

Linear Size Meshes

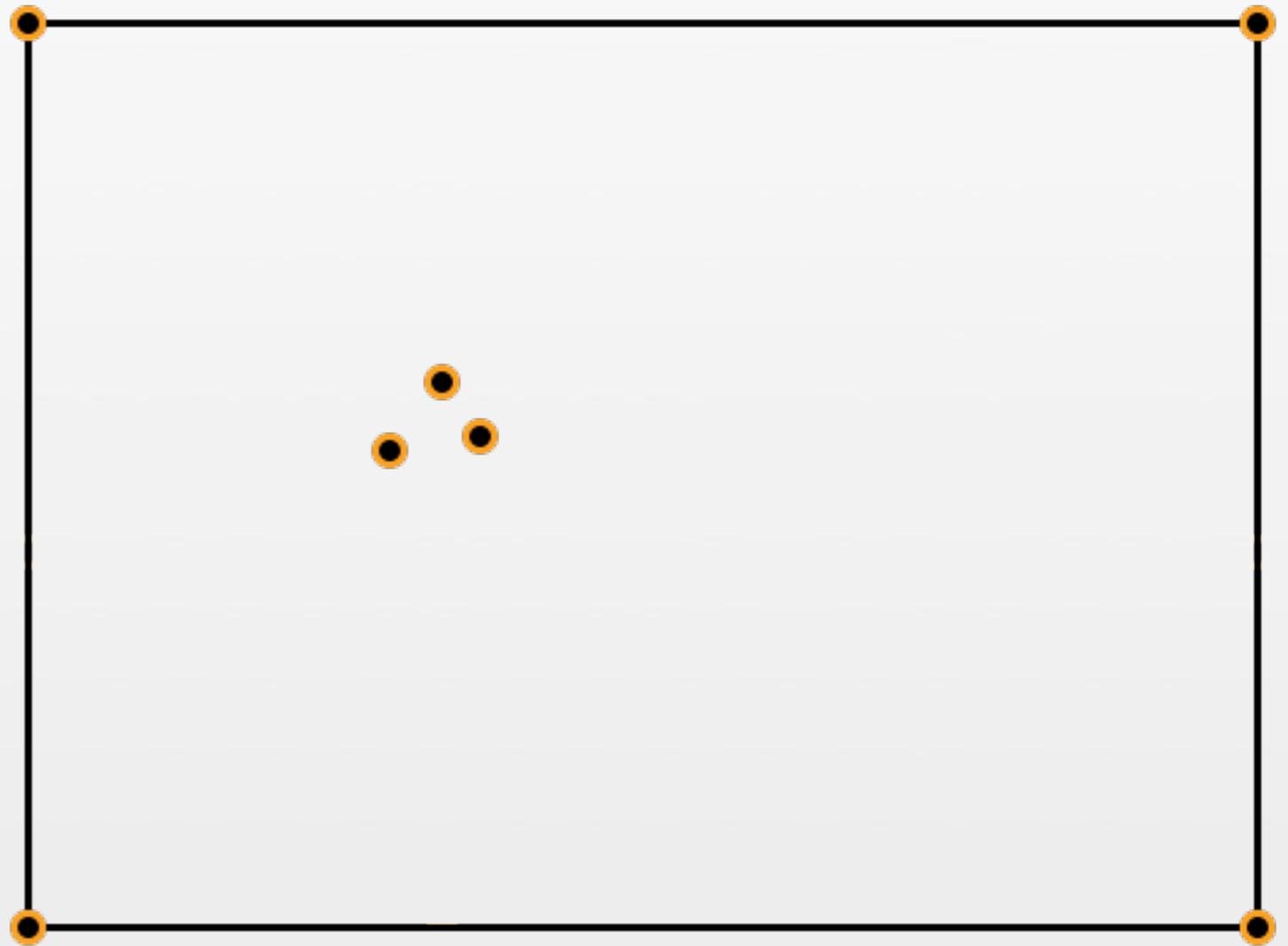
Gary Miller, Todd Phillips, [Don Sheehy](#)

The Meshing Problem

Decompose a geometric domain into simplices.

Recover features.

Guarantee Quality.

