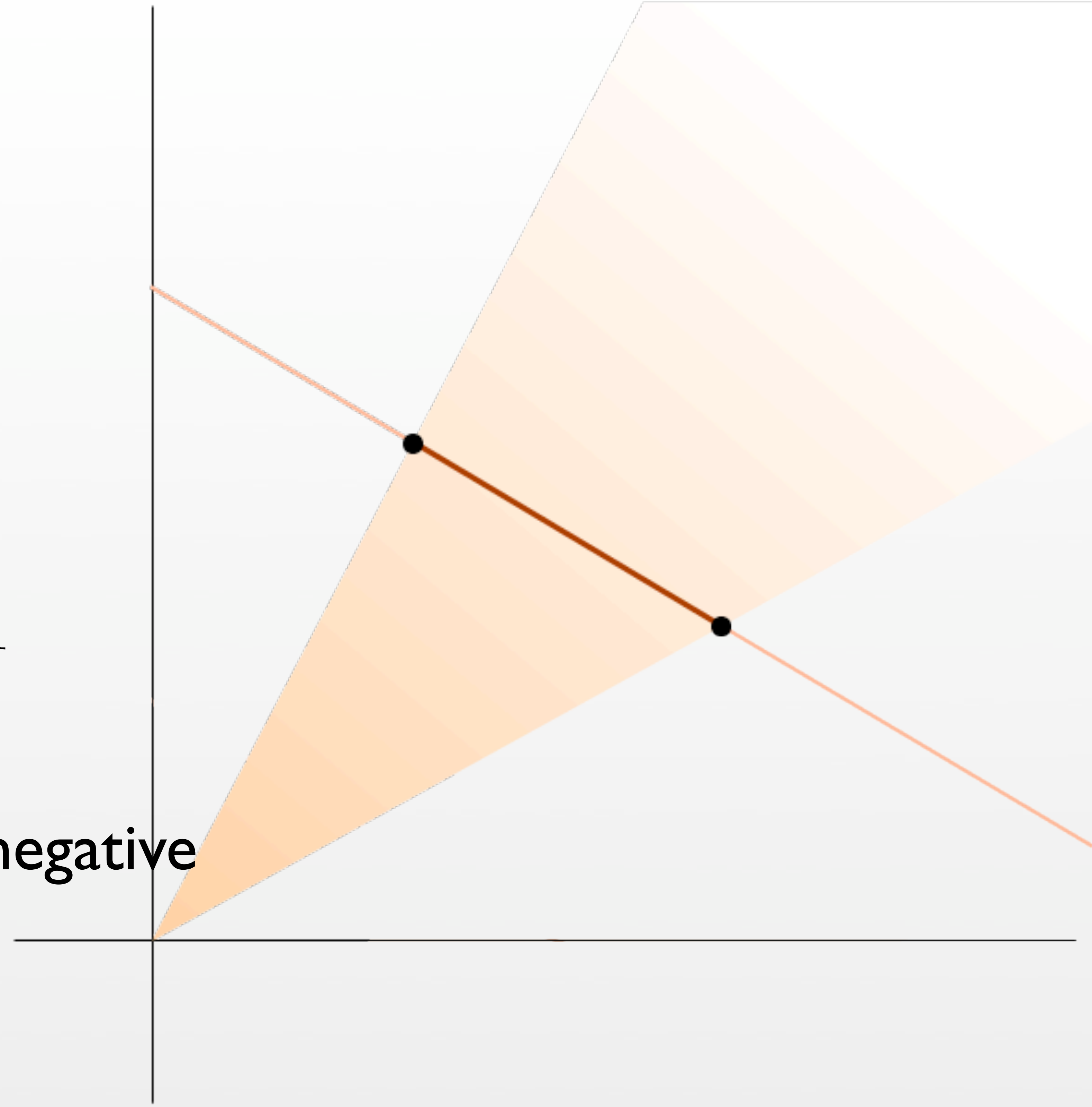


Linear: $\sum_{p_i \in P} c_i p_i$

Nonnegative: $c_i \geq 0$

Affine: $\sum c_i = 1$

Convex: Affine and Nonnegative



Radon => Helly => Center Points Exist.

Radon's Theorem

If $P \in \mathbb{R}^d$ has $d+2$ (or more) points then there is a partition of P into $(U, \overline{U})$ such that $conv(U) \cap conv(\overline{U})$ is nonempty.

Radon's Theorem

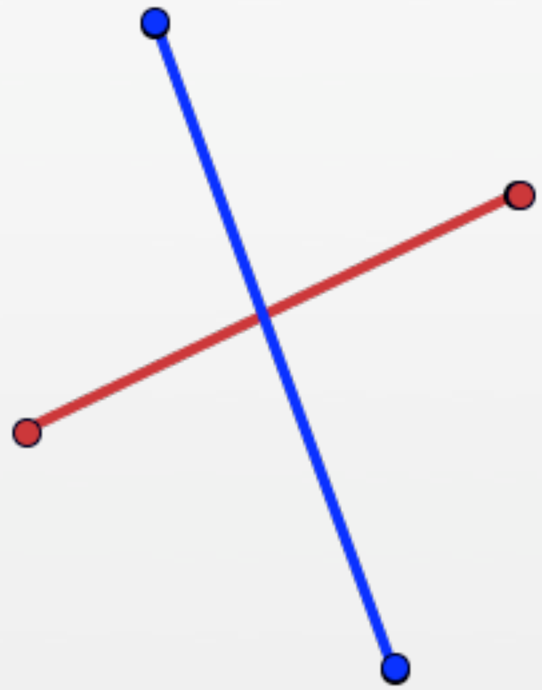
If $P \in \mathbb{R}^d$ has $d+2$ (or more) points then there is a partition of P into $(U, \overline{U})$ such that $conv(U) \cap conv(\overline{U})$ is nonempty.

Radon's Theorem

If $P \in \mathbb{R}^d$ has $d+2$ (or more) points then there is a partition of P into $(U, \overline{U})$ such that $conv(U) \cap conv(\overline{U})$ is nonempty.

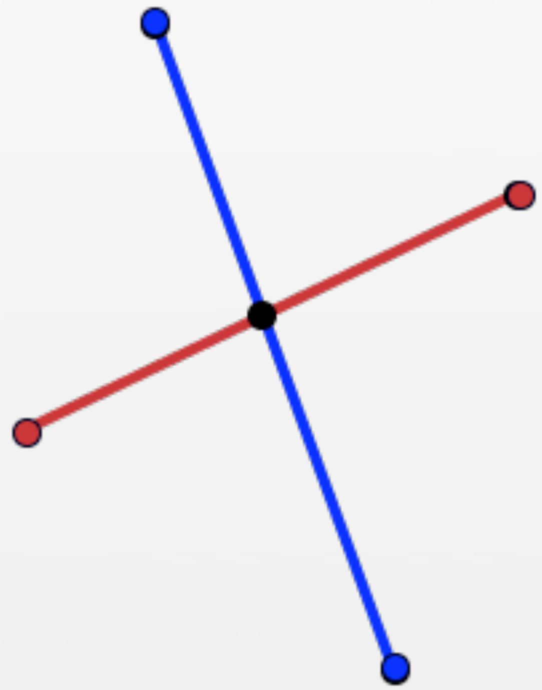
Radon's Theorem

If $P \in \mathbb{R}^d$ has $d+2$ (or more) points then there is a partition of P into $(U, \overline{U})$ such that $conv(U) \cap conv(\overline{U})$ is nonempty.



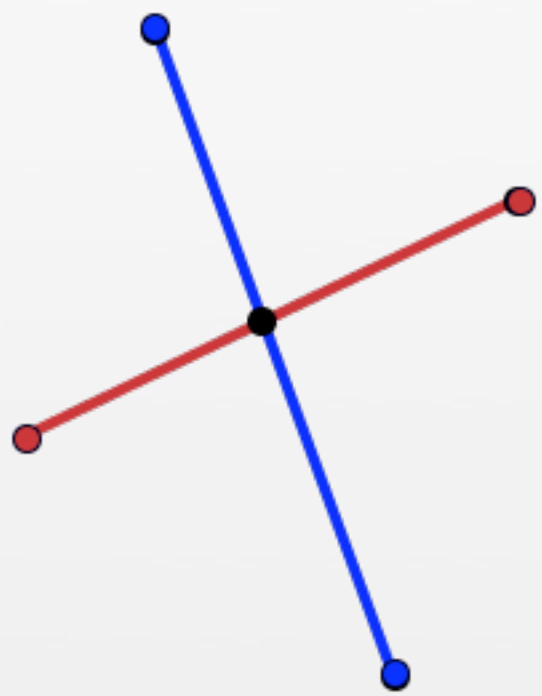
Radon's Theorem

If $P \in \mathbb{R}^d$ has $d+2$ (or more) points then there is a partition of P into $(U, \overline{U})$ such that $conv(U) \cap conv(\overline{U})$ is nonempty.



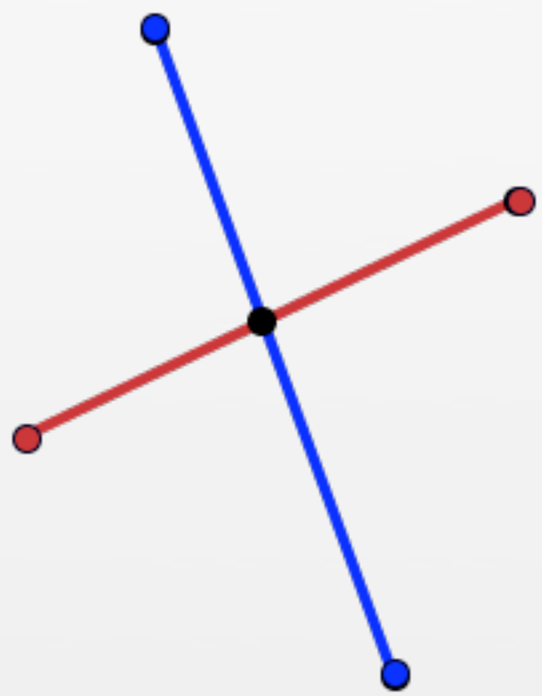
Radon's Theorem

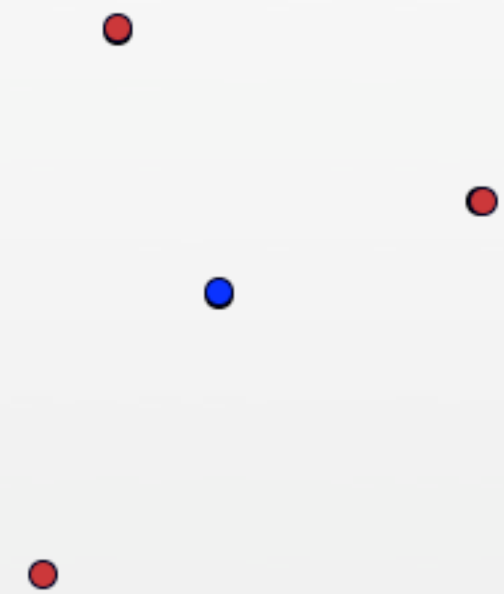
If $P \in \mathbb{R}^d$ has $d+2$ (or more) points then there is a partition of P into $(U, \overline{U})$ such that $conv(U) \cap conv(\overline{U})$ is nonempty.



Radon's Theorem

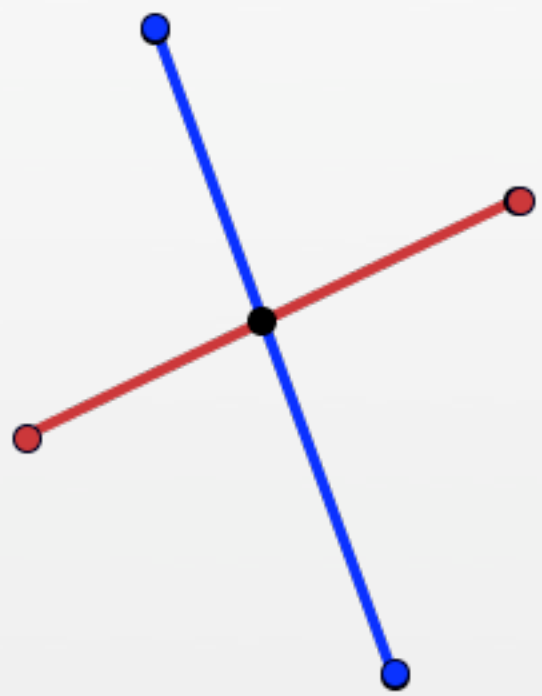
If $P \in \mathbb{R}^d$ has $d+2$ (or more) points then there is a partition of P into $(U, \overline{U})$ such that $conv(U) \cap conv(\overline{U})$ is nonempty.

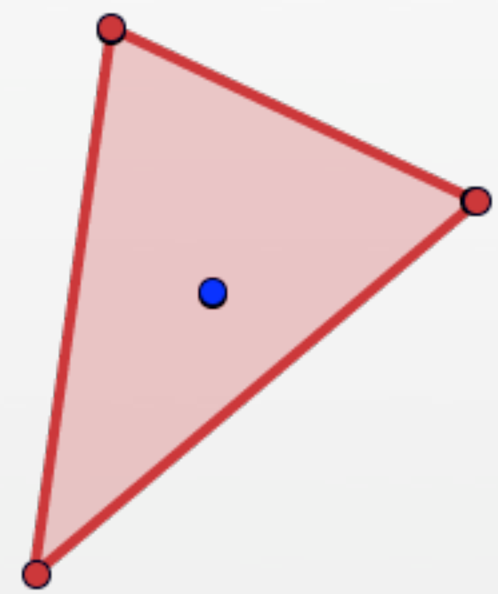




Radon's Theorem

If $P \in \mathbb{R}^d$ has $d+2$ (or more) points then there is a partition of P into $(U, \overline{U})$ such that $conv(U) \cap conv(\overline{U})$ is nonempty.





Radon's Theorem

If $P \in \mathbb{R}^d$ has $d+2$ (or more) points then there is a partition of P into $(U, \overline{U})$ such that $conv(U) \cap conv(\overline{U})$ is nonempty.

Radon's Theorem

If $P \in \mathbb{R}^d$ has $d+2$ (or more) points then there is a partition of P into $(U, \overline{U})$ such that $conv(U) \cap conv(\overline{U})$ is nonempty.

$$\sum_{i=1}^{d+2} c_i p_i = 0$$

$$\sum_{i=1}^{d+2} c_i = 0$$

Radon's Theorem

If $P \in \mathbb{R}^d$ has $d+2$ (or more) points then there is a partition of P into $(U, \overline{U})$ such that $conv(U) \cap conv(\overline{U})$ is nonempty.

$$I^+ = \{i : c_i > 0\}$$
$$I^- = \{i : c_i < 0\}$$

$$\sum_{i=1}^{d+2} c_i p_i = 0$$

$$\sum_{i \in I^+} c_i p_i = \sum_{i \in I^-} (-c_i) p_i$$

$$\sum_{i=1}^{d+2} c_i = 0$$

Radon's Theorem

If $P \in \mathbb{R}^d$ has $d+2$ (or more) points then there is a partition of P into $(U, \overline{U})$ such that $conv(U) \cap conv(\overline{U})$ is nonempty.

$$I^+ = \{i : c_i > 0\}$$

$$I^- = \{i : c_i < 0\}$$

$$\sum_{i=1}^{d+2} c_i p_i = 0$$

$$\sum_{i \in I^+} c_i p_i = \sum_{i \in I^-} (-c_i) p_i$$

$$\sum_{i=1}^{d+2} c_i = 0$$

$$\sum_{i \in I^+} c_i = \sum_{i \in I^-} (-c_i)$$

Radon's Theorem

If $P \in \mathbb{R}^d$ has $d+2$ (or more) points then there is a partition of P into $(U, \overline{U})$ such that $conv(U) \cap conv(\overline{U})$ is nonempty.

$$I^+ = \{i : c_i > 0\}$$

$$I^- = \{i : c_i < 0\}$$

$$\sum_{i=1}^{d+2} c_i p_i = 0$$

$$\sum_{i \in I^+} c_i p_i = \sum_{i \in I^-} (-c_i) p_i$$

$$\sum_{i=1}^{d+2} c_i = 0$$

$$\sum_{i \in I^+} c_i = \sum_{i \in I^-} (-c_i)$$

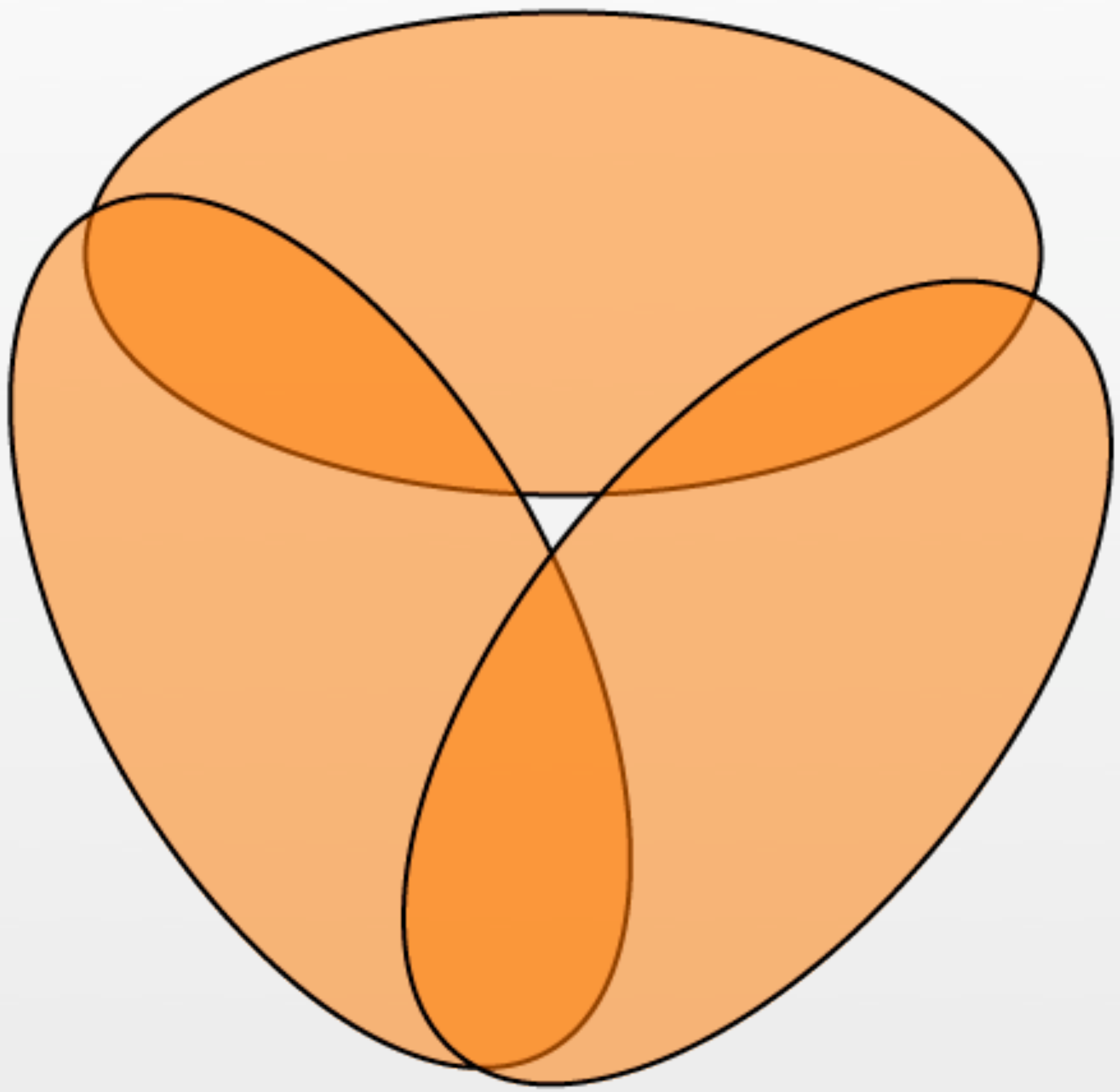
$$x = \sum_{i \in I^+} \left(\frac{c_i}{\sum_{j \in I^+} c_j} \right) p_i = \sum_{i \in I^-} \left(\frac{-c_i}{\sum_{j \in I^-} -c_j} \right) p_i$$

Helly's Theorem

Given some convex sets in $\mathbb{R}^d$ such that every $d+1$ sets have common intersection, then the whole collection of sets has a common intersection.