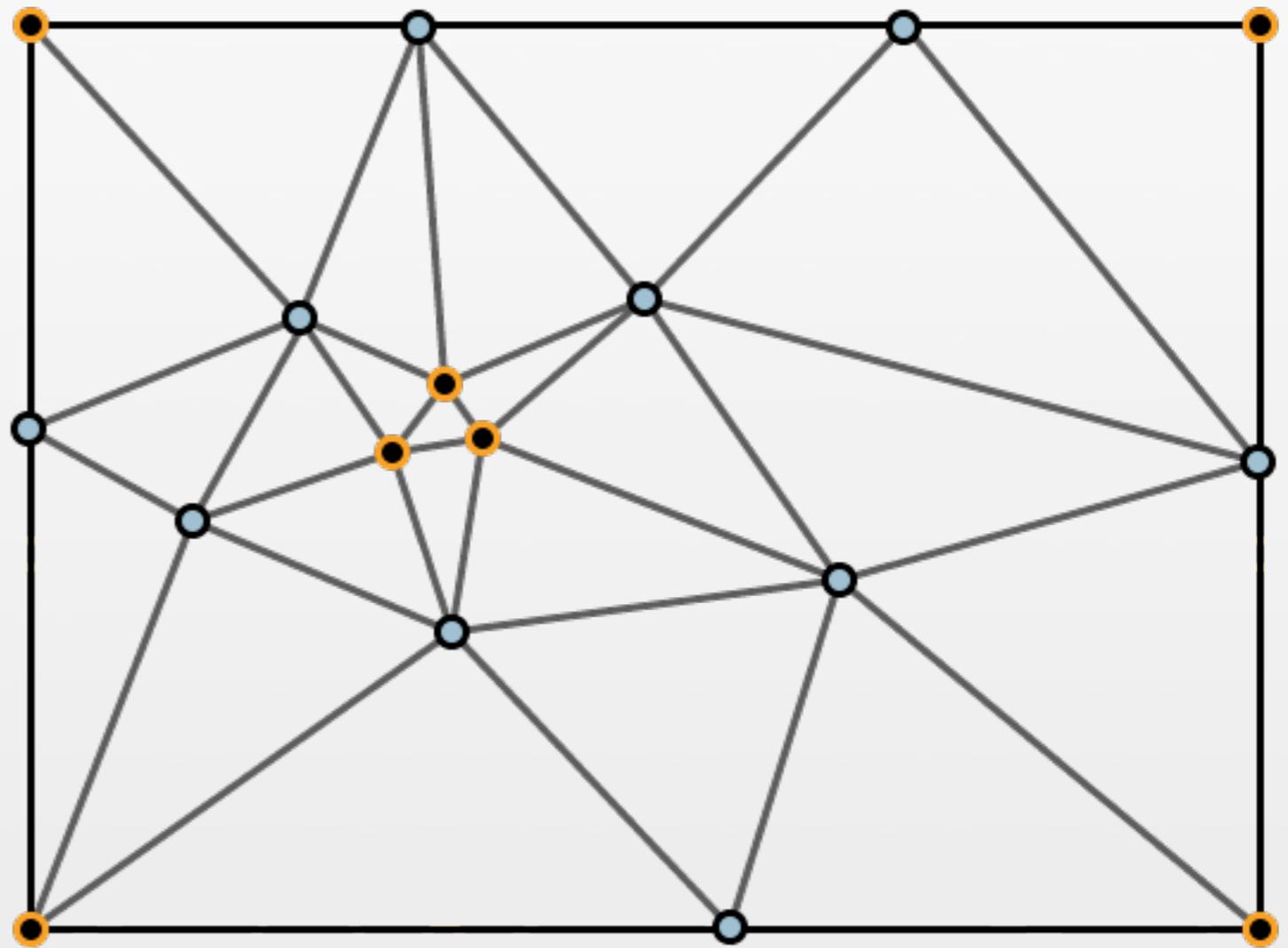


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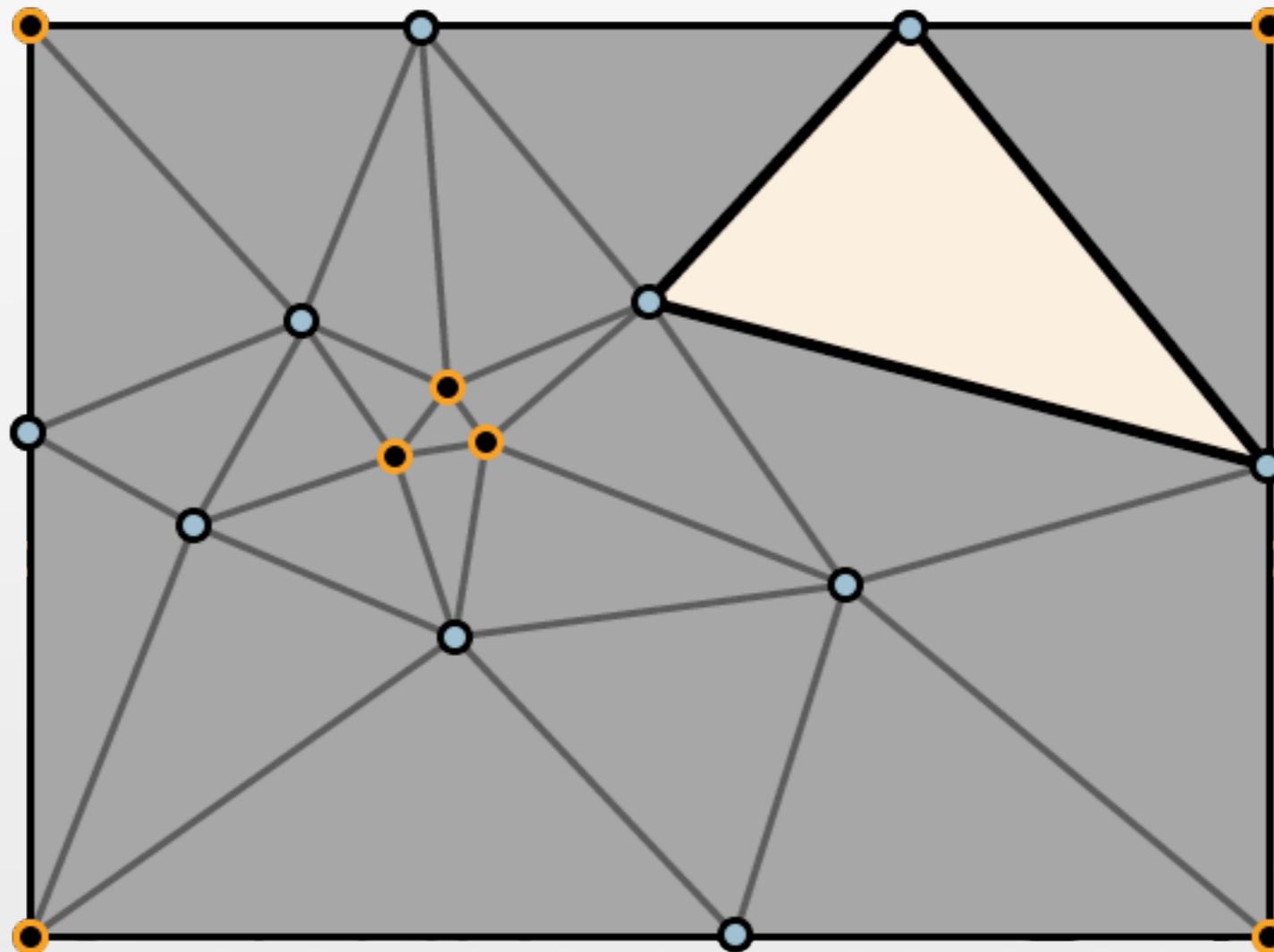
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Meshes Represent Functions

Store $f(v)$ for all mesh vertices v .