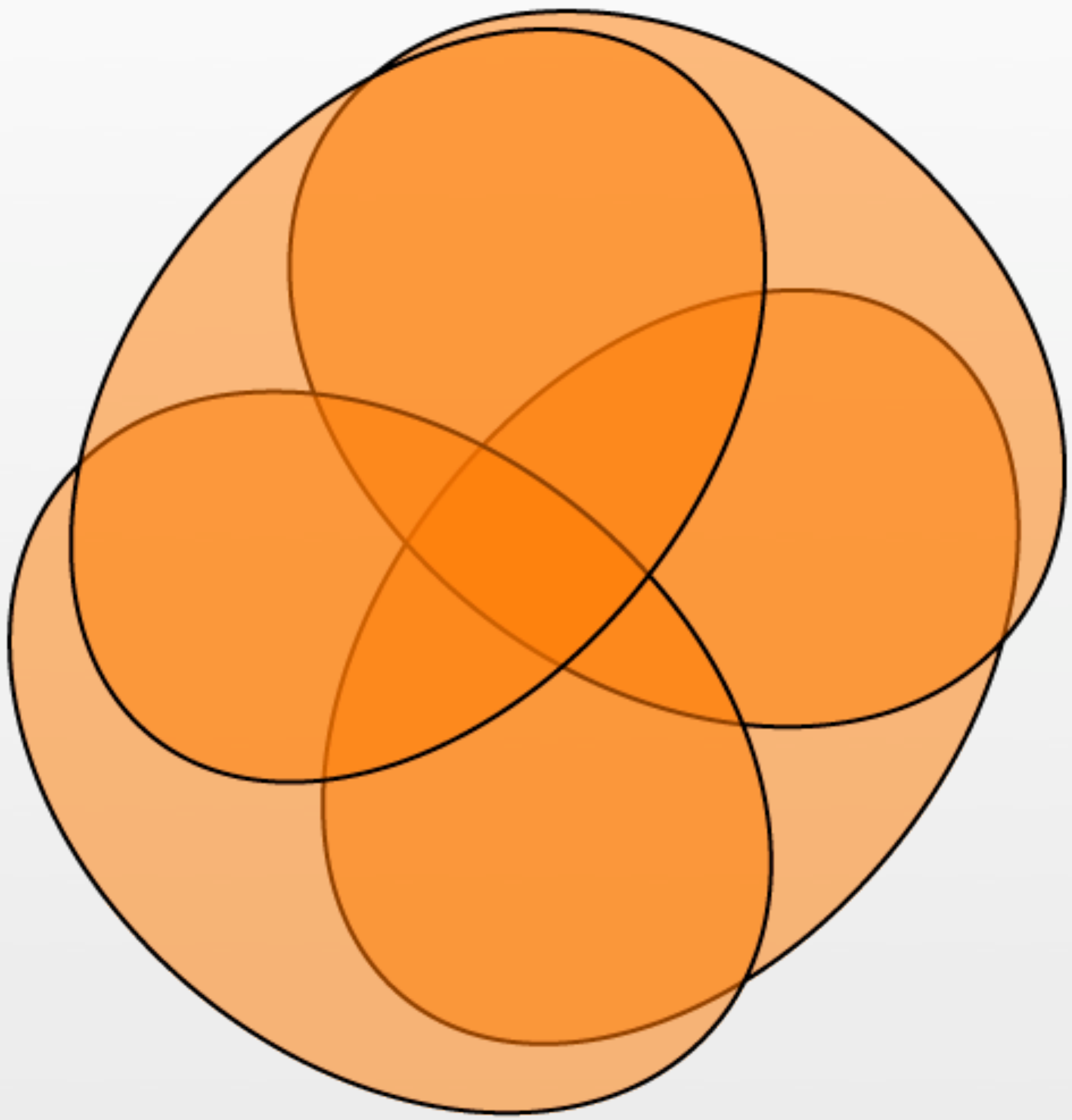
Helly's Theorem

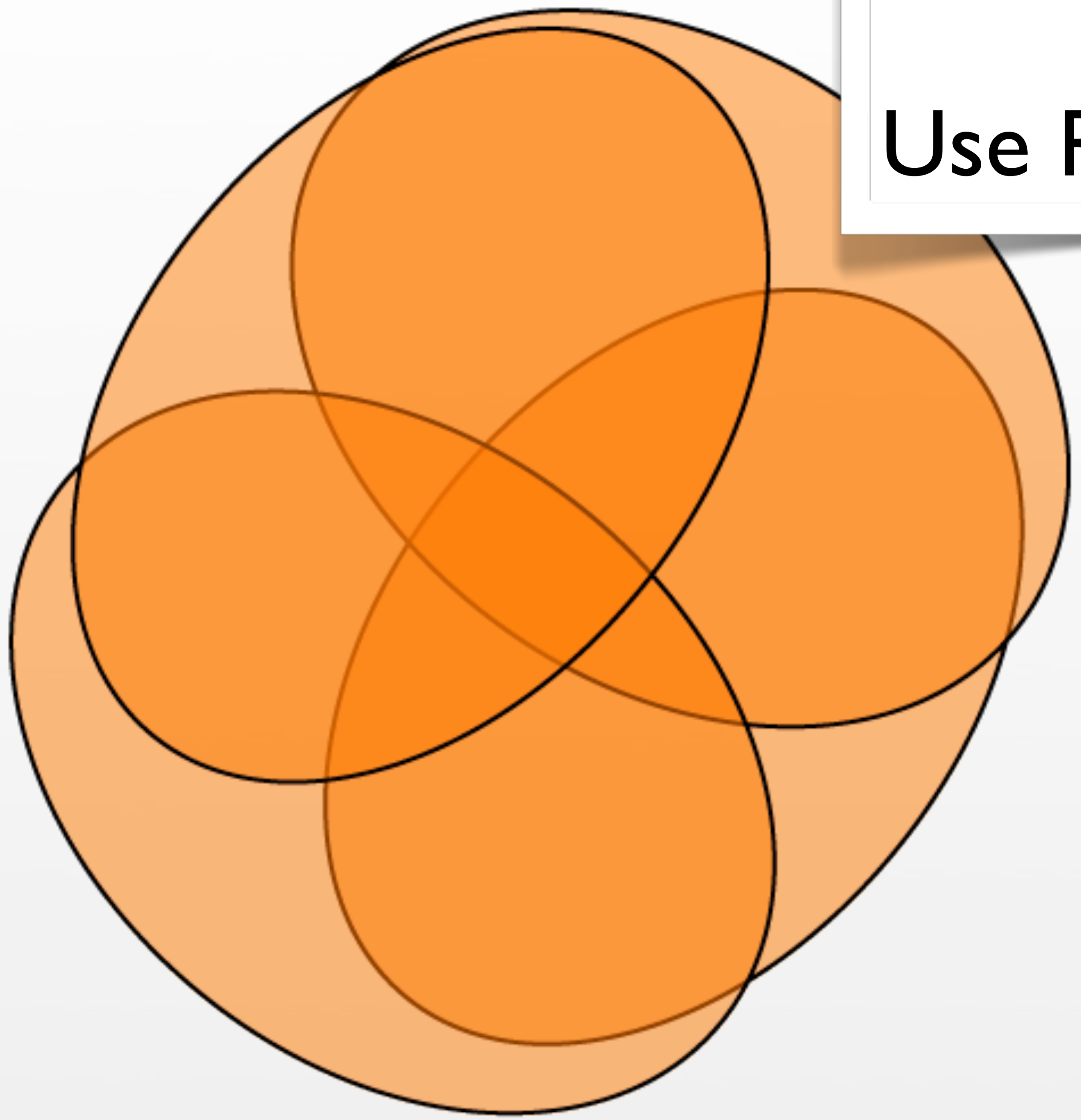
Given some convex sets in $\mathbb{R}^d$ such that every $d+1$ sets have common intersection, then the whole collection of sets has a common intersection.



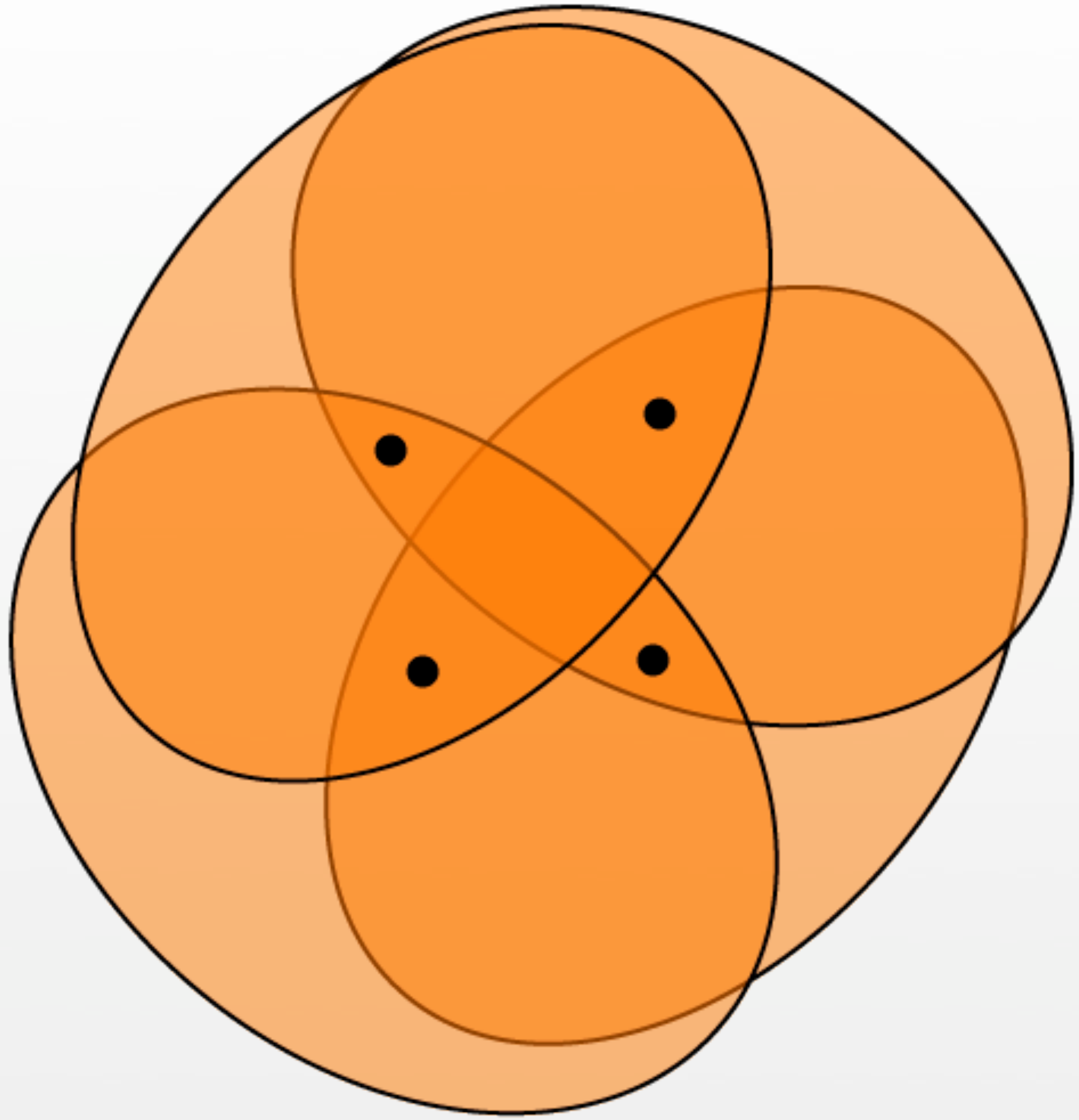
Helly's Theorem

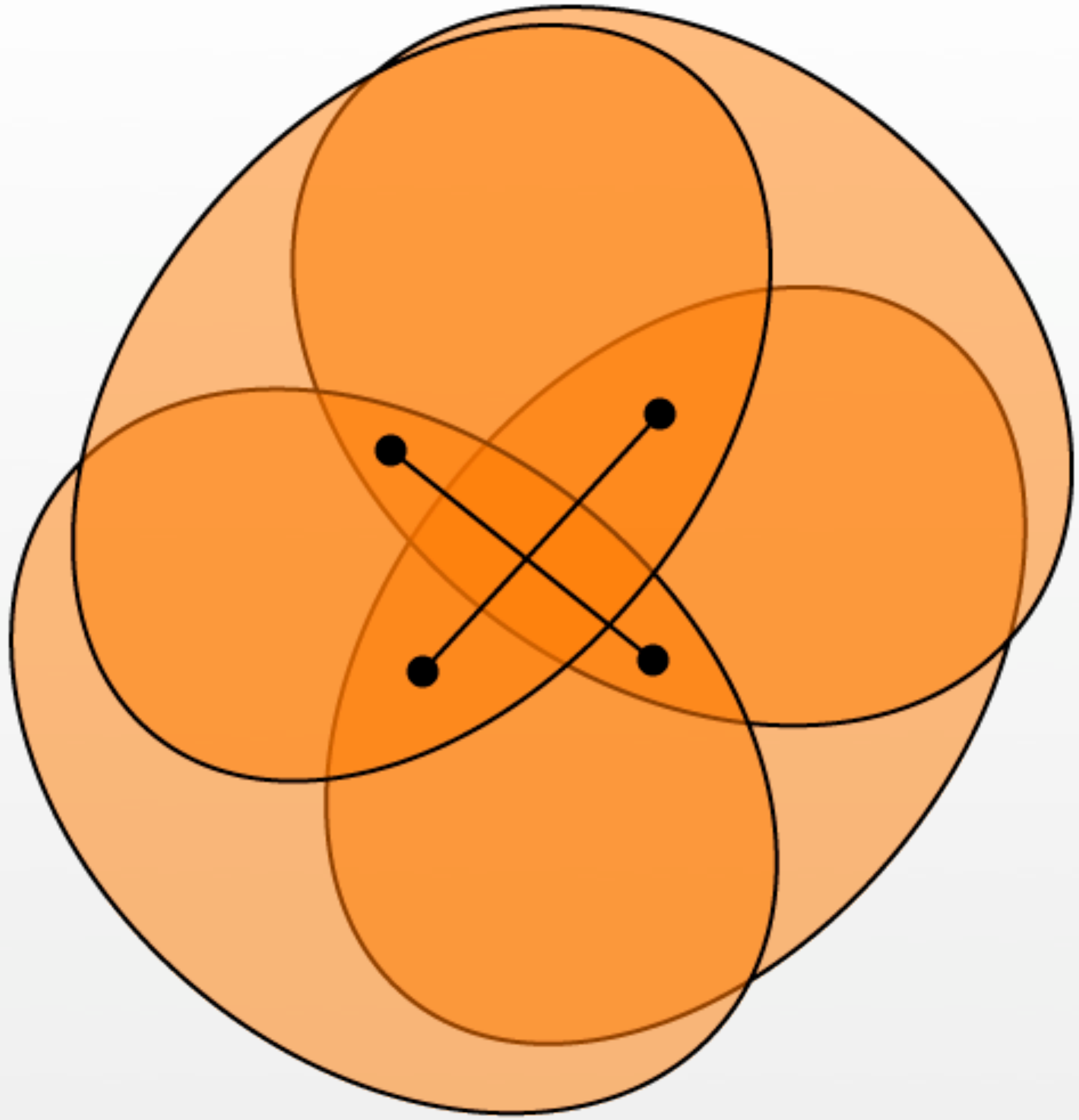
Given some convex sets in $\mathbb{R}^d$ such that every $d+1$ sets have common intersection, then the whole collection of sets has a common intersection.

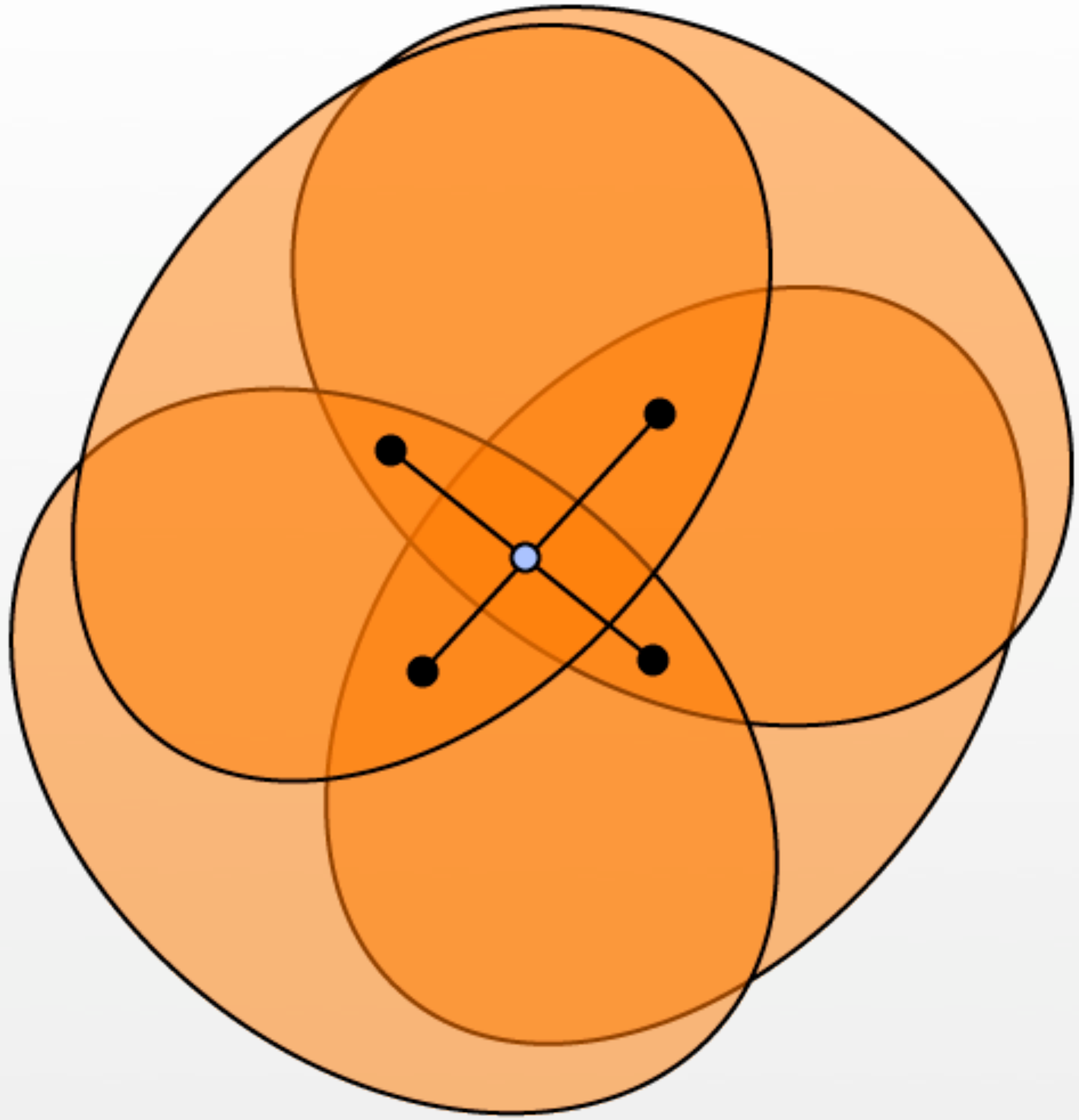


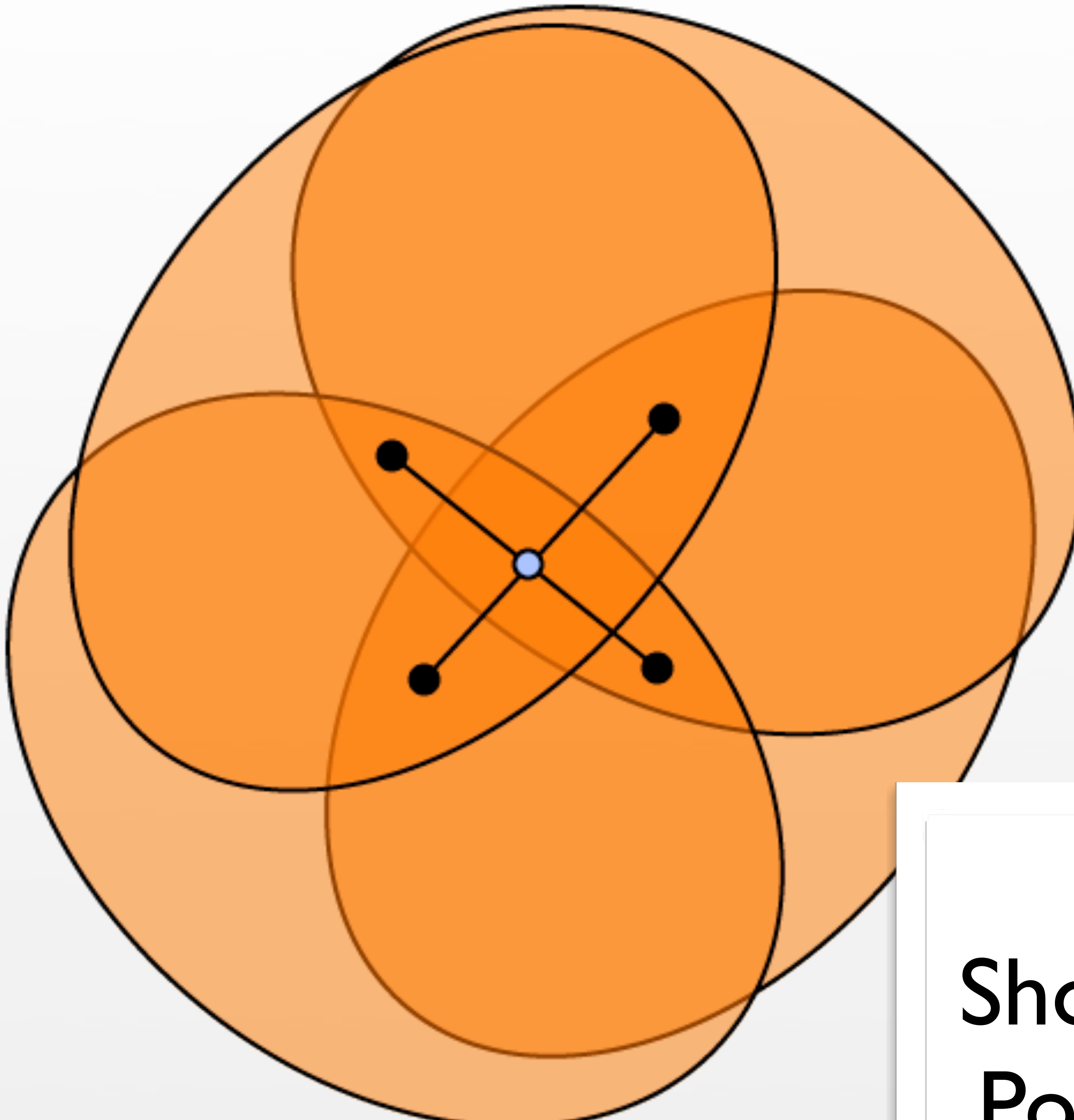


Proof Hint:
Use Radon’s Theorem!



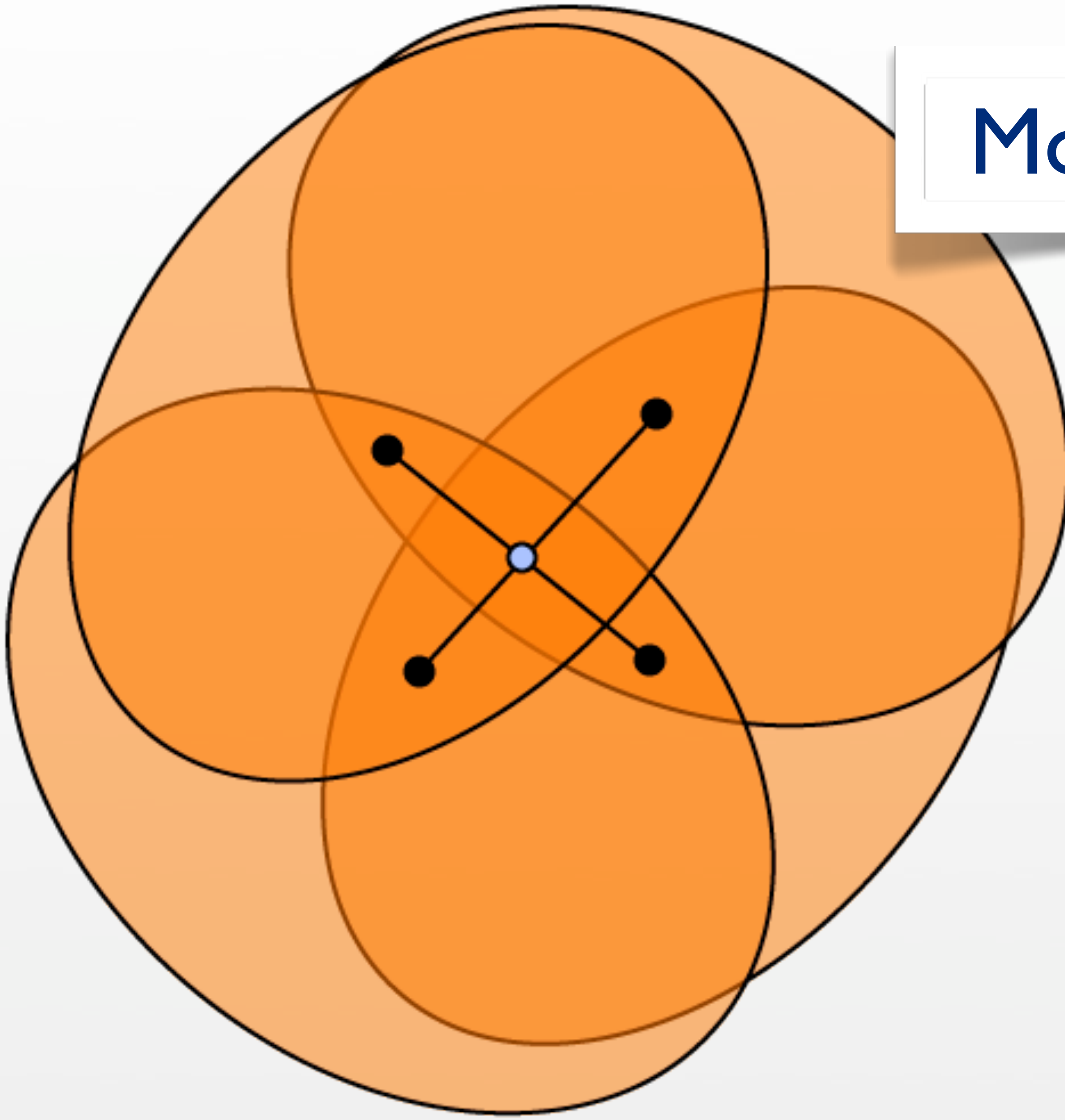






Fun Exercise:

Show that the Radon Point is in every set.



More than d+2 sets?

The Center Point Theorem

Consider the set of all minimal halfspaces containing at least $\frac{dn}{d+1} + 1$ points.

Observe that every $d + 1$ have a common intersection.

Helly's Theorem implies that all the halfspaces have a common intersection. The intersection is the set of center points.

Tverberg's Theorem

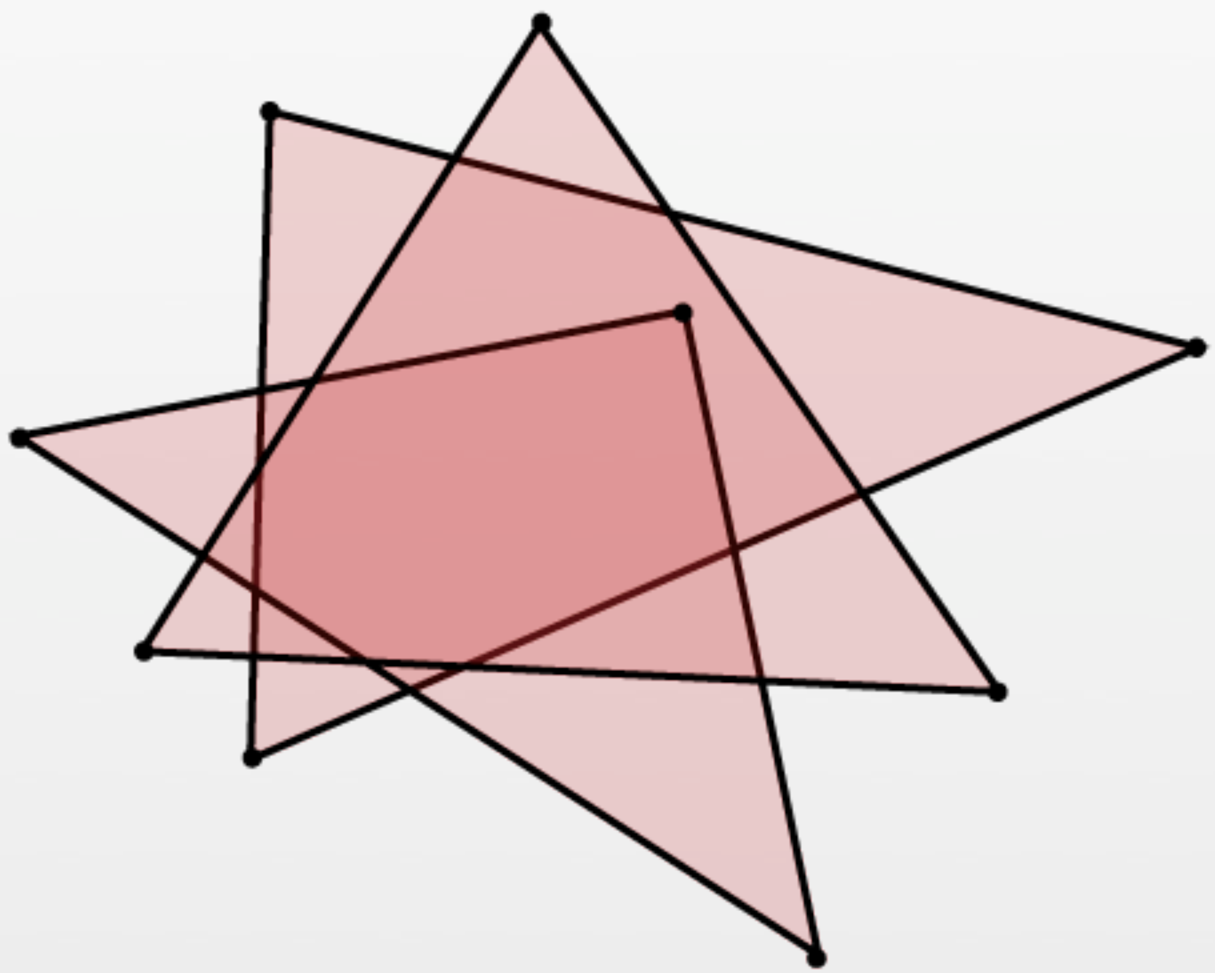
Let S be a set of at least $(d+1)(r-1)+1$ points in $\mathbb{R}^d$. There exists a partition of S into r subsets $X_1, \ldots, X_r$ such that $\bigcap_{i=1}^{r} conv(X_i) \neq \emptyset$

Tverberg's Theorem

Let S be a set of at least $(d+1)(r-1)+1$ points in $\mathbb{R}^d$. There exists a partition of S into r subsets $X_1, \ldots, X_r$ such that $\bigcap_{i=1}^{r} conv(X_i) \neq \emptyset$

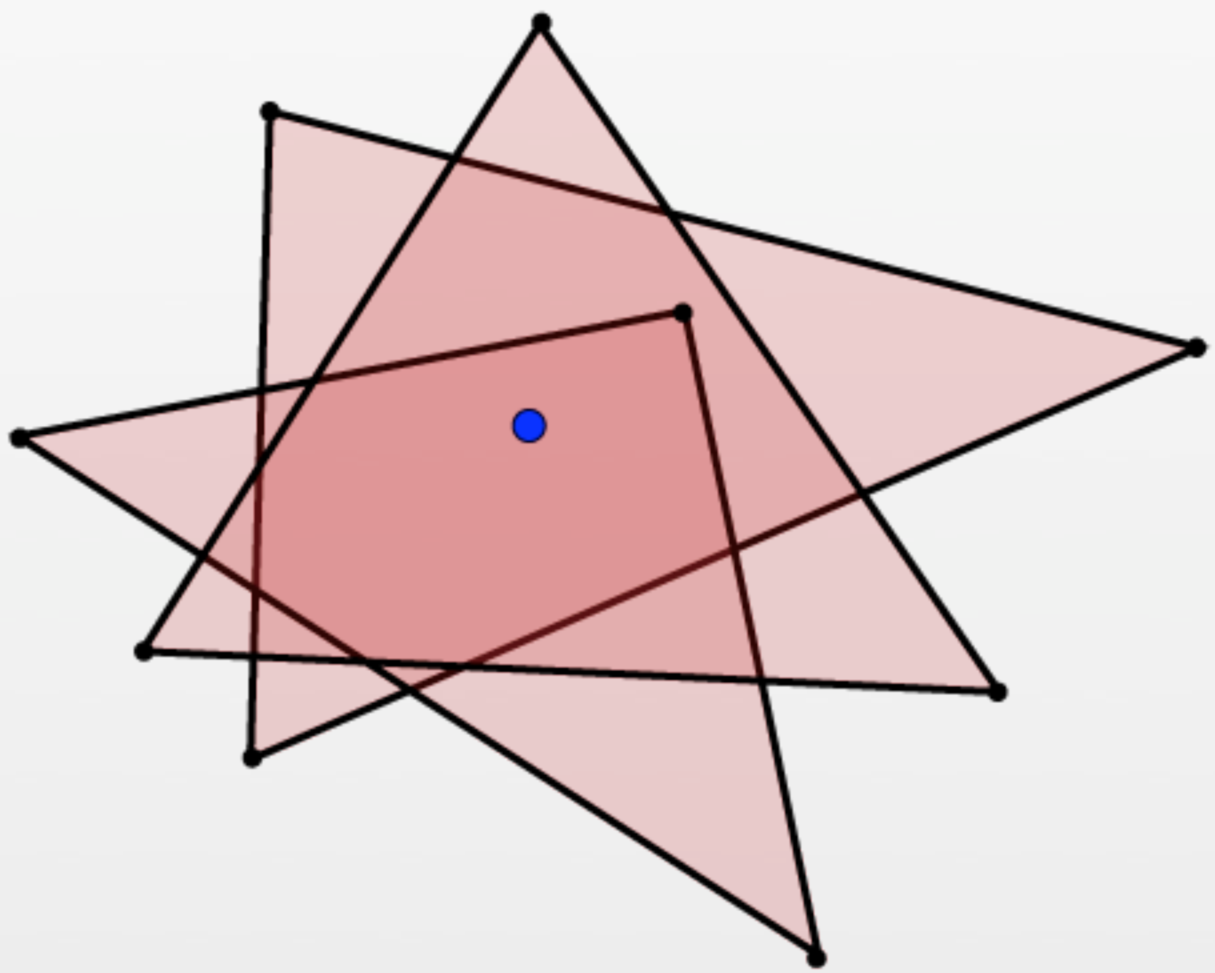
Tverberg's Theorem

Let S be a set of at least $(d+1)(r-1)+1$ points in $\mathbb{R}^d$. There exists a partition of S into r subsets $X_1, \ldots, X_r$ such that $\bigcap_{i=1}^{r} conv(X_i) \neq \emptyset$



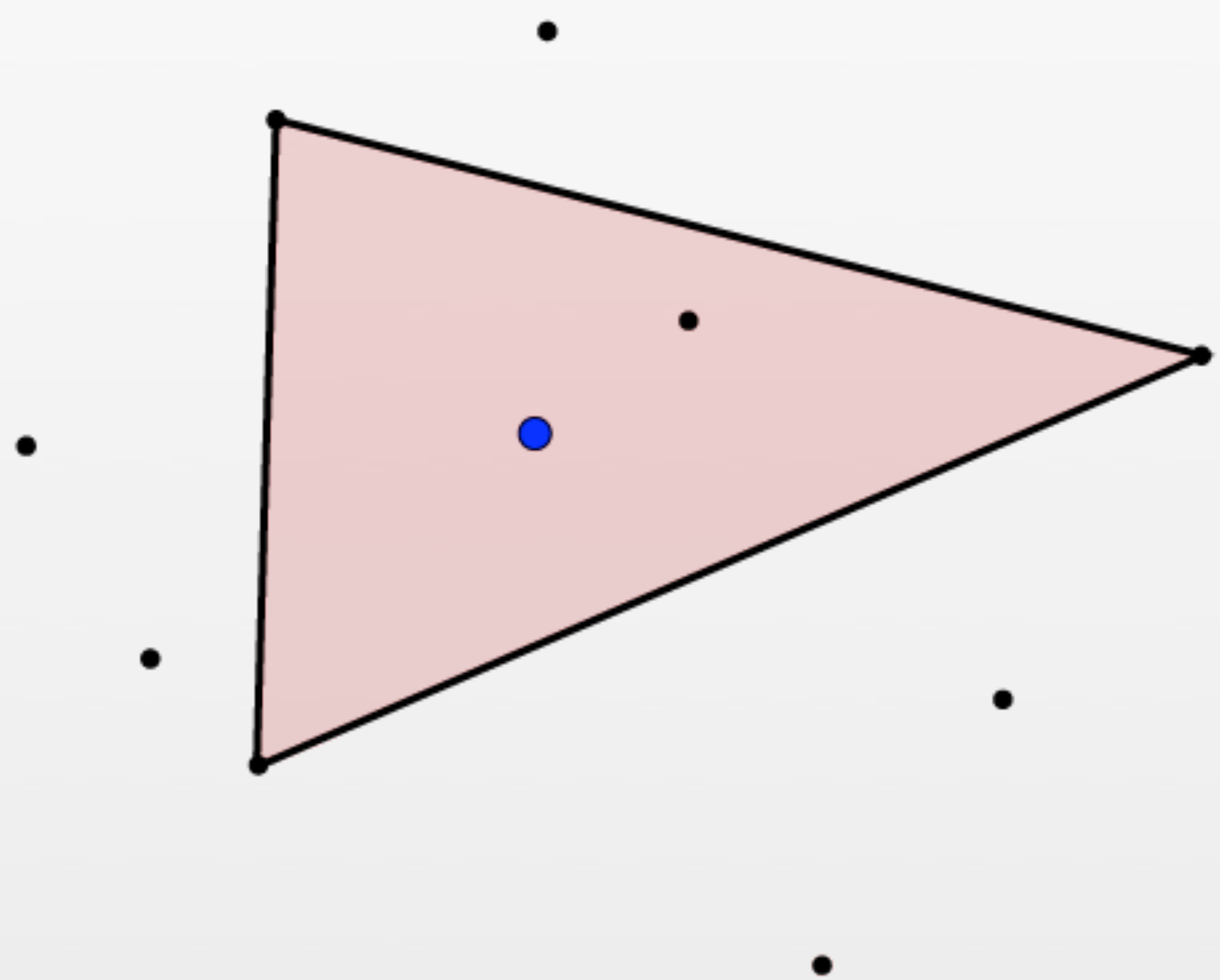
Tverberg's Theorem

Let S be a set of at least $(d+1)(r-1)+1$ points in $\mathbb{R}^d$. There exists a partition of S into r subsets $X_1, \ldots, X_r$ such that $\bigcap_{i=1}^{r} conv(X_i) \neq \emptyset$