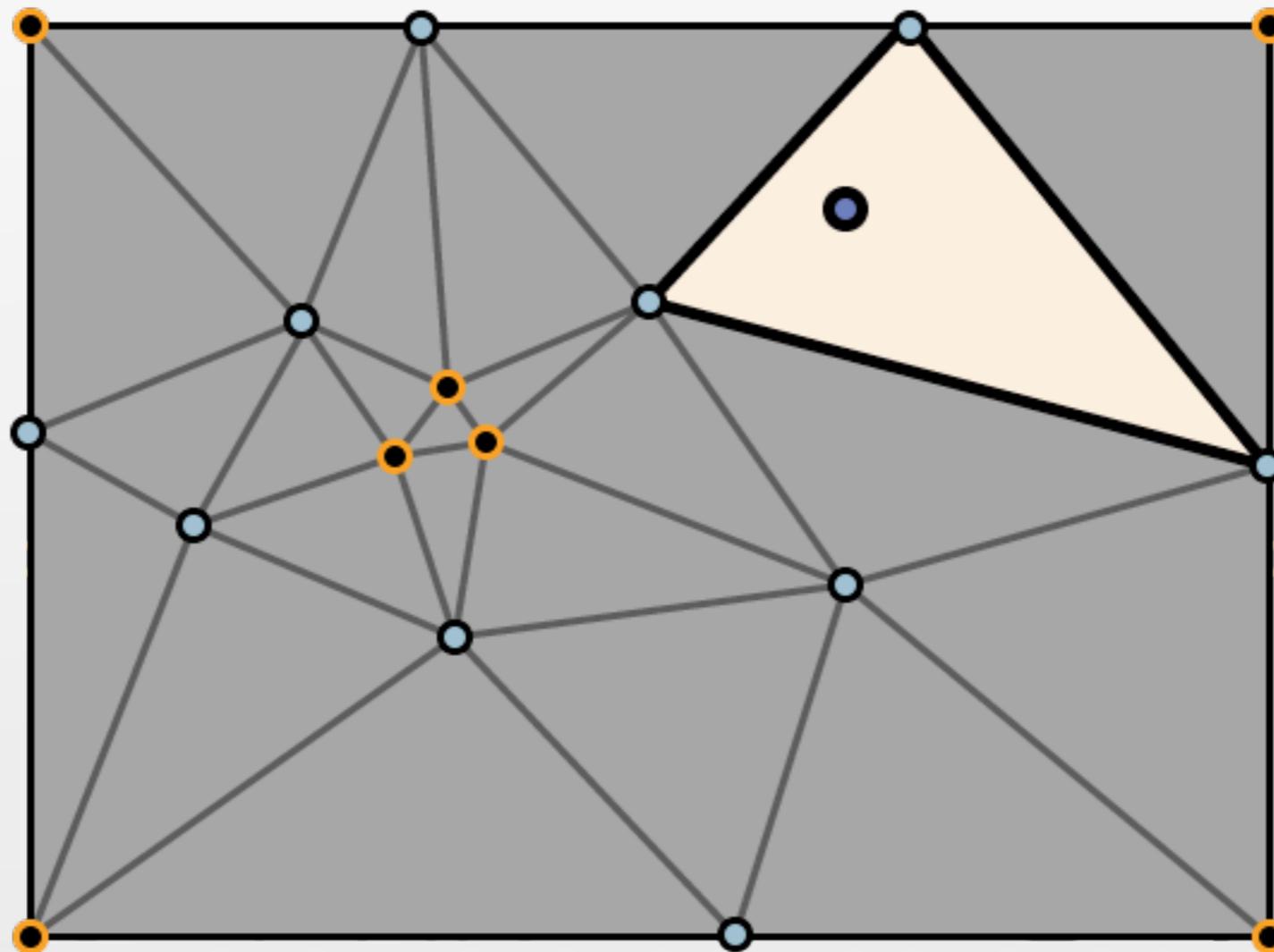
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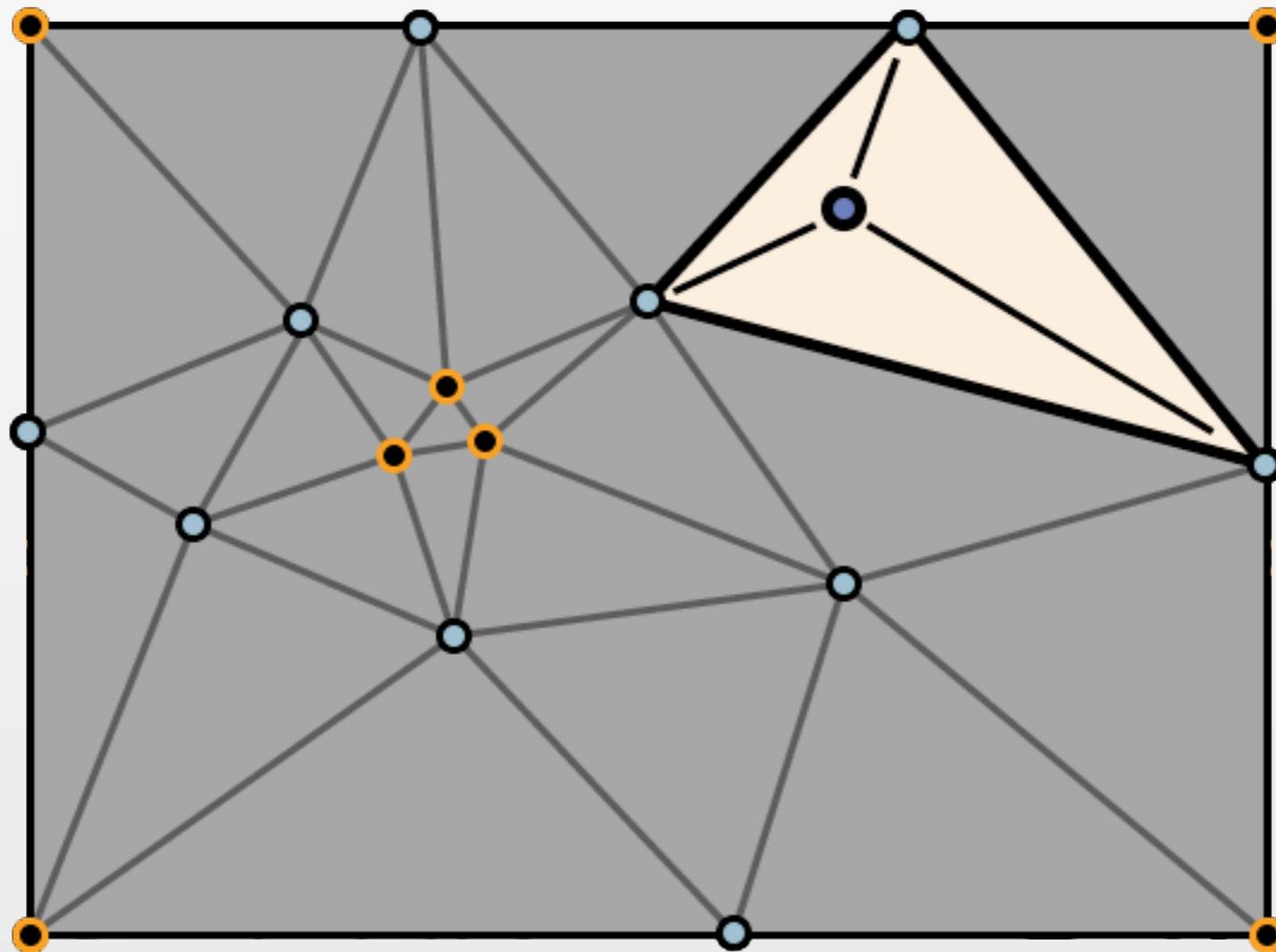
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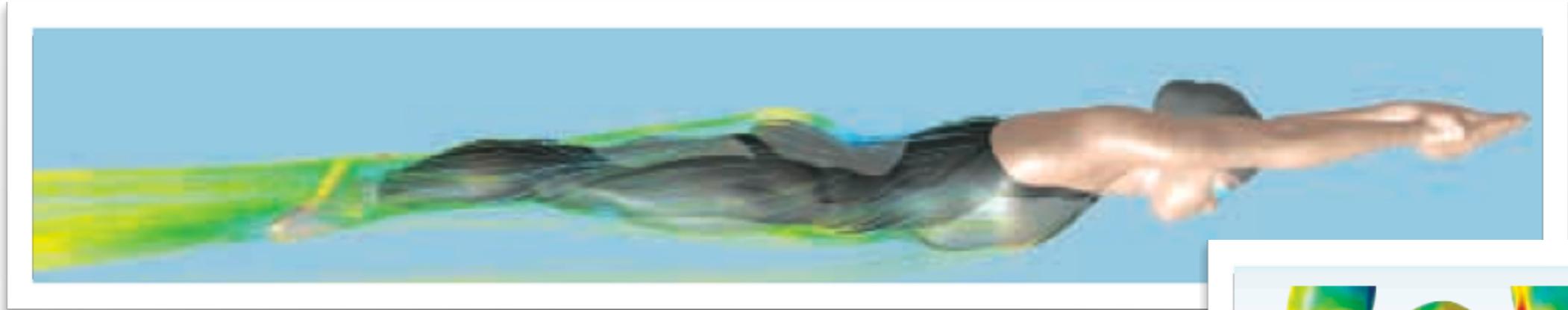