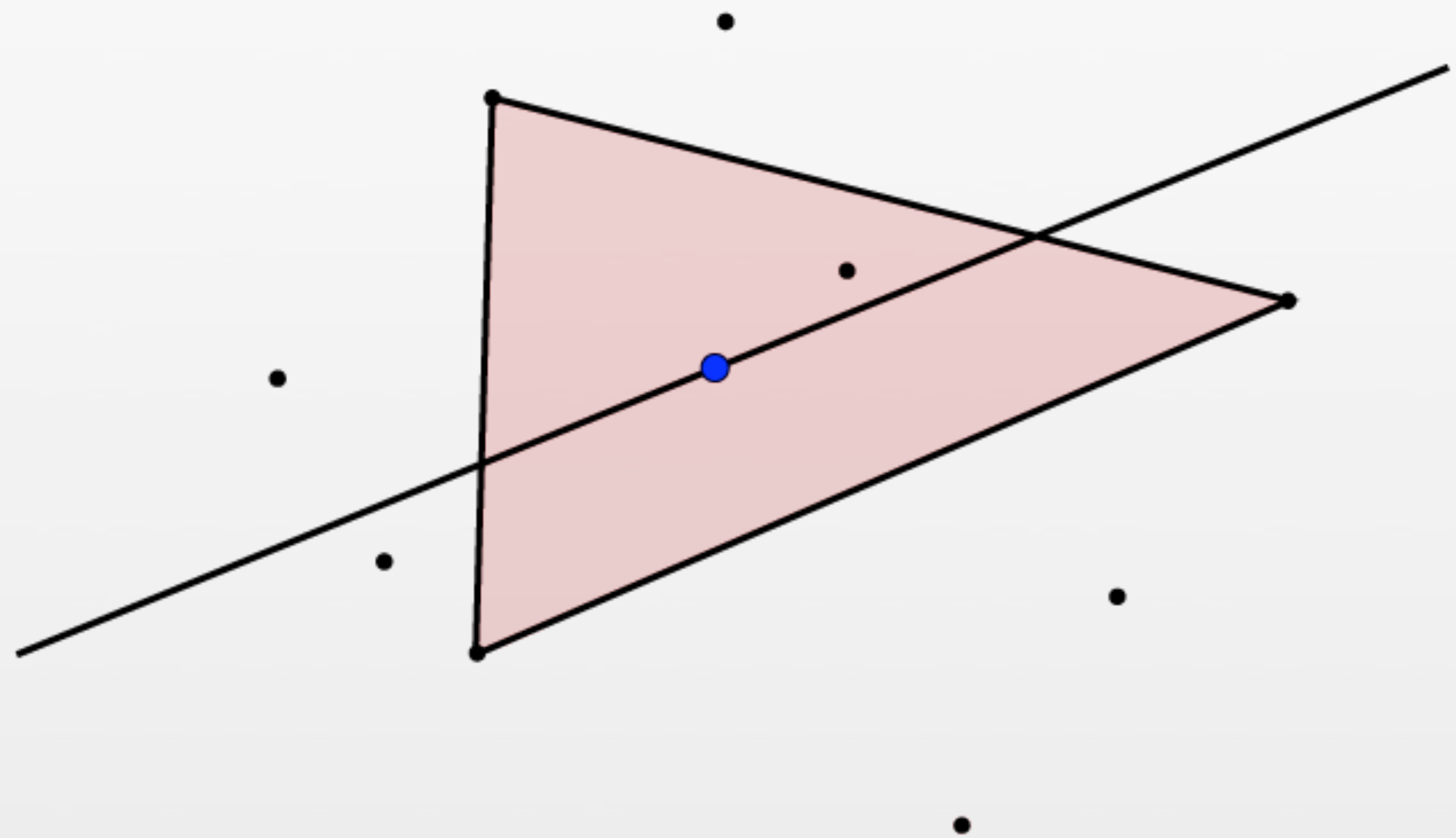
Tverberg's Theorem

Let S be a set of at least $(d+1)(r-1)+1$ points in $\mathbb{R}^d$. There exists a partition of S into r subsets $X_1, \ldots, X_r$ such that $\bigcap_{i=1}^{r} conv(X_i) \neq \emptyset$



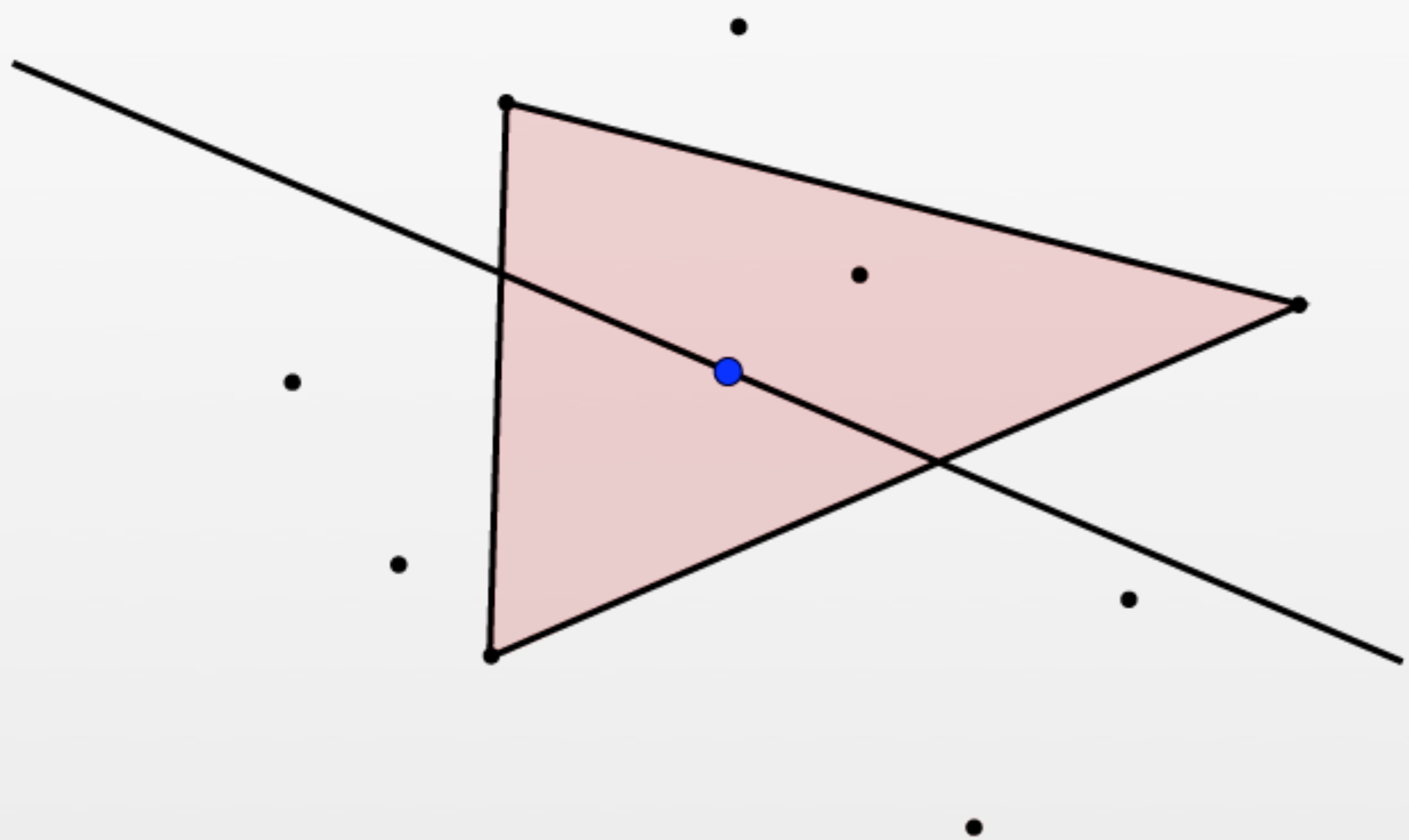
Tverberg's Theorem

Let S be a set of at least $(d+1)(r-1)+1$ points in $\mathbb{R}^d$. There exists a partition of S into r subsets $X_1, \ldots, X_r$ such that $\bigcap_{i=1}^{r} conv(X_i) \neq \emptyset$



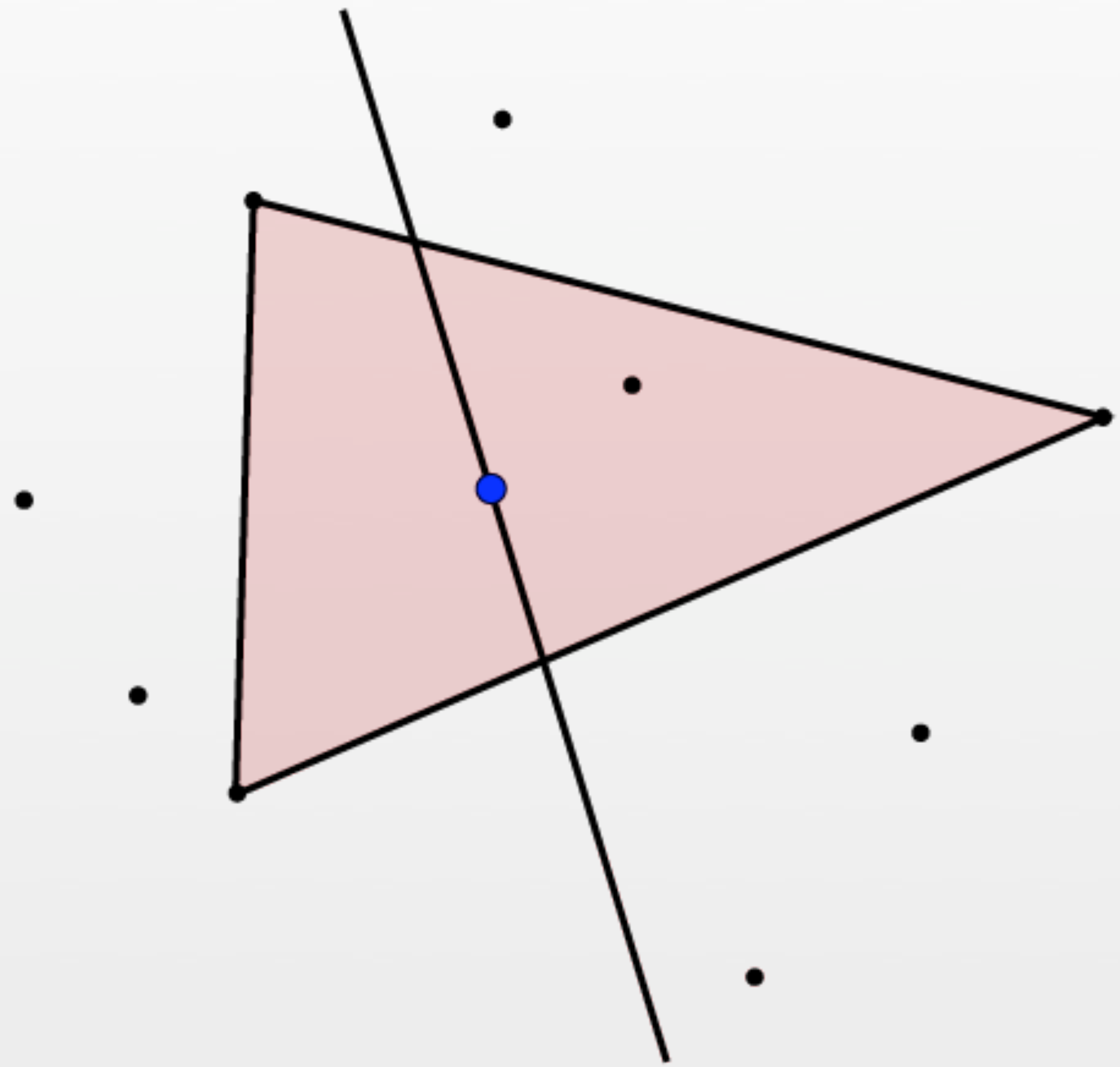
Tverberg's Theorem

Let S be a set of at least $(d+1)(r-1)+1$ points in $\mathbb{R}^d$. There exists a partition of S into r subsets $X_1, \ldots, X_r$ such that $\bigcap_{i=1}^{r} conv(X_i) \neq \emptyset$



Tverberg's Theorem

Let S be a set of at least $(d+1)(r-1)+1$ points in $\mathbb{R}^d$. There exists a partition of S into r subsets $X_1, \ldots, X_r$ such that $\bigcap_{i=1}^{r} conv(X_i) \neq \emptyset$

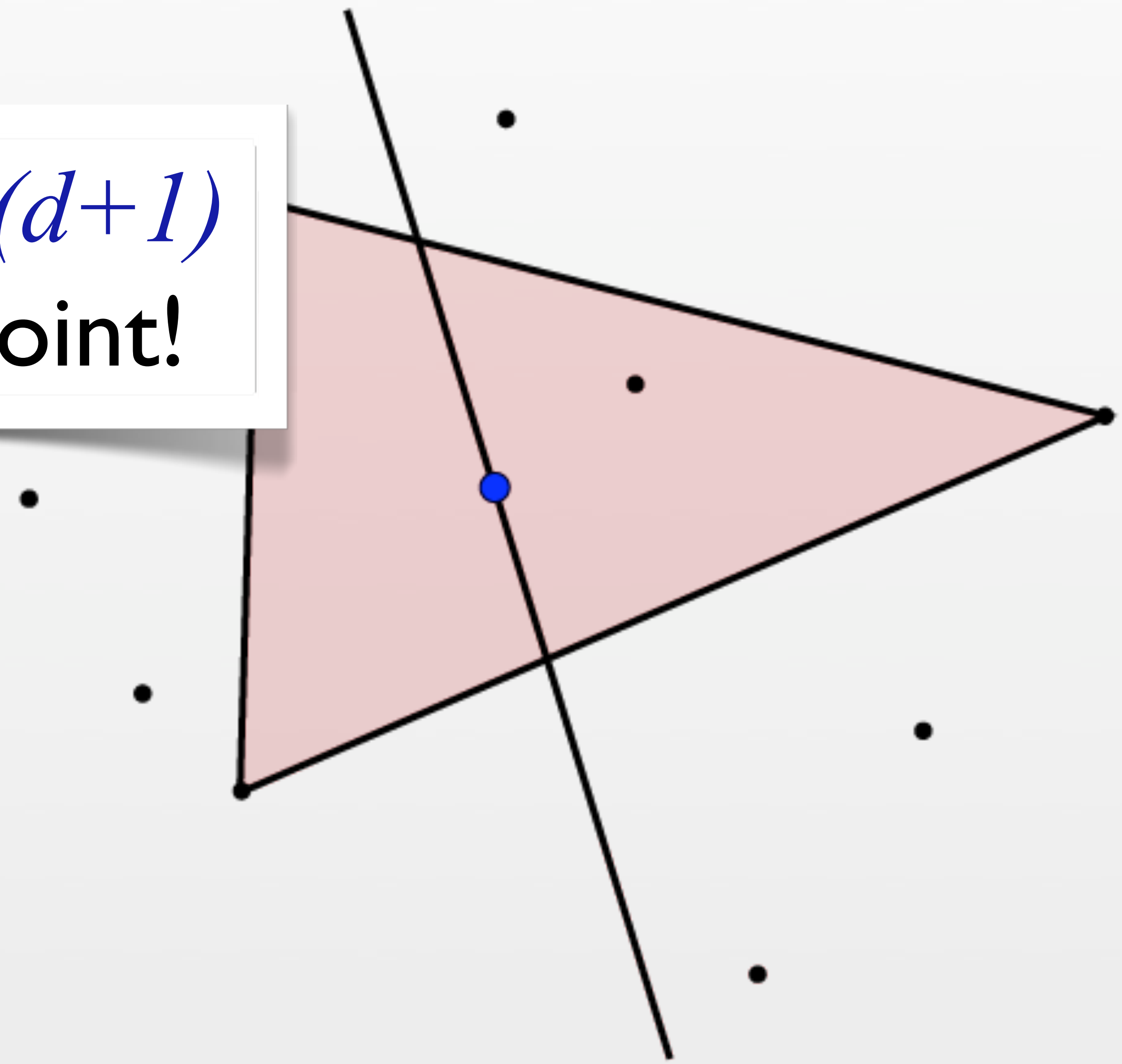


Tverberg's Theorem

Let S be a set of at least $(d+1)(r-1)+1$ points in $\mathbb{R}^d$. There exists a partition of S into r subsets $X_1, \ldots, X_r$ such that $\bigcap_{i=1}^{r} conv(X_i) \neq \emptyset$

Choose $r = n/(d+1)$

It's a center point!



Proof via Helly's Theorem

Proof via Tverberg's Theorem

Proof via Helly's Theorem

coNP

Proof via Tverberg's Theorem

NP

An Algorithm

Approximating Center Points with Iterated Radon Points

[Clarkson, Eppstein, Miller, Sturtivant, Teng, 1993]

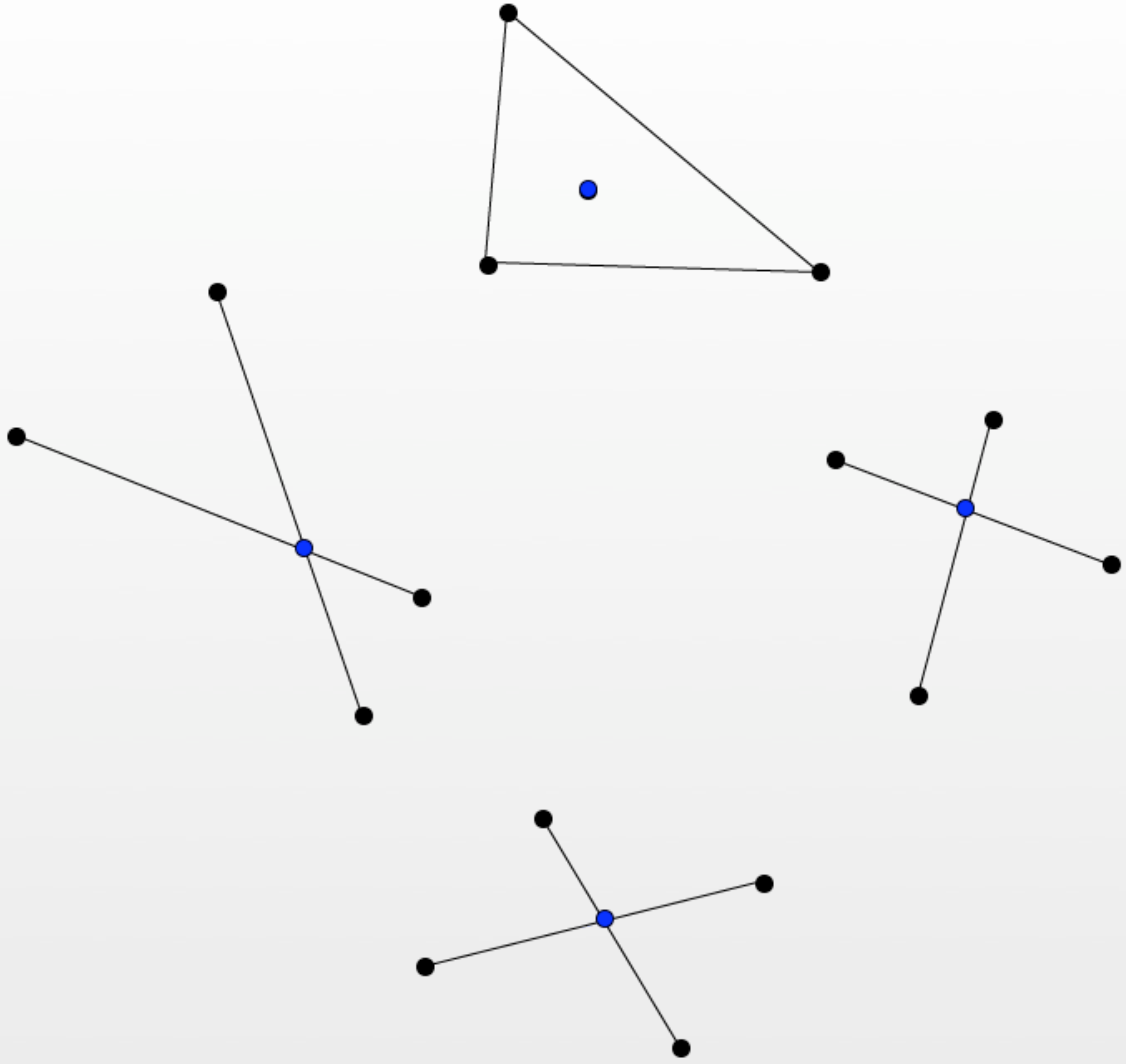
Approximating Center Points with Iterated Radon Points
[Clarkson, Eppstein, Miller, Sturtivant, Teng, 1993]

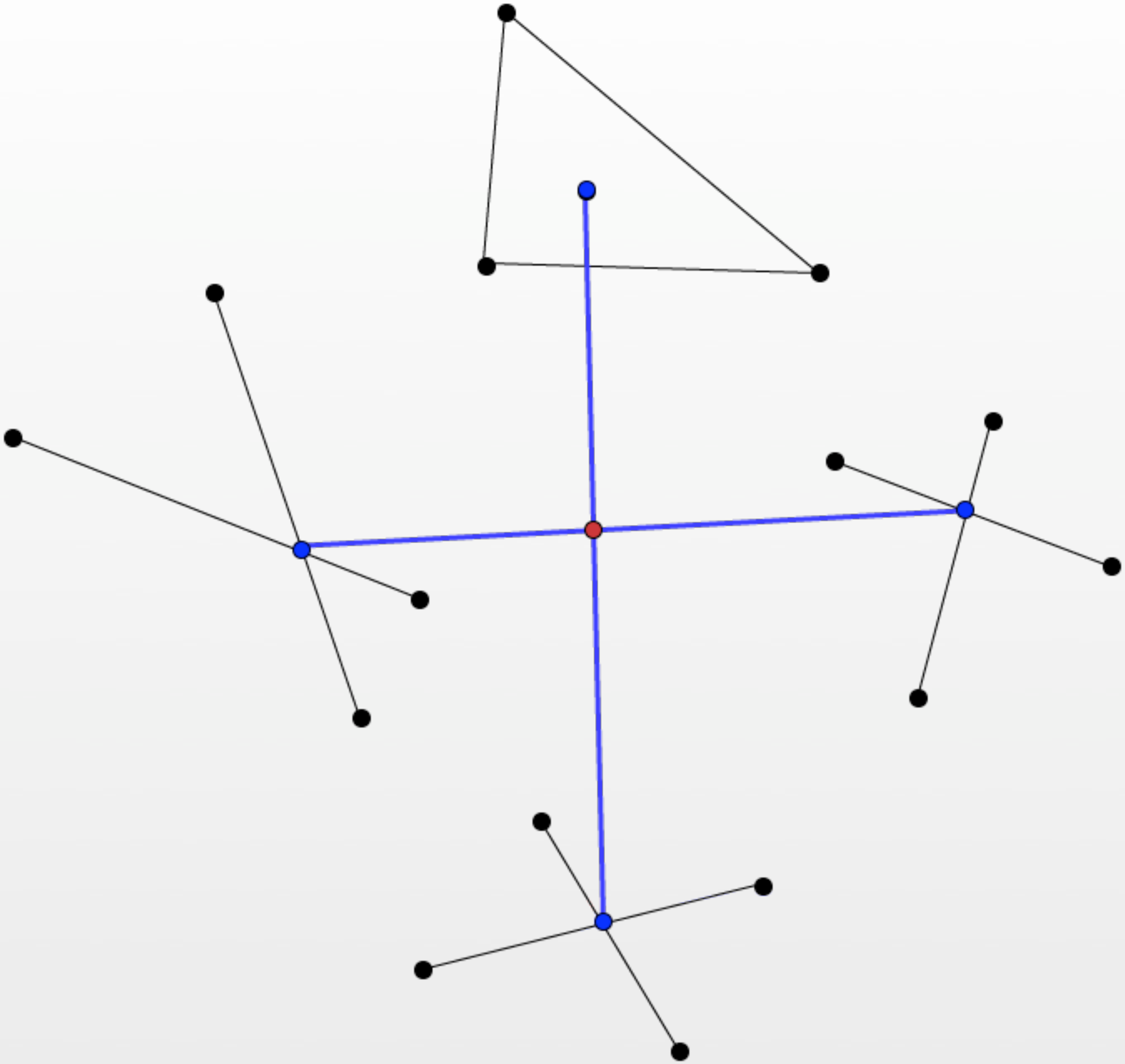
1. Randomly sample points into sets of d+2.
2. Compute the Radon point for each set.
3. Compute the Radon points of the Radon points
4. Continue until only one point remains.
5. Return that point.