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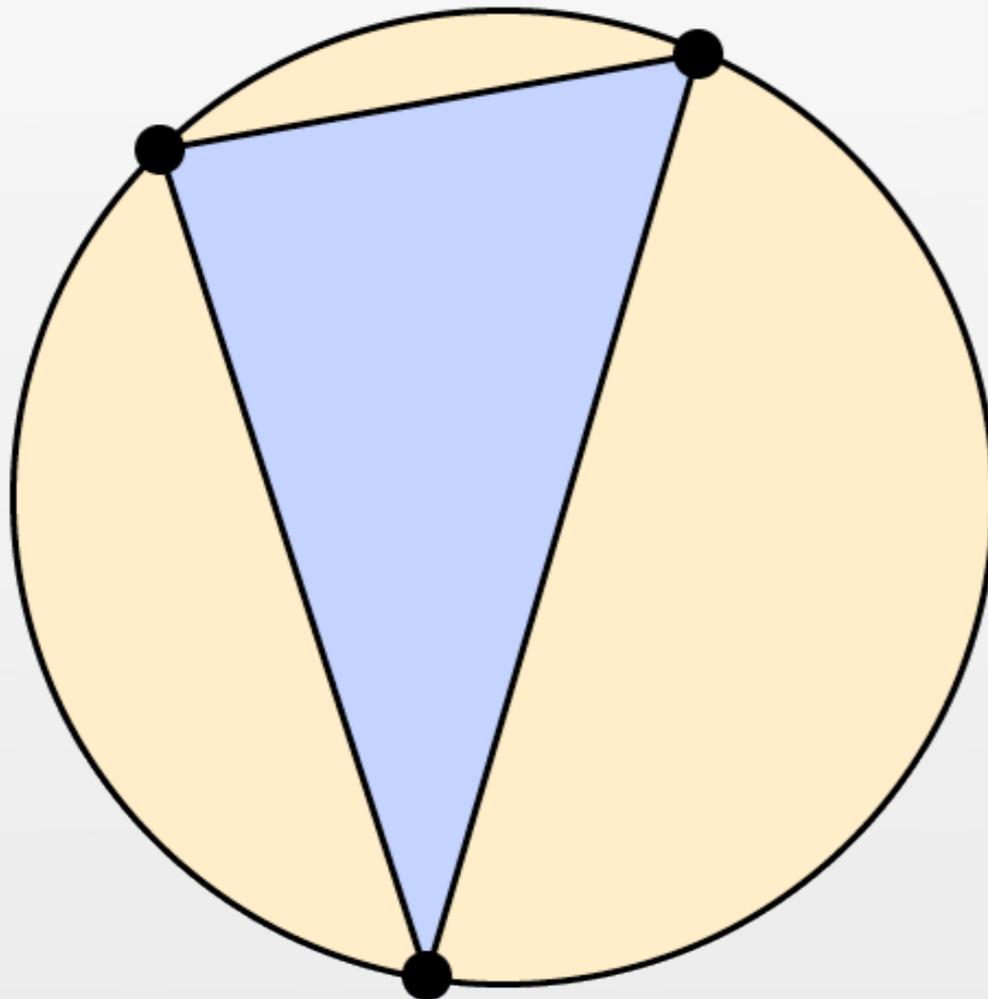