Approximating Center Points with Iterated Radon Points
[Clarkson, Eppstein, Miller, Sturtivant, Teng, 1993]

1. Randomly sample points into sets of d+2.
2. Compute the Radon point for each set.
3. Compute the Radon points of the Radon points
4. Continue until only one point remains.
5. Return that point.

$O\left(\frac{n}{d^2}\right)$-center with high probability.

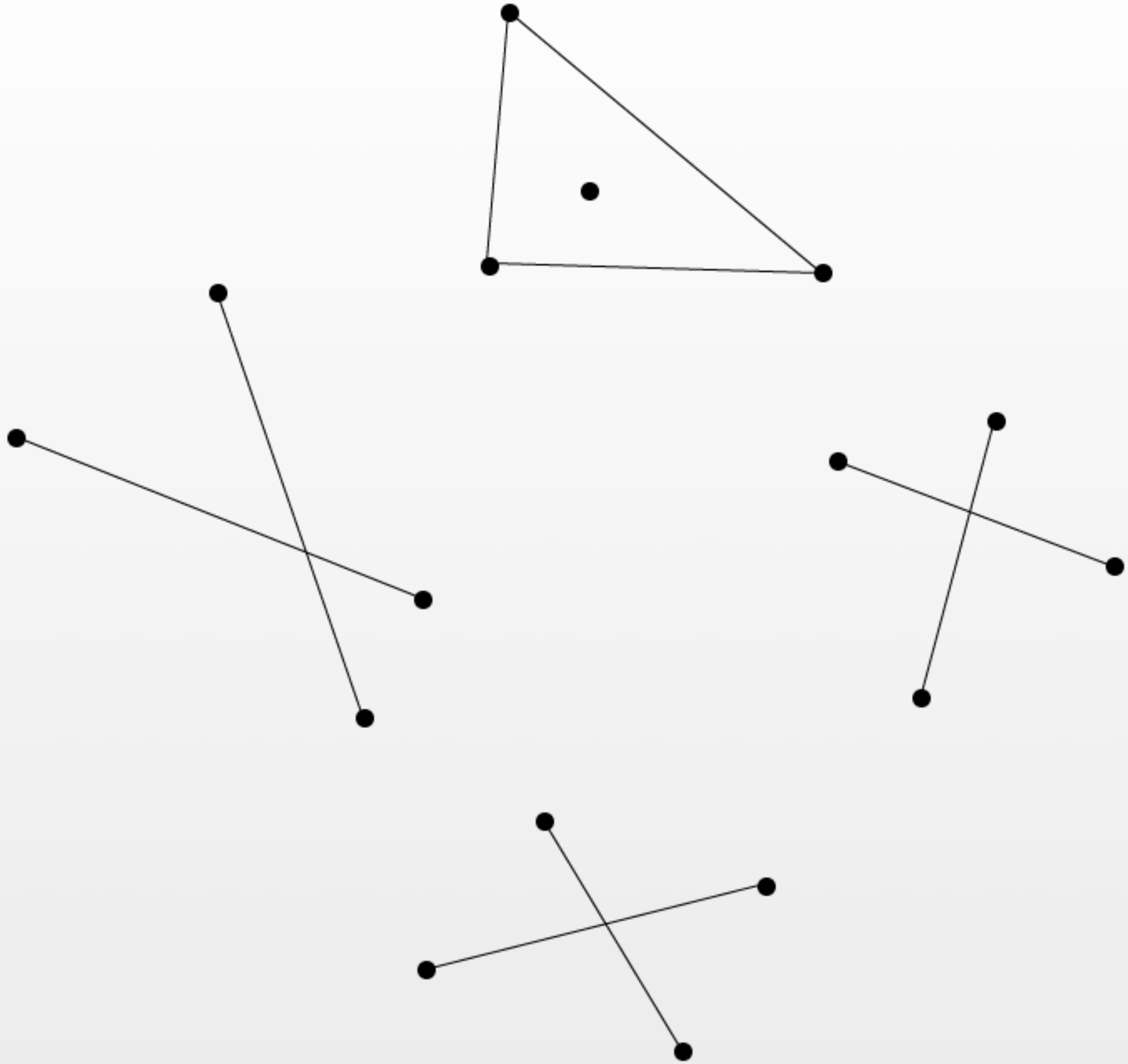


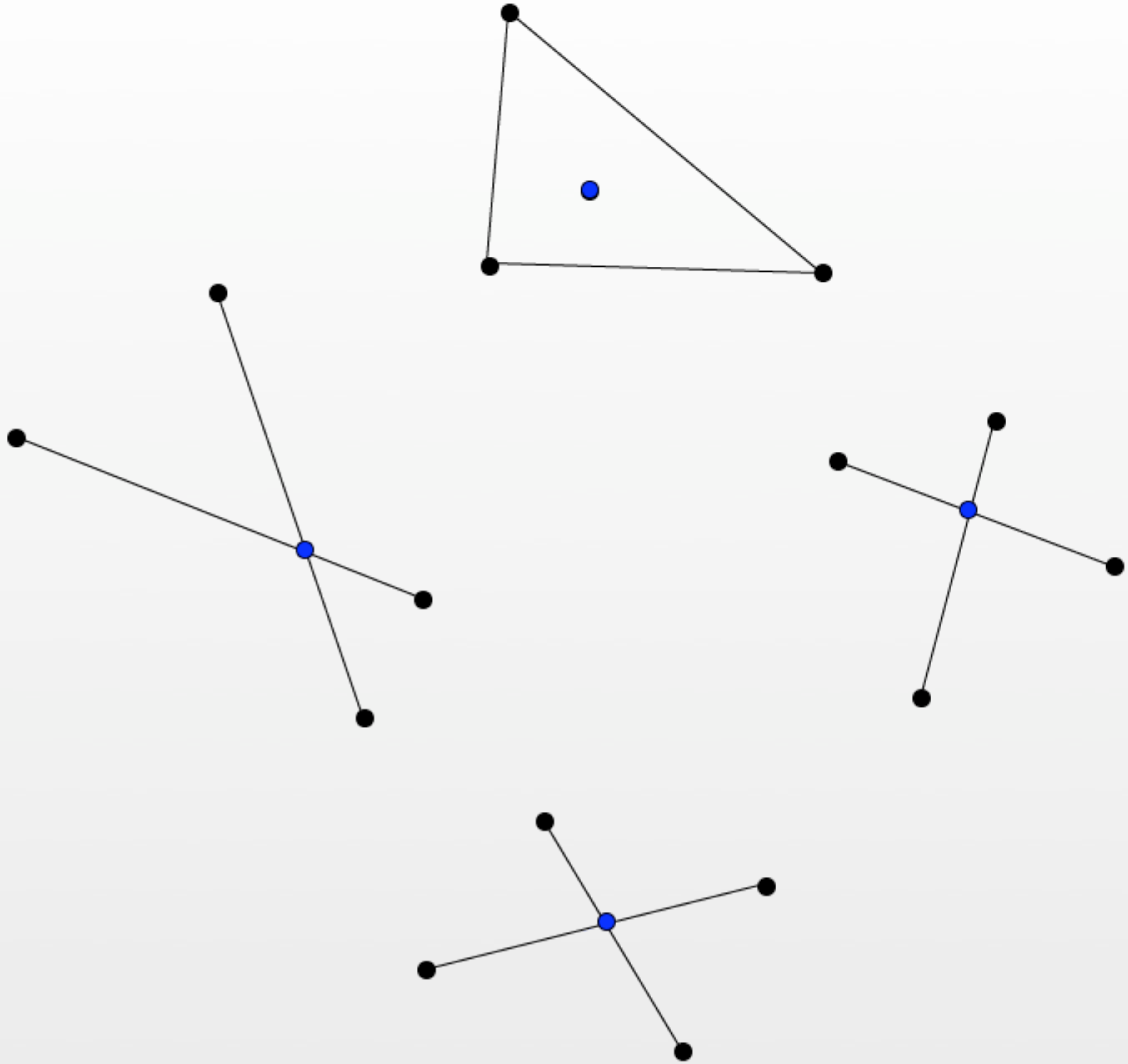


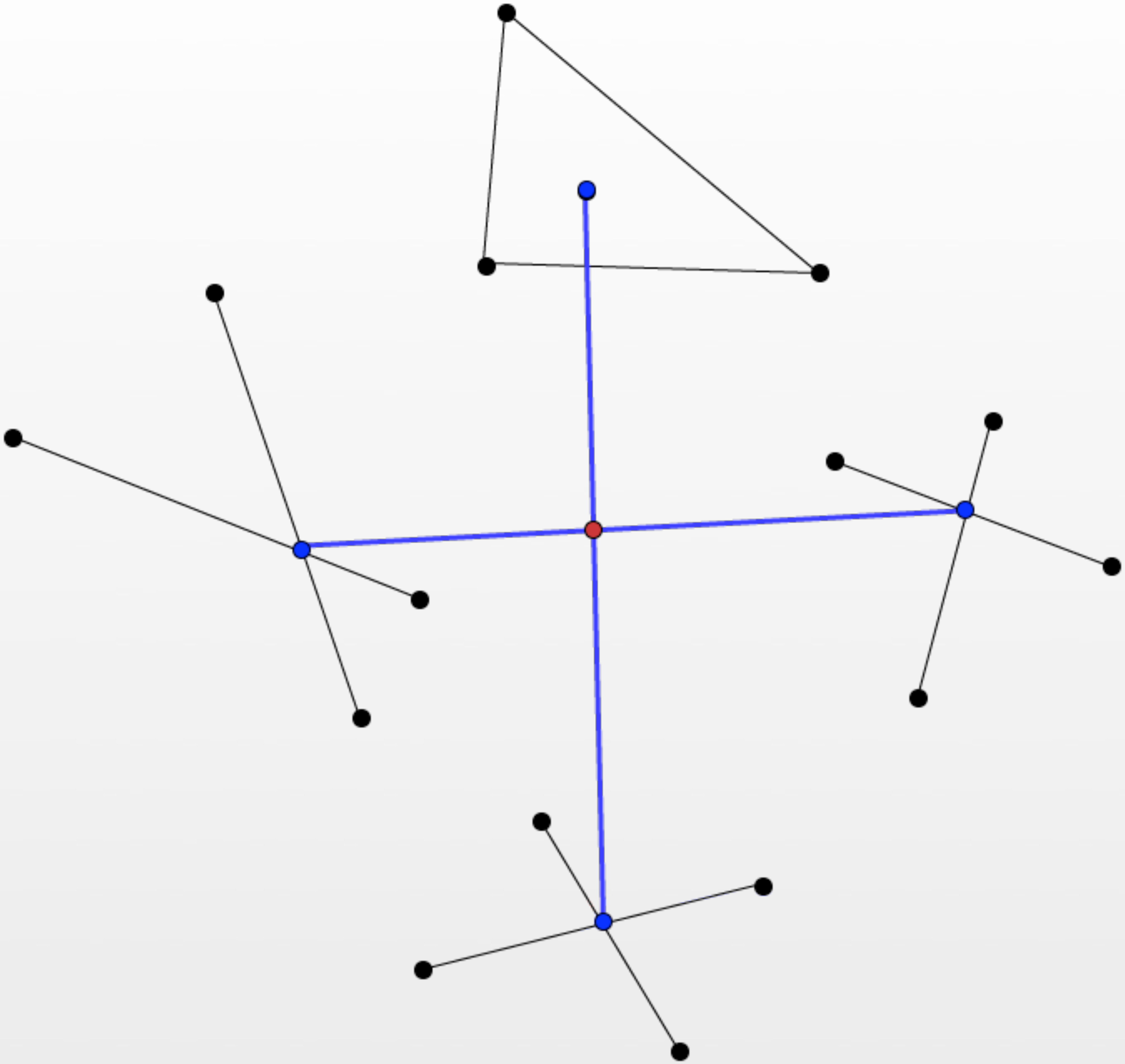
Analysis looks like Helly-type proof.

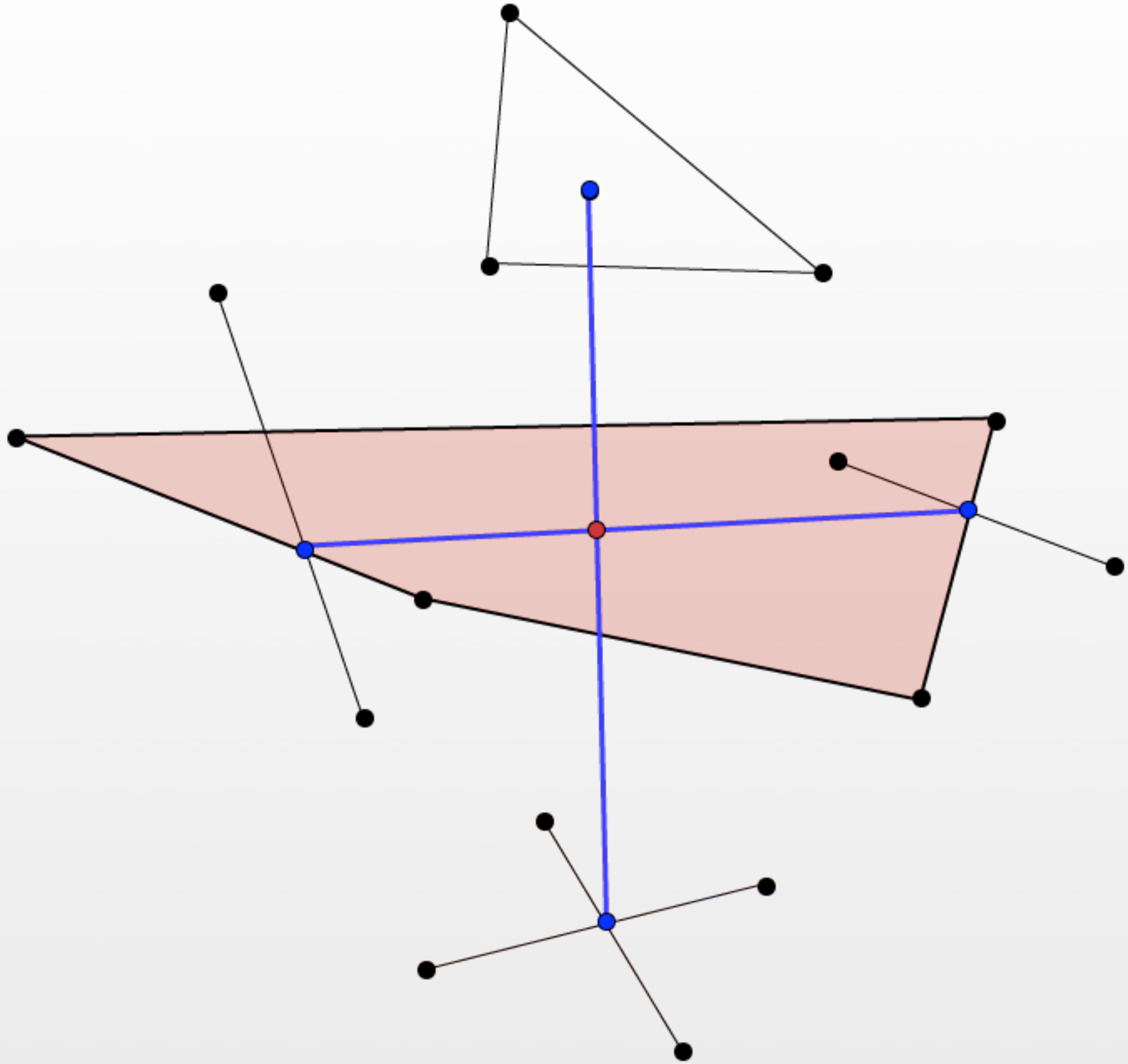
Look at all projections to one dimension at the same time.

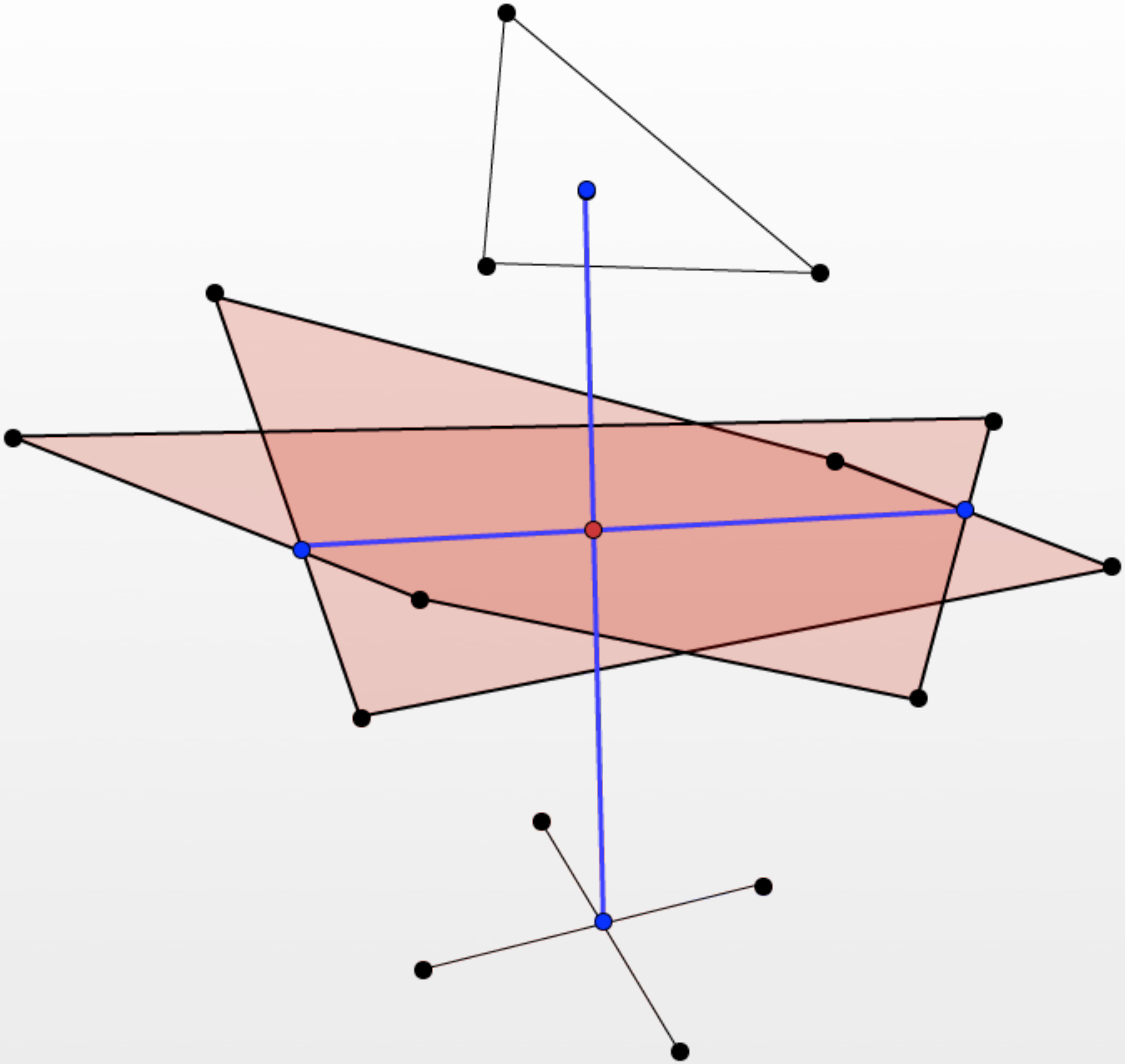
Let's build an algorithm so that the analysis will look less like Helly and more like Tverberg.

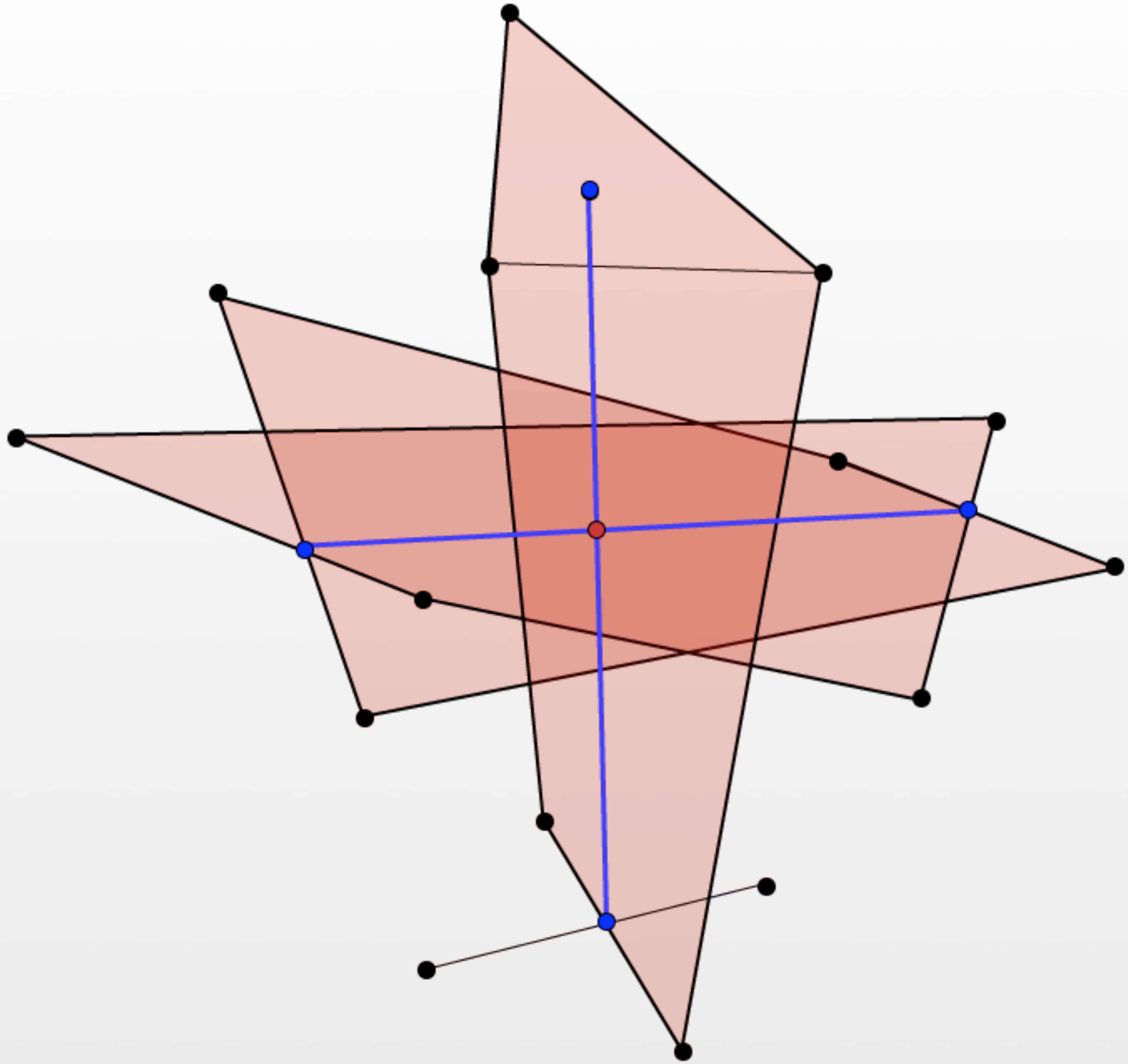


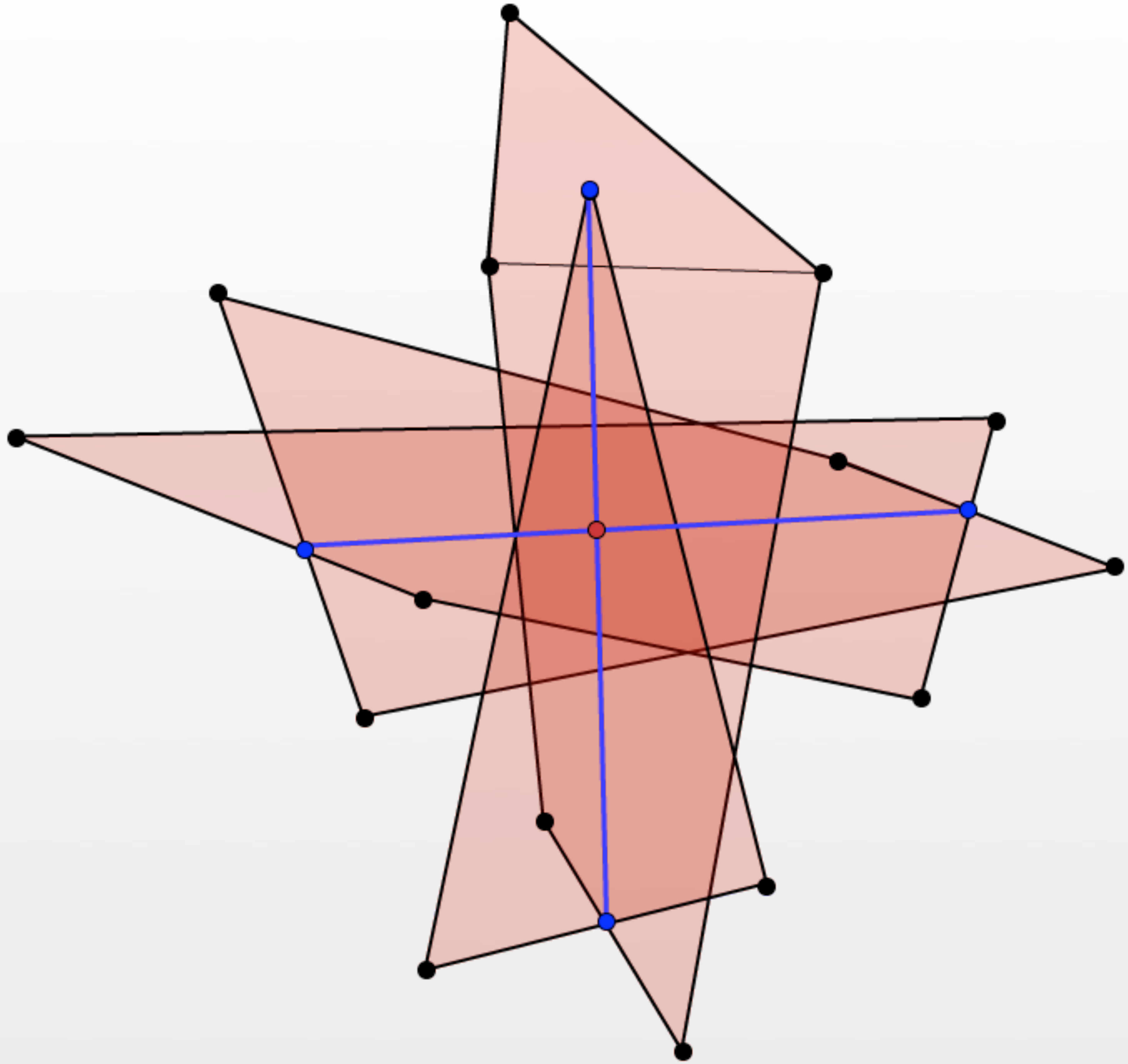




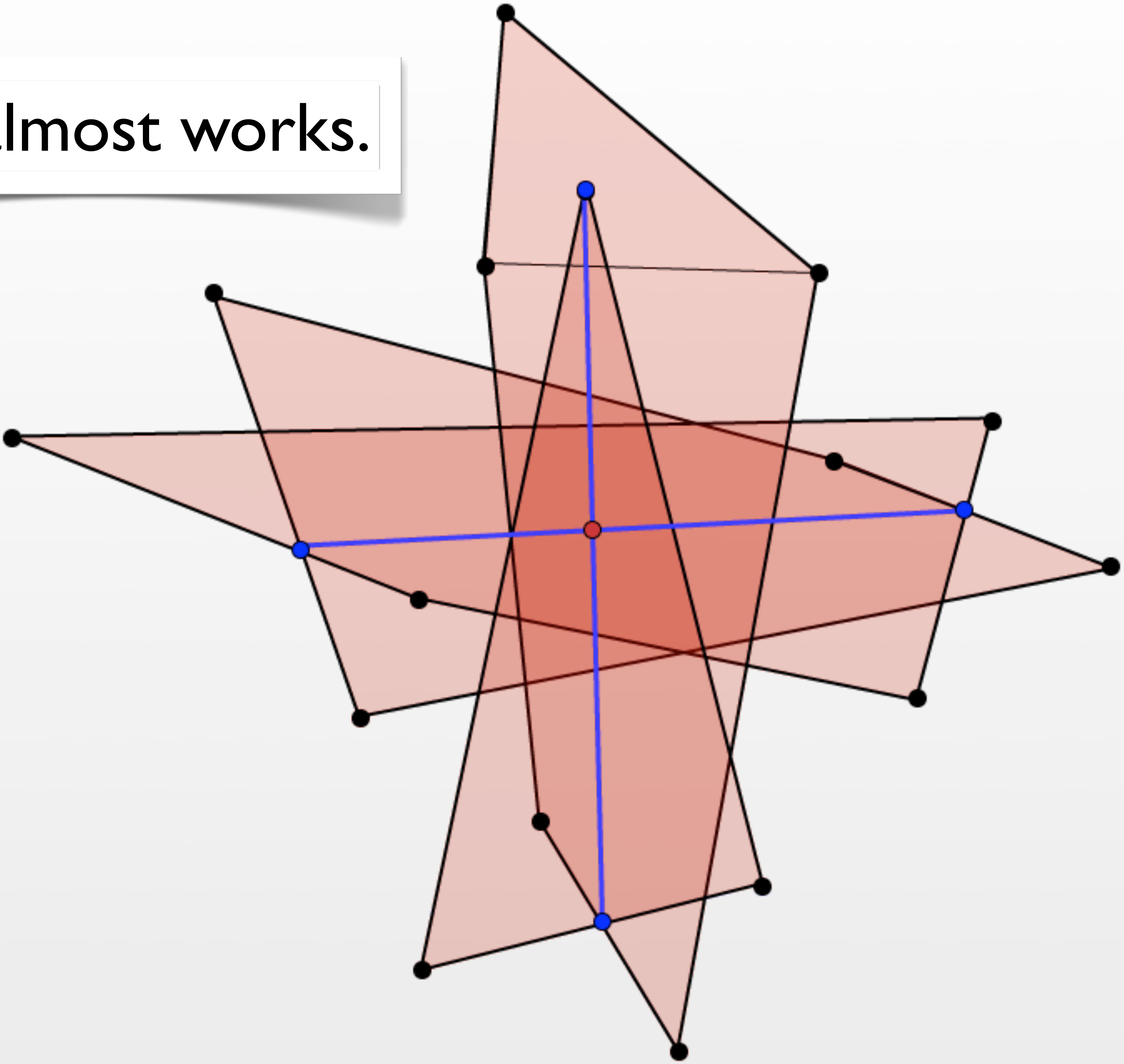


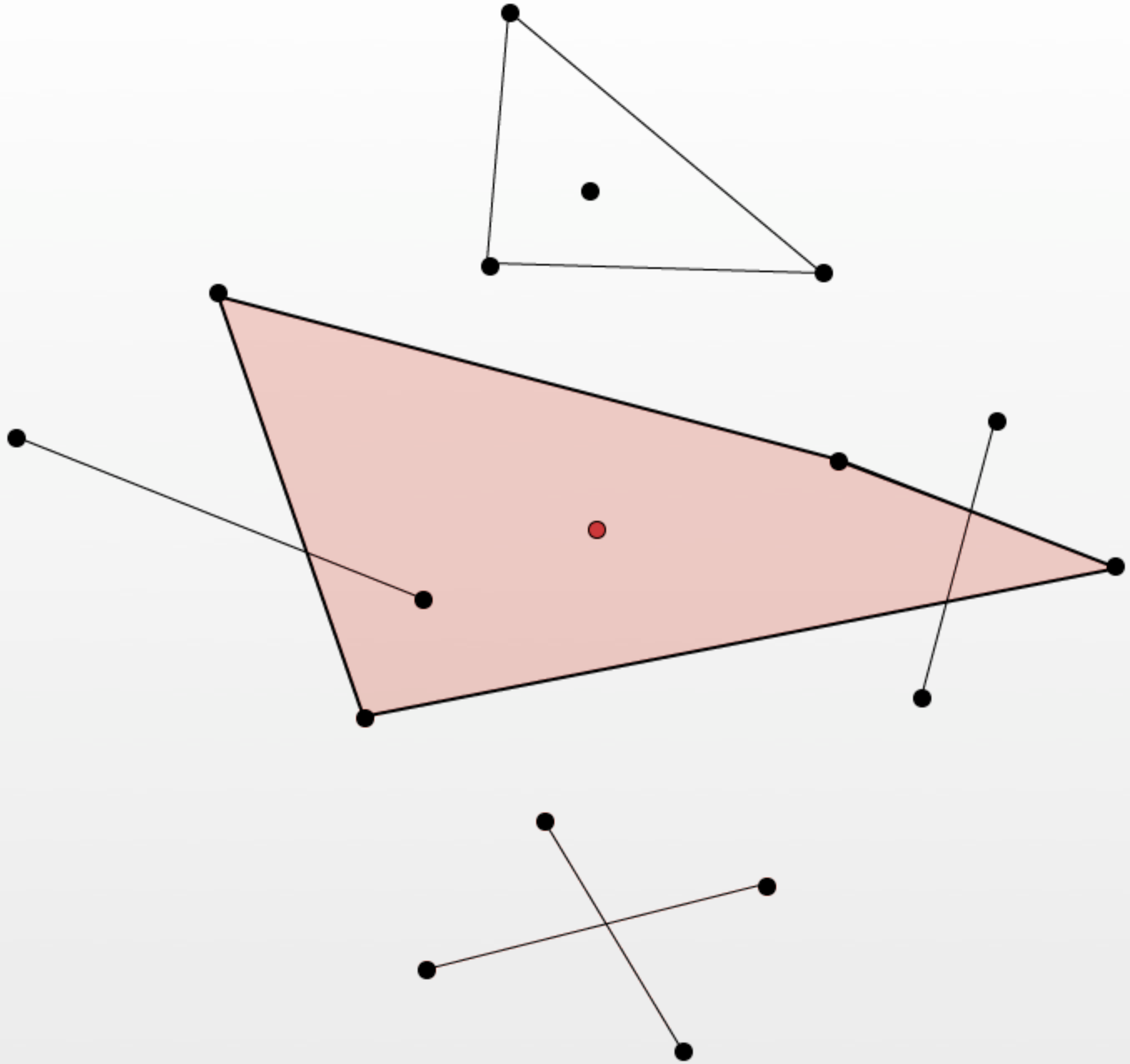


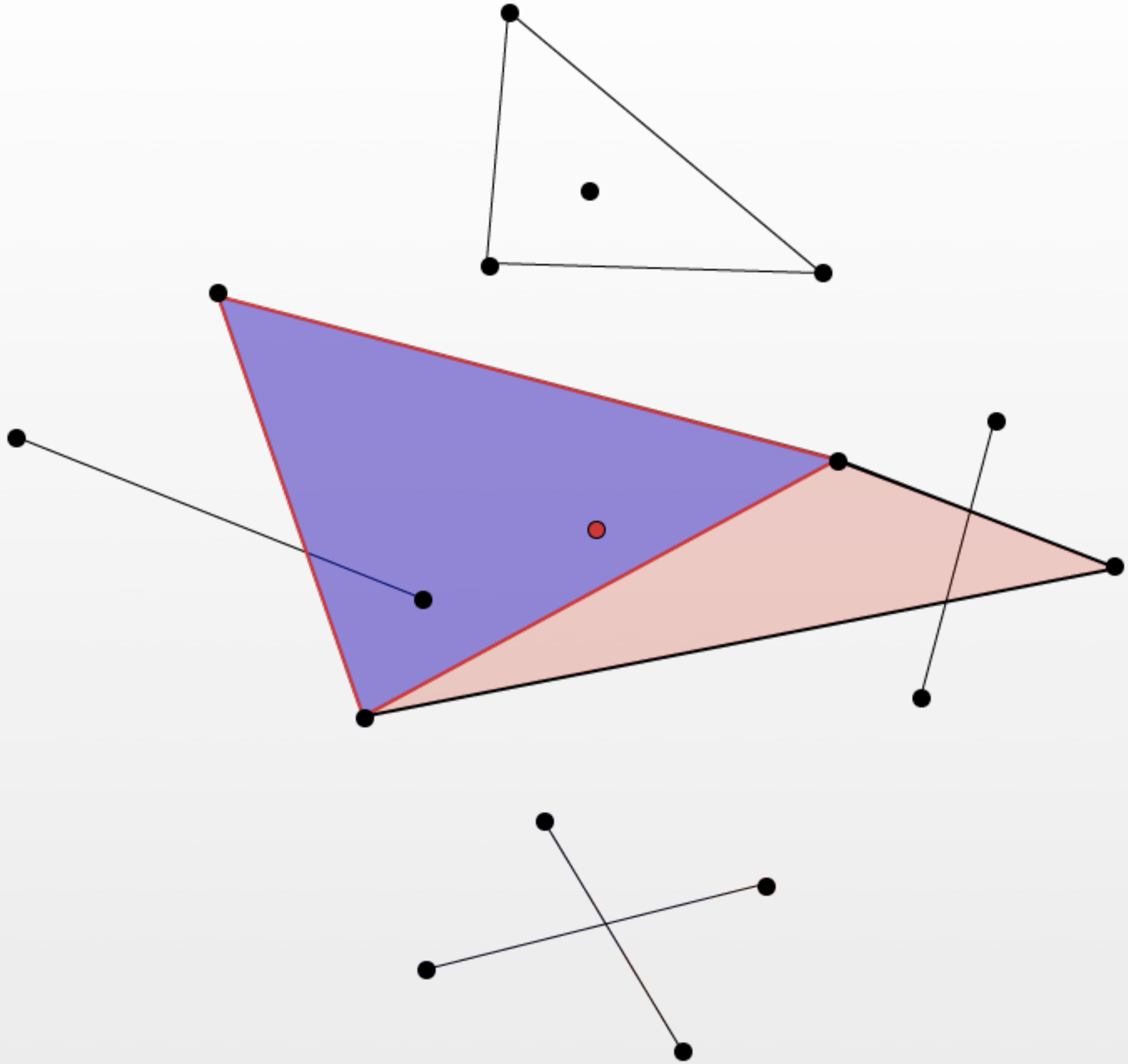


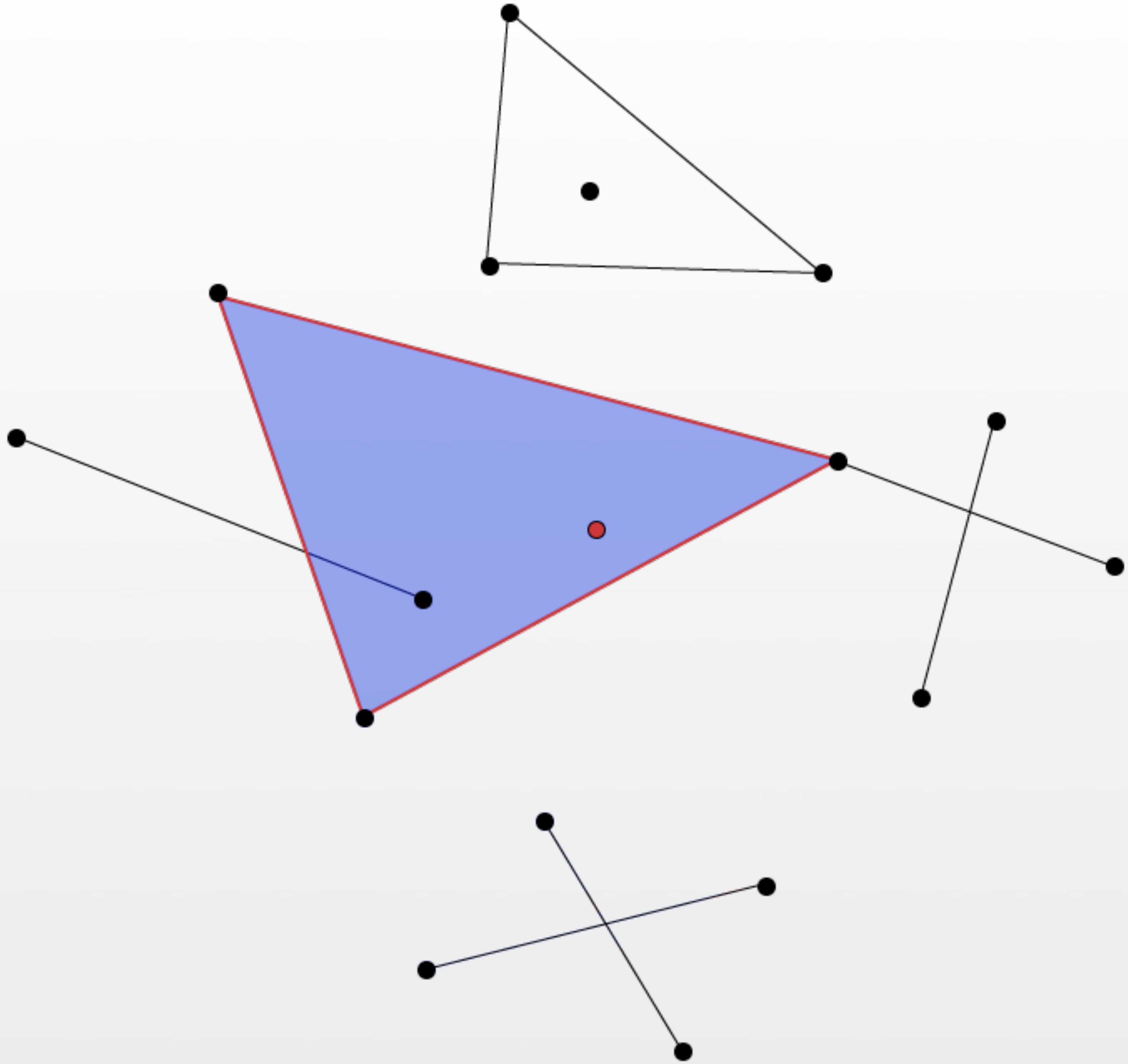


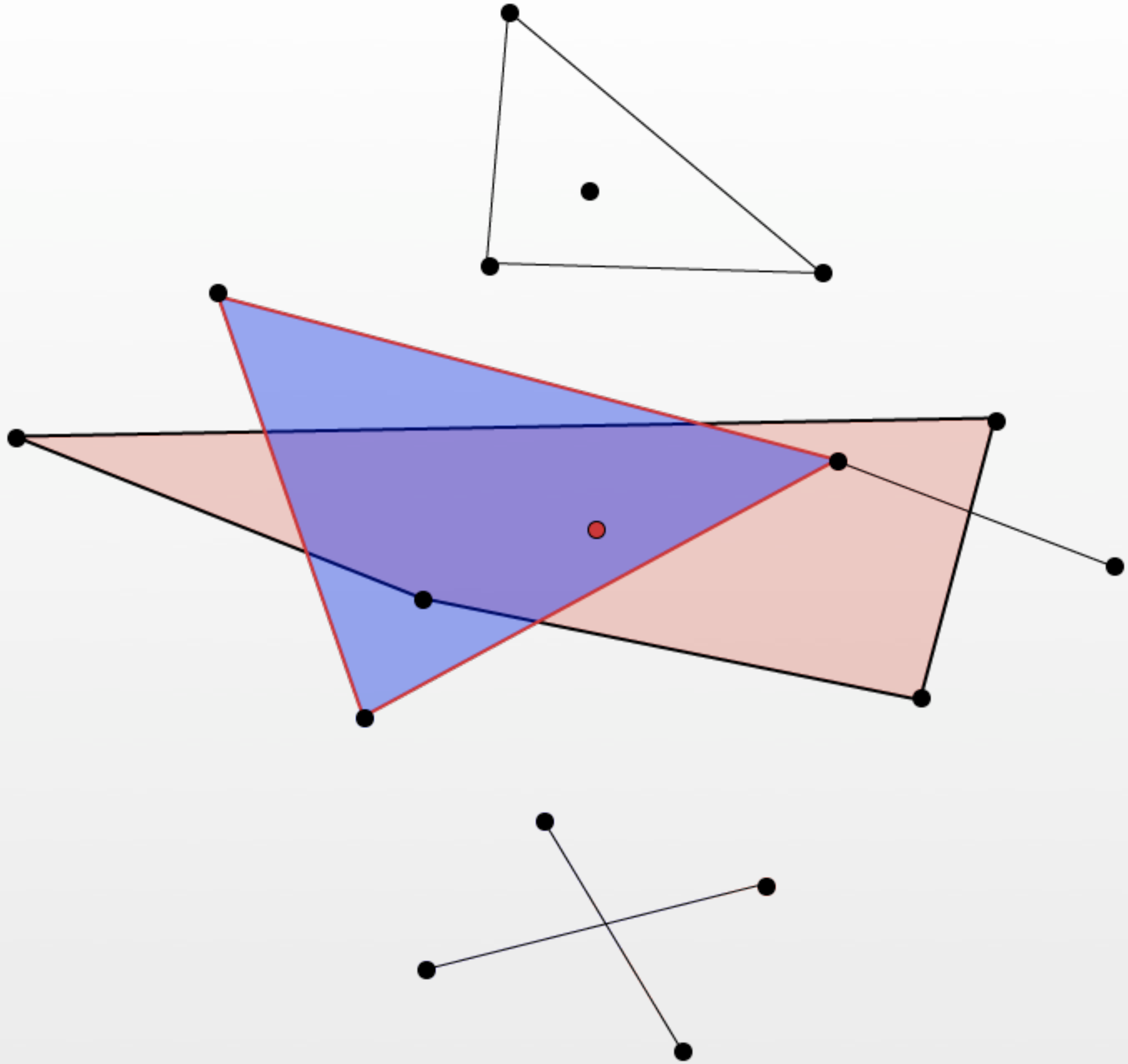
This almost works.