Meshes for fluid flow simulation.



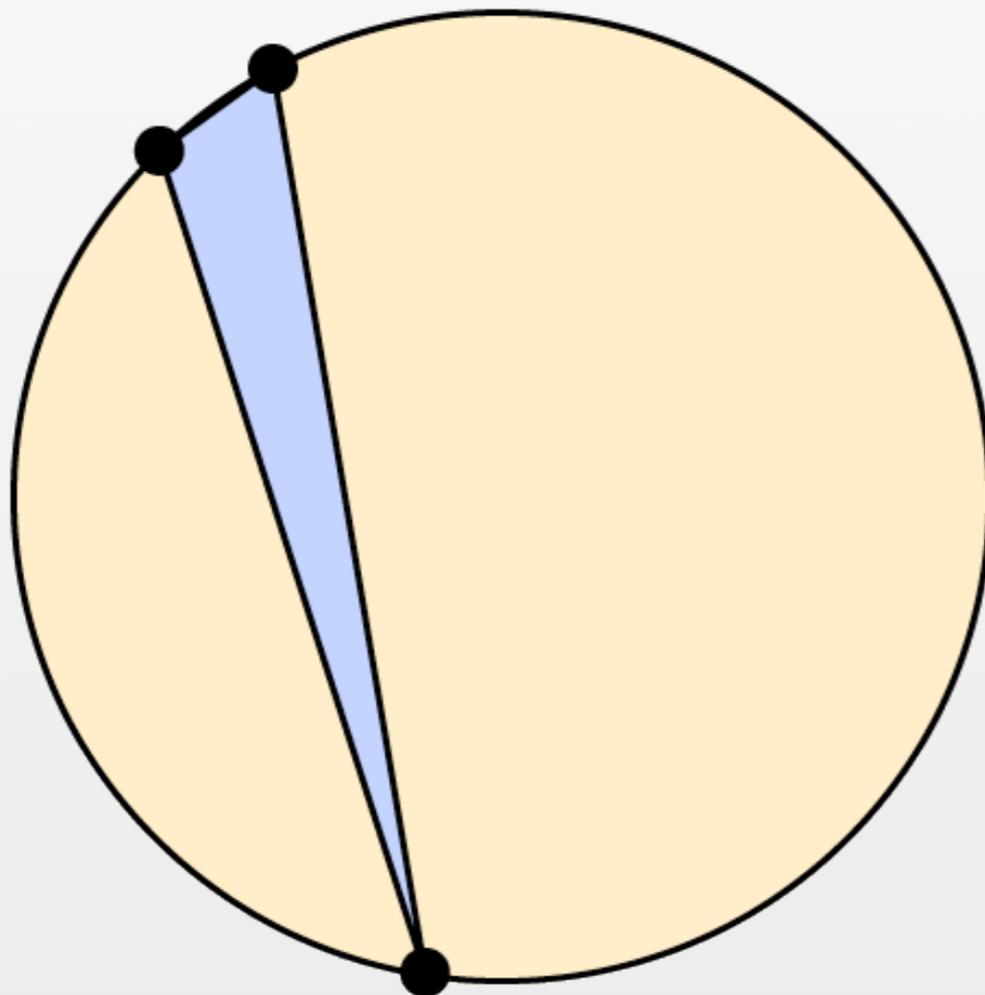
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The number of points
we add matters.

Size Optimality

m : The size of our mesh.

m_{OPT} : The size of the smallest possible mesh achieving similar quality guarantees.

Size Optimal: $m = O(m_{\text{OPT}})$

The Ruppert Bound

Theorem 1 *The number of number of vertices in any optimal-size quality mesh of a domain $\Omega \subseteq \mathbb{R}^d$ is $\Theta\left(\int_{x \in \Omega} \frac{1}{\text{fs}(x)^d} dx\right)$.*

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Note: This bound is tight.

[BernEppsteinGilbert94, Ruppert95, Ungor04, HudsonMillerPhillips06]