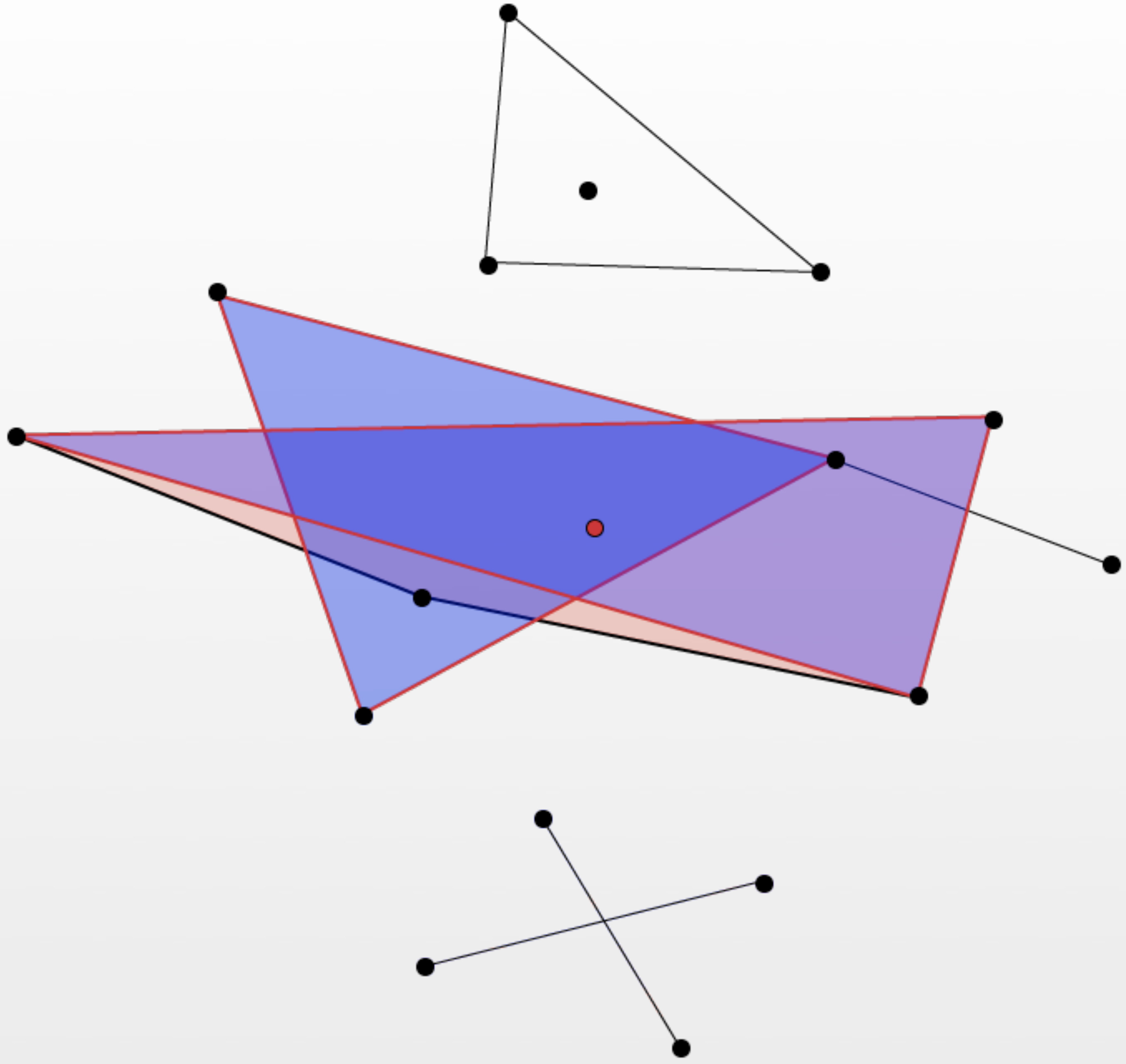


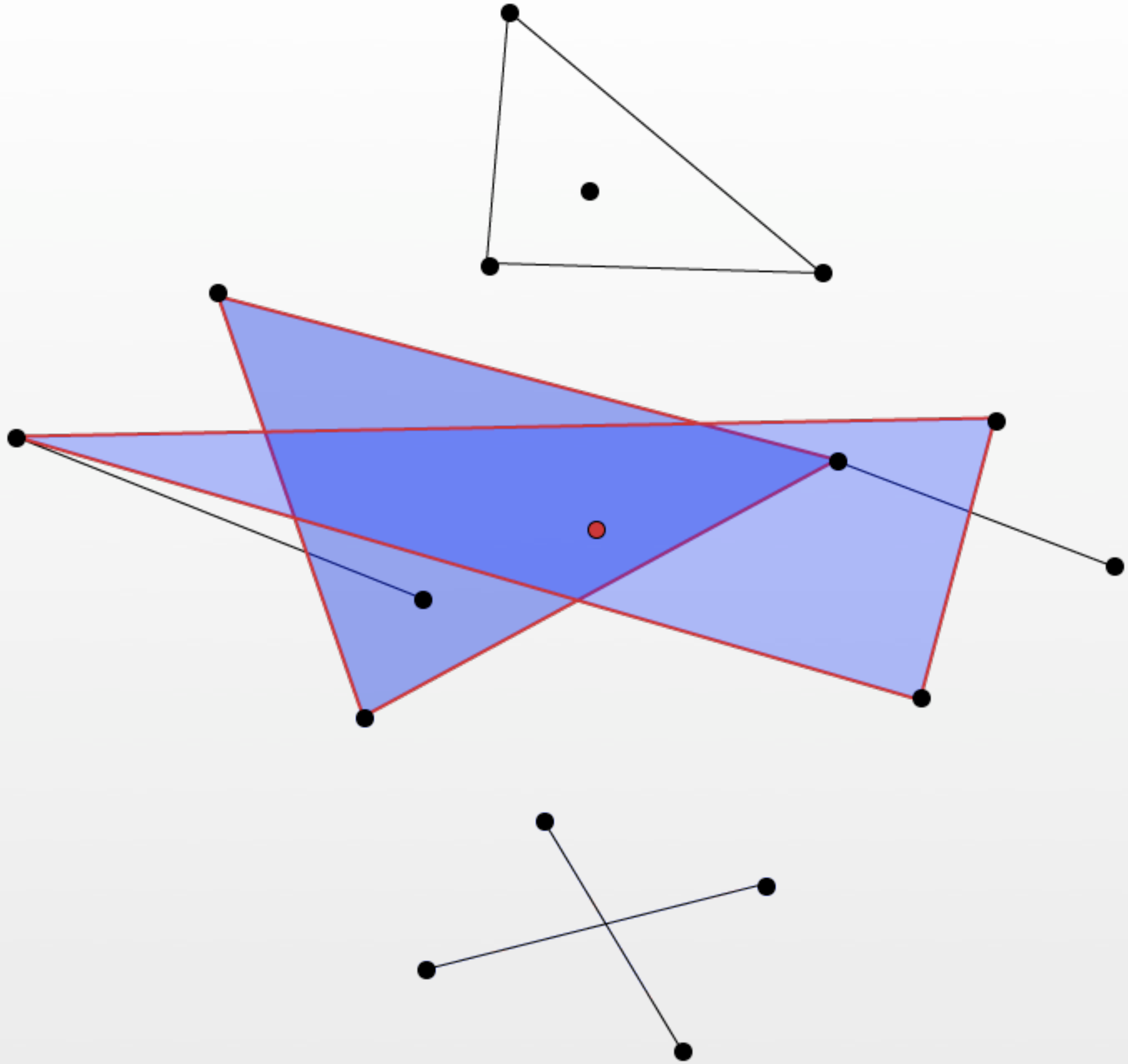


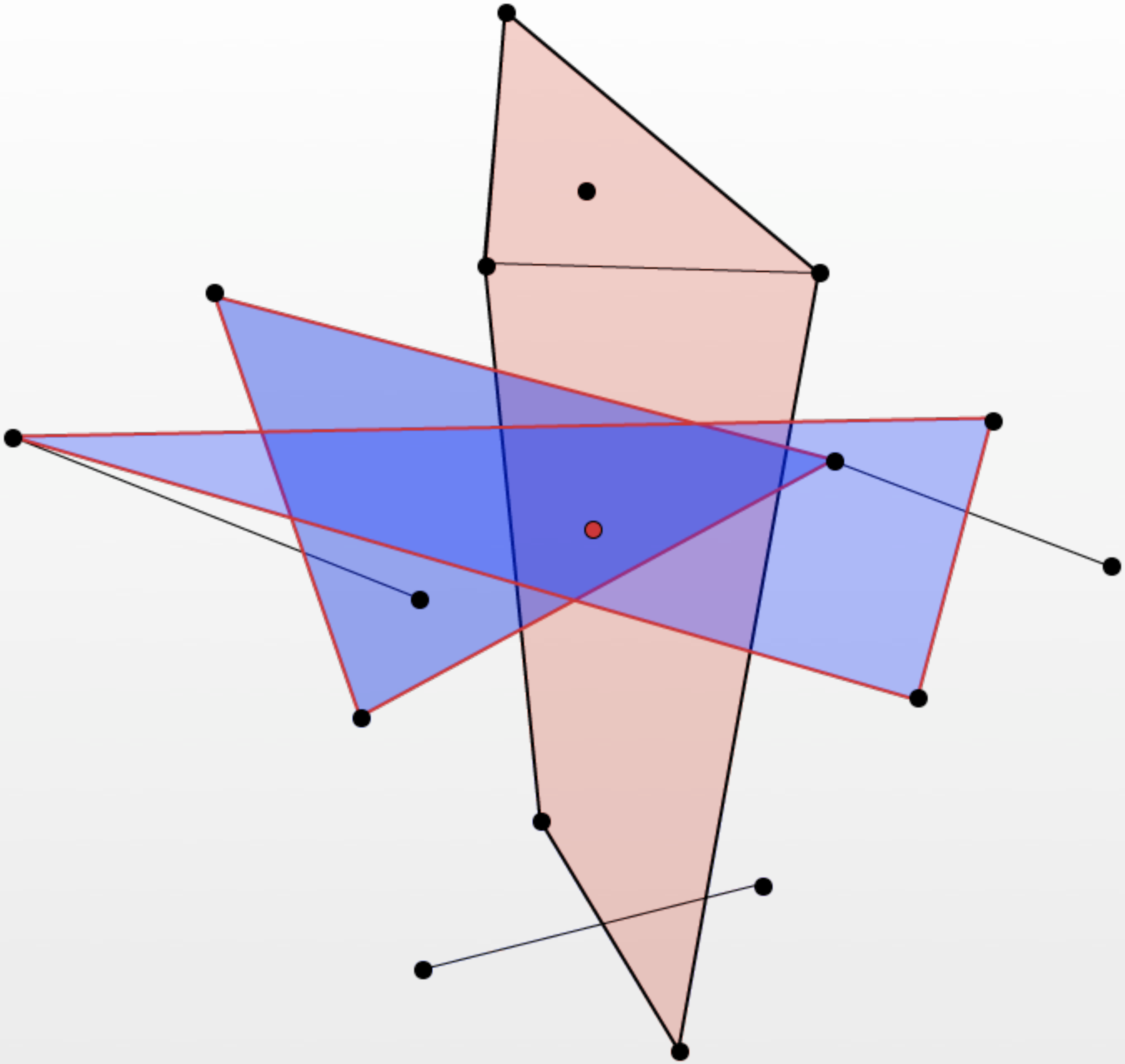


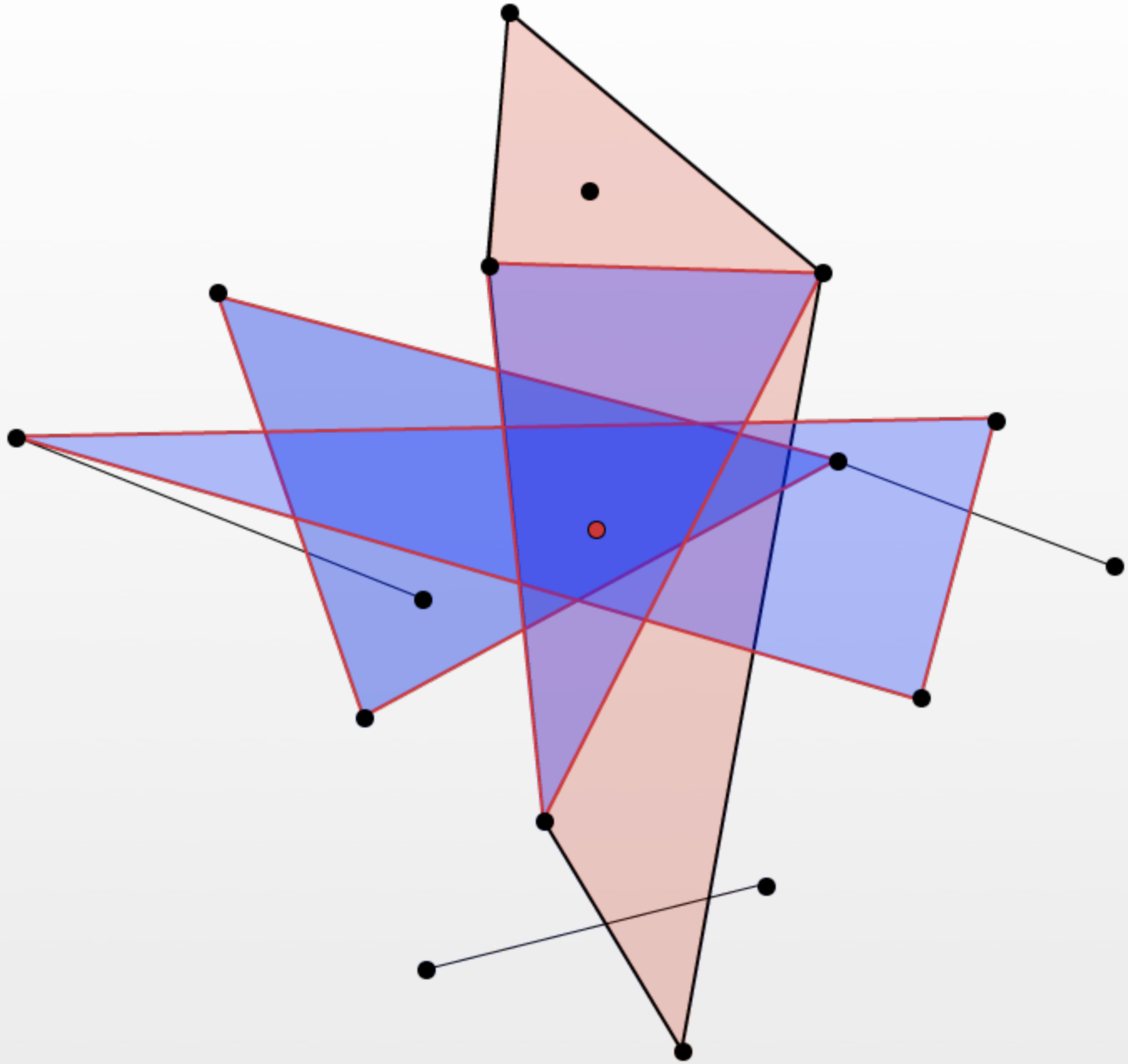


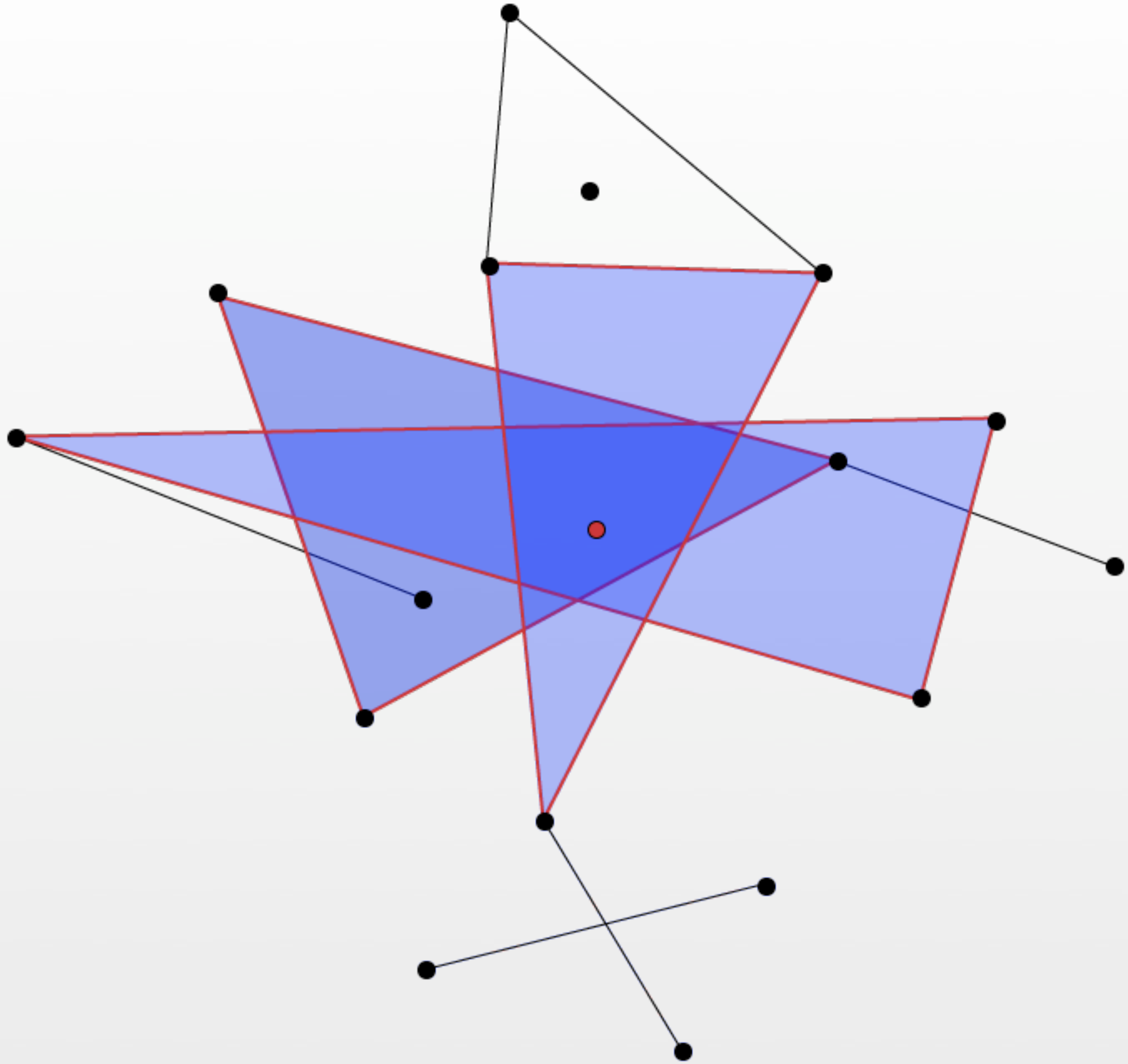


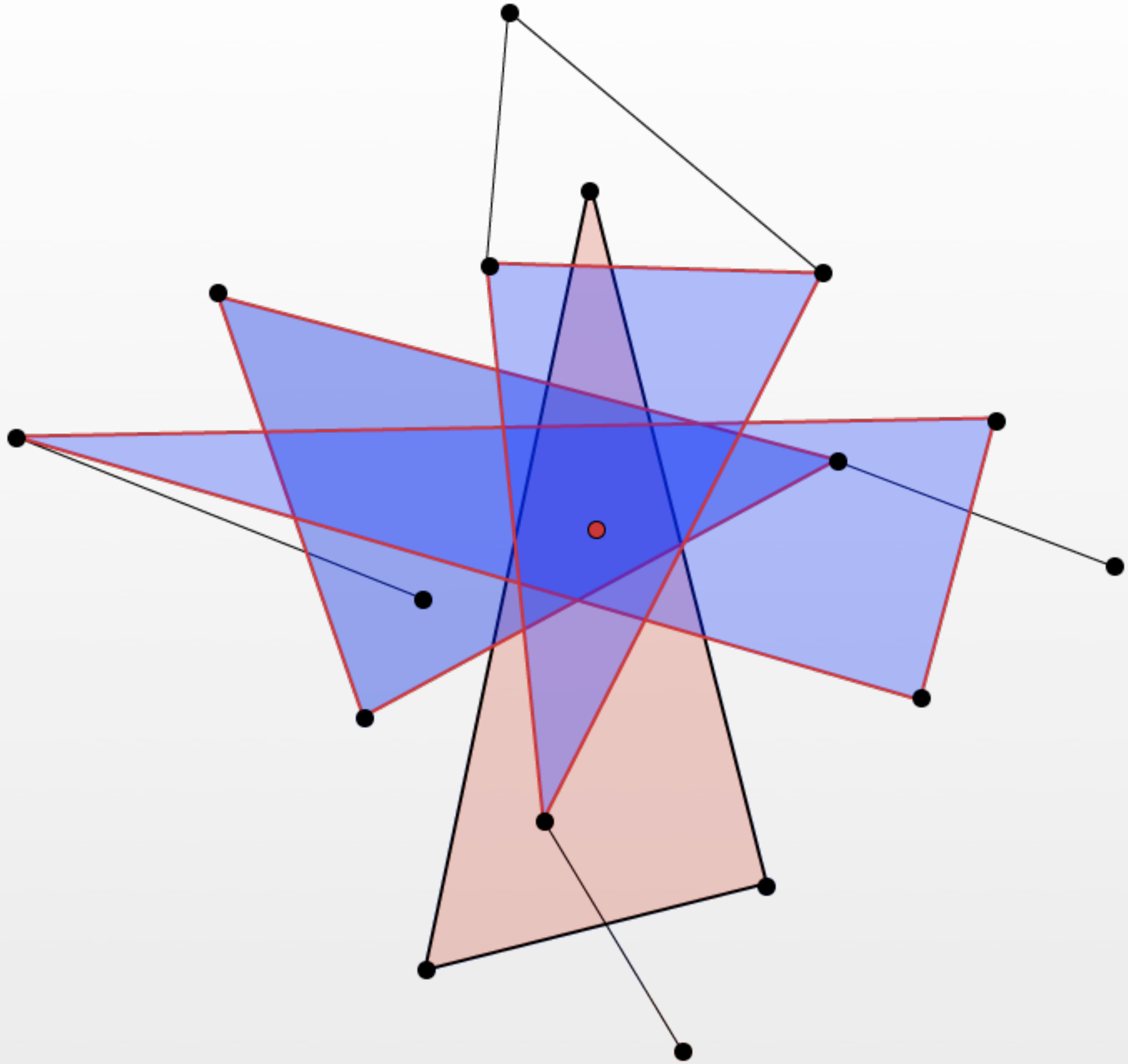


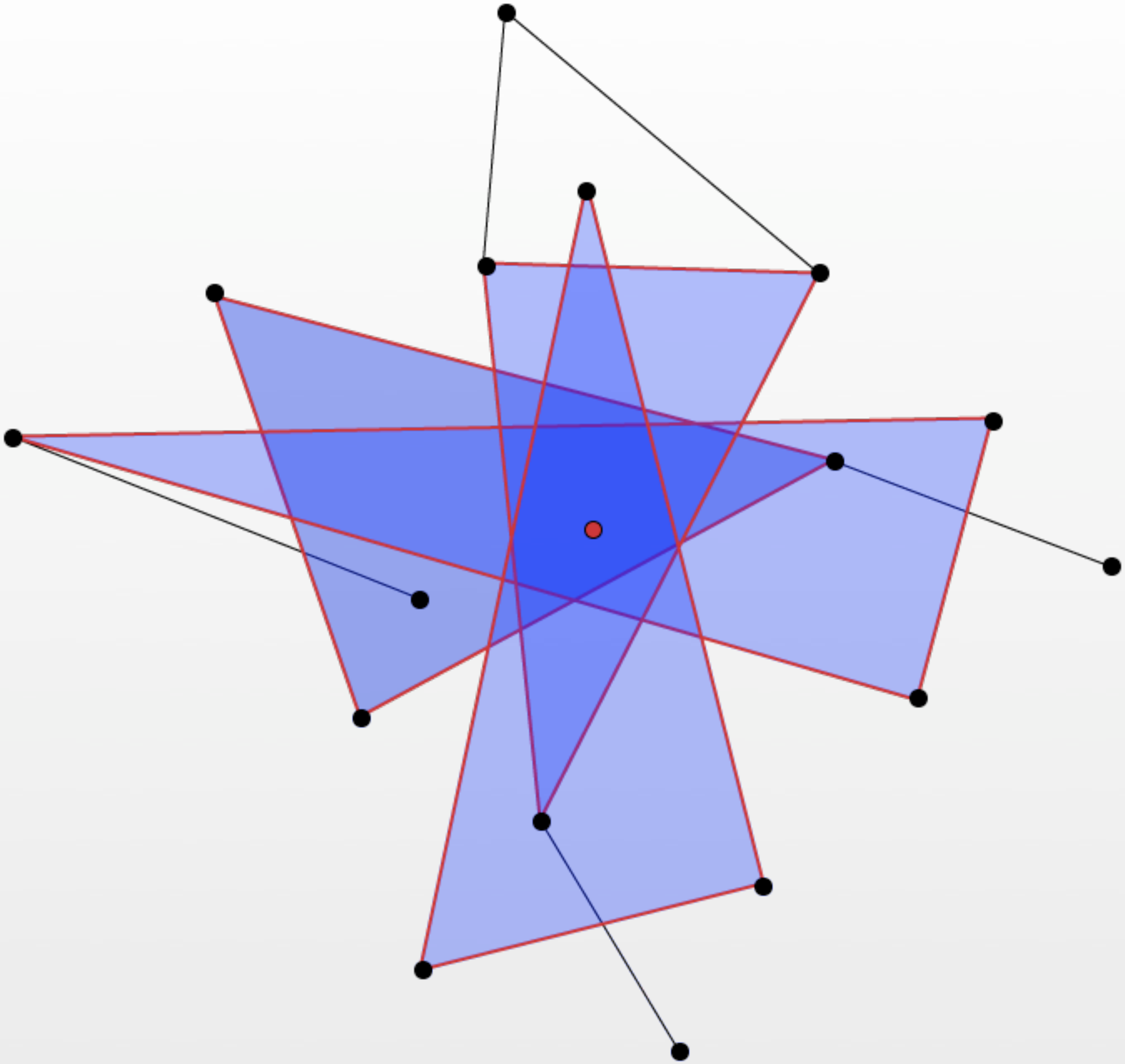


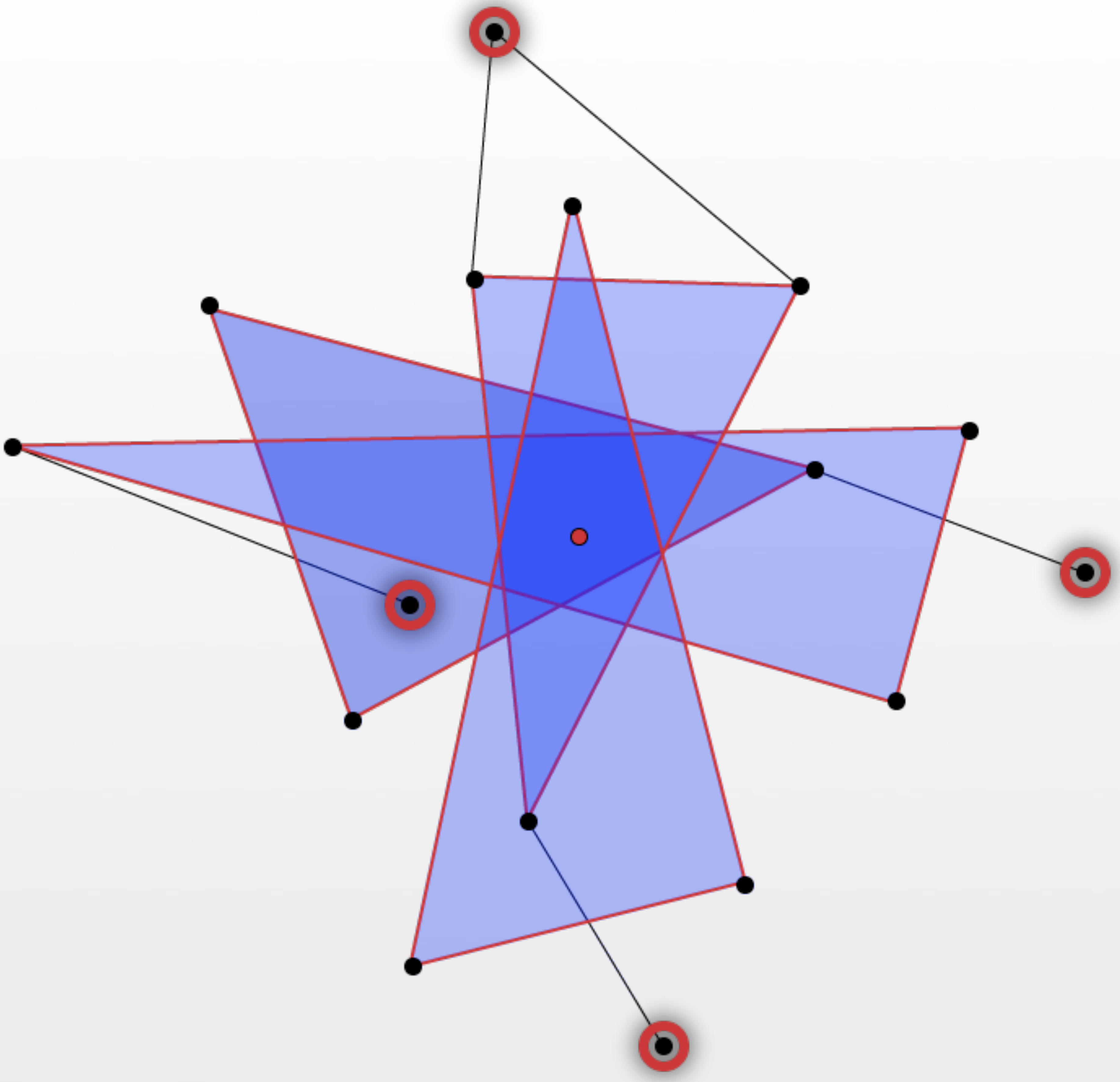












Analysis: How good is the resulting center point?

Analysis: How good is the resulting center point?

Say g_n is the minimum guaranteed partition size on n points

Analysis: How good is the resulting center point?

Say g_n is the minimum guaranteed partition size on n points

Suppose for contradiction that $g_n < \frac{n}{(d+1)^2}$.

Analysis: How good is the resulting center point?

Say g_n is the minimum guaranteed partition size on n points

Suppose for contradiction that $g_n < \frac{n}{(d+1)^2}$.

Then, $g_{n/2} < \frac{n}{2(d+1)^2}$ and the corresponding partition uses less than $\frac{n}{2(d+1)}$ points.

Analysis: How good is the resulting center point?

Say g_n is the minimum guaranteed partition size on n points

Suppose for contradiction that $g_n < \frac{n}{(d+1)^2}$.

Then, $g_{n/2} < \frac{n}{2(d+1)^2}$ and the corresponding partition uses less than $\frac{n}{2(d+1)}$ points.

So, with n points, we can construct $d+2$ points with partitions of size $g_{n/2}$.

Analysis: How good is the resulting center point?

Say g_n is the minimum guaranteed partition size on n points

Suppose for contradiction that $g_n < \frac{n}{(d+1)^2}$.

Then, $g_{n/2} < \frac{n}{2(d+1)^2}$ and the corresponding partition uses less than $\frac{n}{2(d+1)}$ points.

So, with n points, we can construct $d+2$ points with partitions of size $g_{n/2}$.

This means we can iterate the algorithm, and $g_n \geq 2g_{n/2}$

Analysis: How good is the resulting center point?

Say g_n is the minimum guaranteed partition size on n points

Suppose for contradiction that $g_n < \frac{n}{(d+1)^2}$.

Then, $g_{n/2} < \frac{n}{2(d+1)^2}$ and the corresponding partition uses less than $\frac{n}{2(d+1)}$ points.

So, with n points, we can construct $d+2$ points with partitions of size $g_{n/2}$.

This means we can iterate the algorithm, and $g_n \geq 2g_{n/2}$

Base case: $g_{d+2} = 2$.

Analysis: How good is the resulting center point?

Say g_n is the minimum guaranteed partition size on n points

Suppose for contradiction that $g_n < \frac{n}{(d+1)^2}$.

Then, $g_{n/2} < \frac{n}{2(d+1)^2}$ and the corresponding partition uses less than $\frac{n}{2(d+1)}$ points.

So, with n points, we can construct $d+2$ points with partitions of size $g_{n/2}$.

This means we can iterate the algorithm, and $g_n \geq 2g_{n/2}$

Base case: $g_{d+2} = 2$. $\quad \implies g_n \geq 2^{\log \frac{n}{d+2}} = \dfrac{n}{d+2}$

Thank you.