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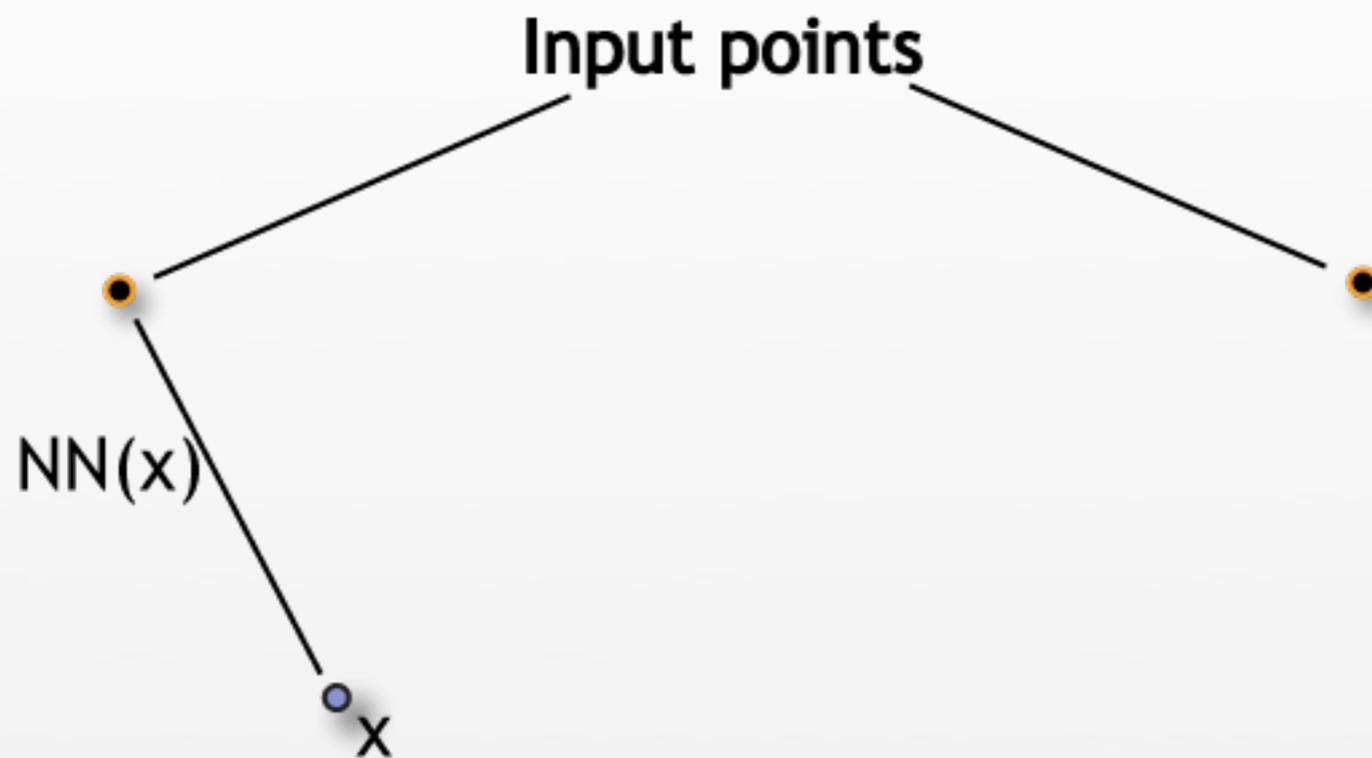
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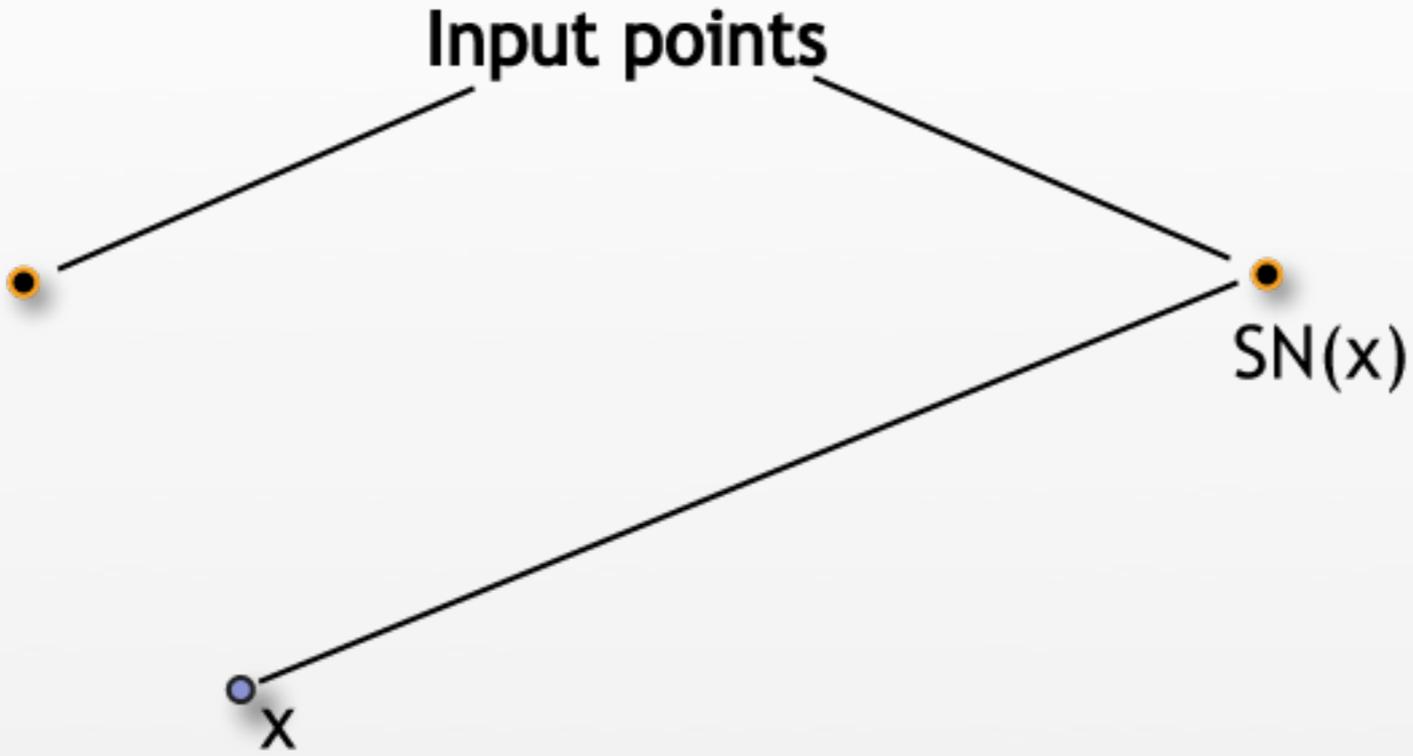


$NN(x)$

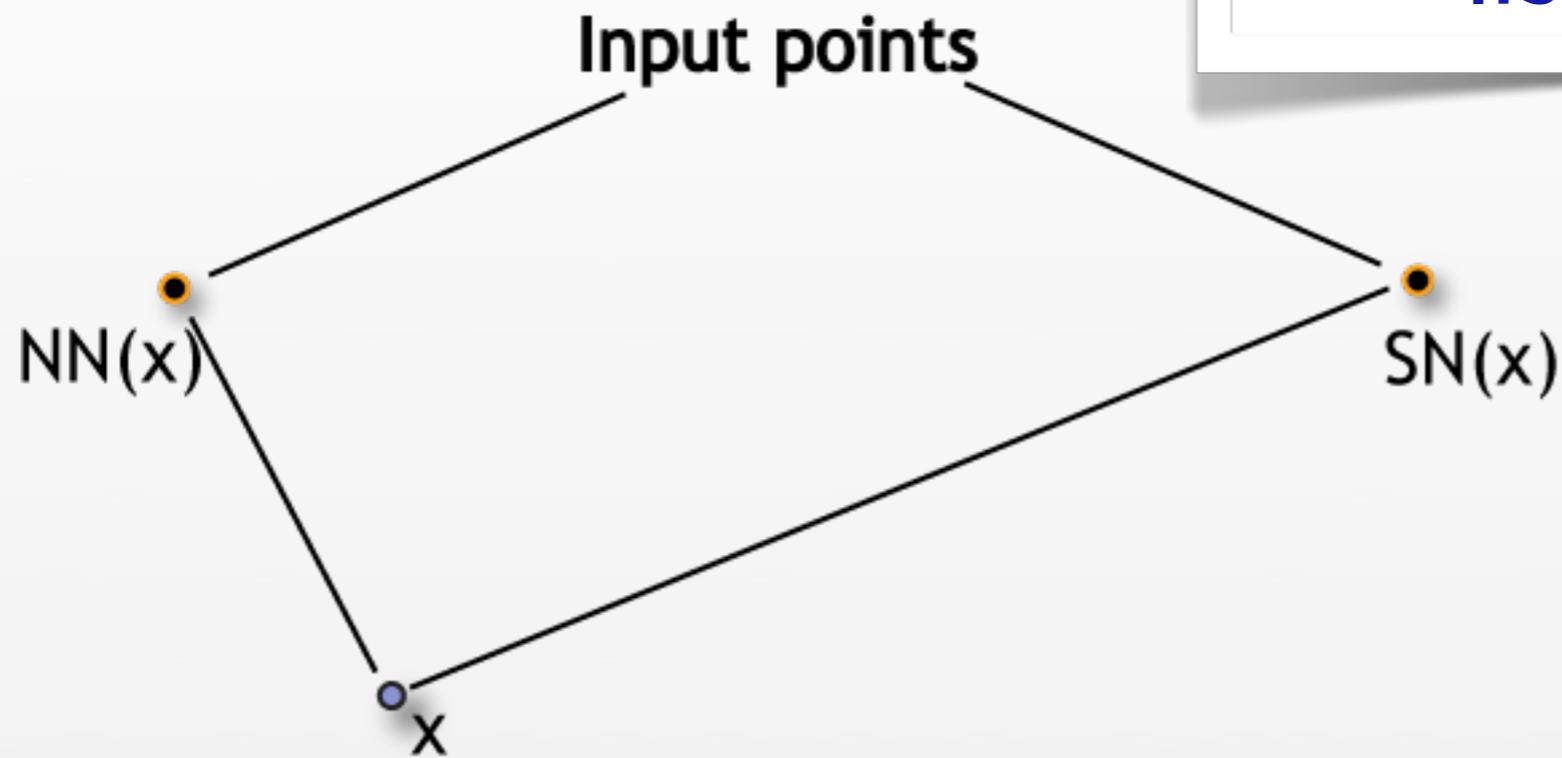


x

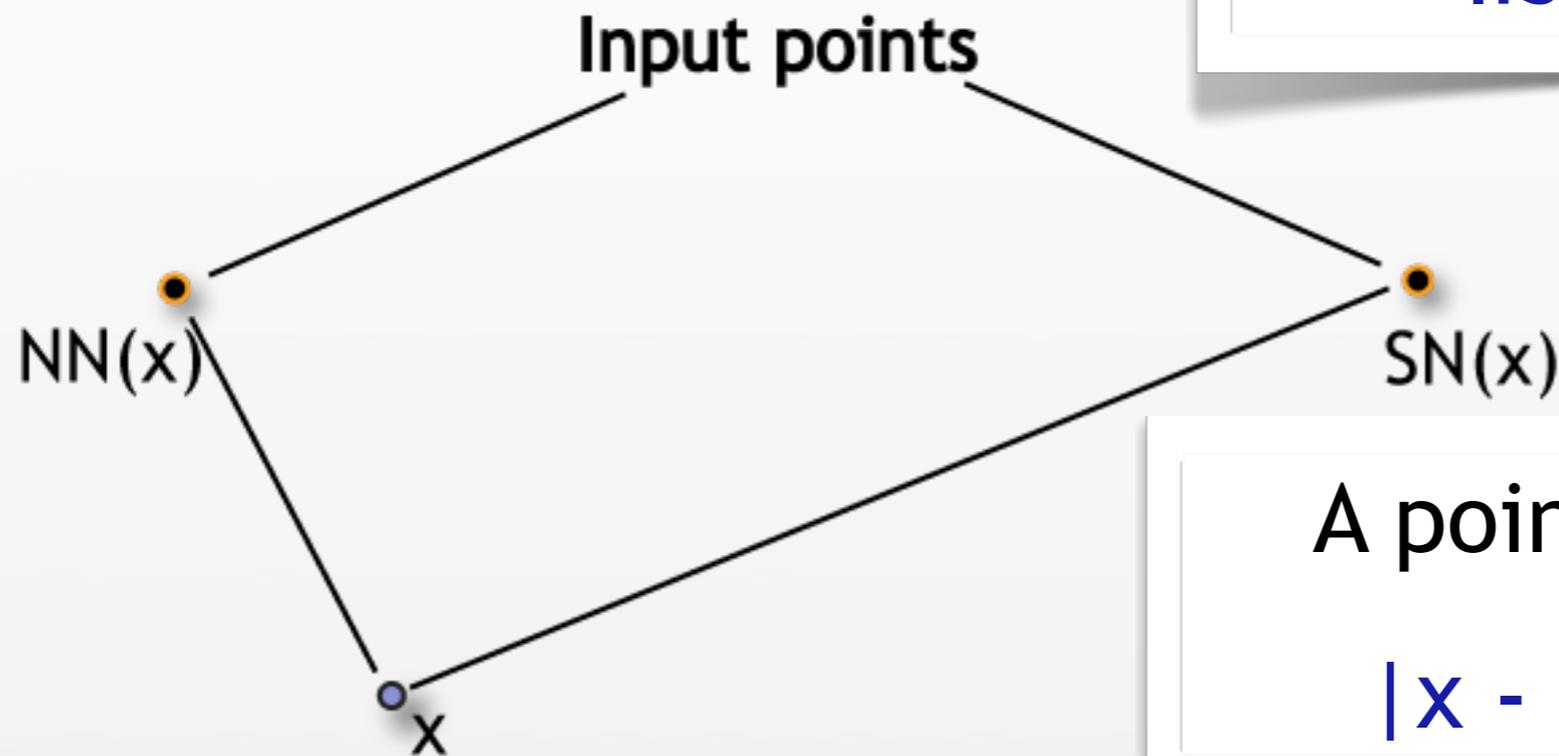




The **local feature size** at x is
 $lfs(x) = |x - SN(x)|$.

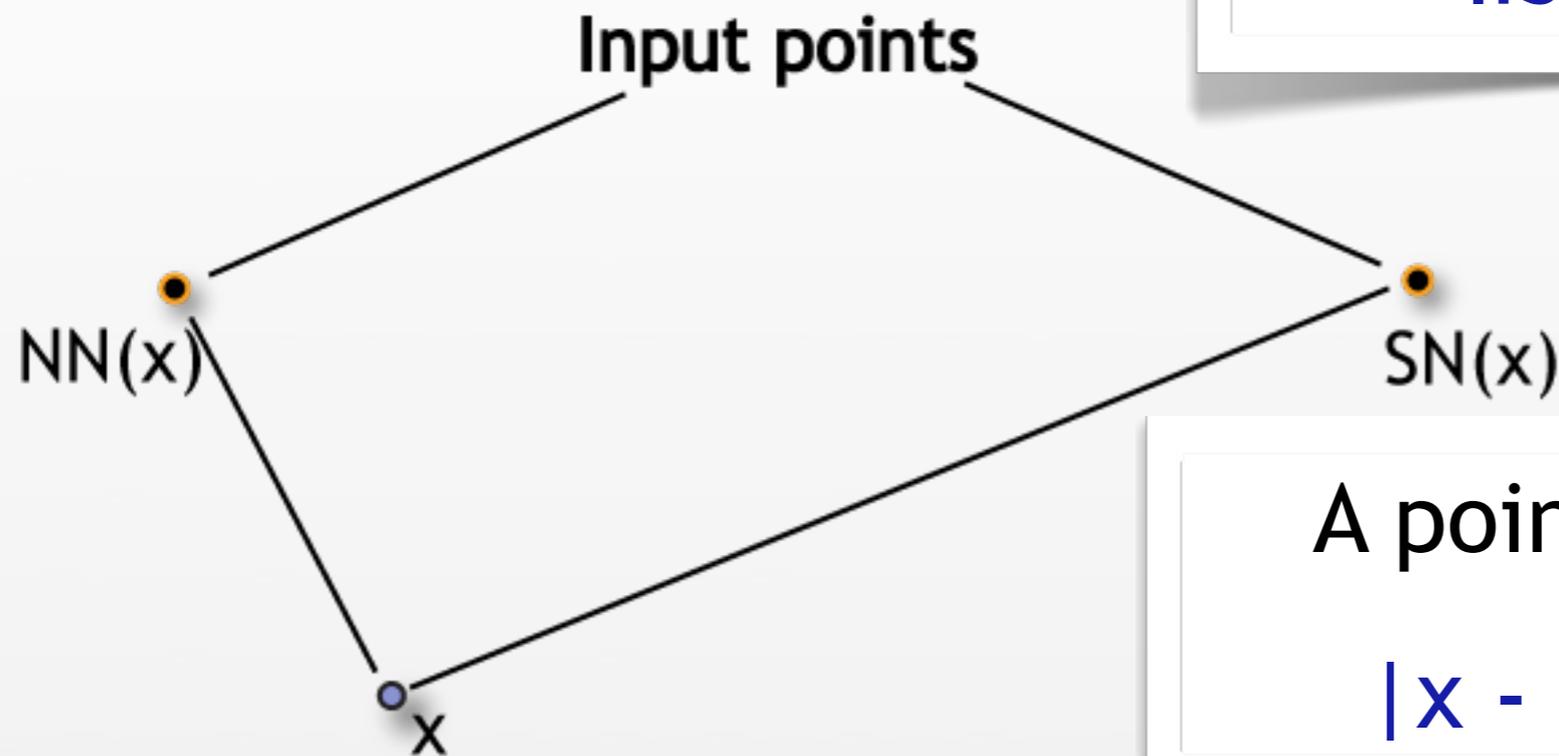


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An ordered set of points P is a **well-paced extension** of a point set Q if p_i is θ -medial w.r.t. $Q \cup \{p_1, \dots, p_{i-1}\}$.

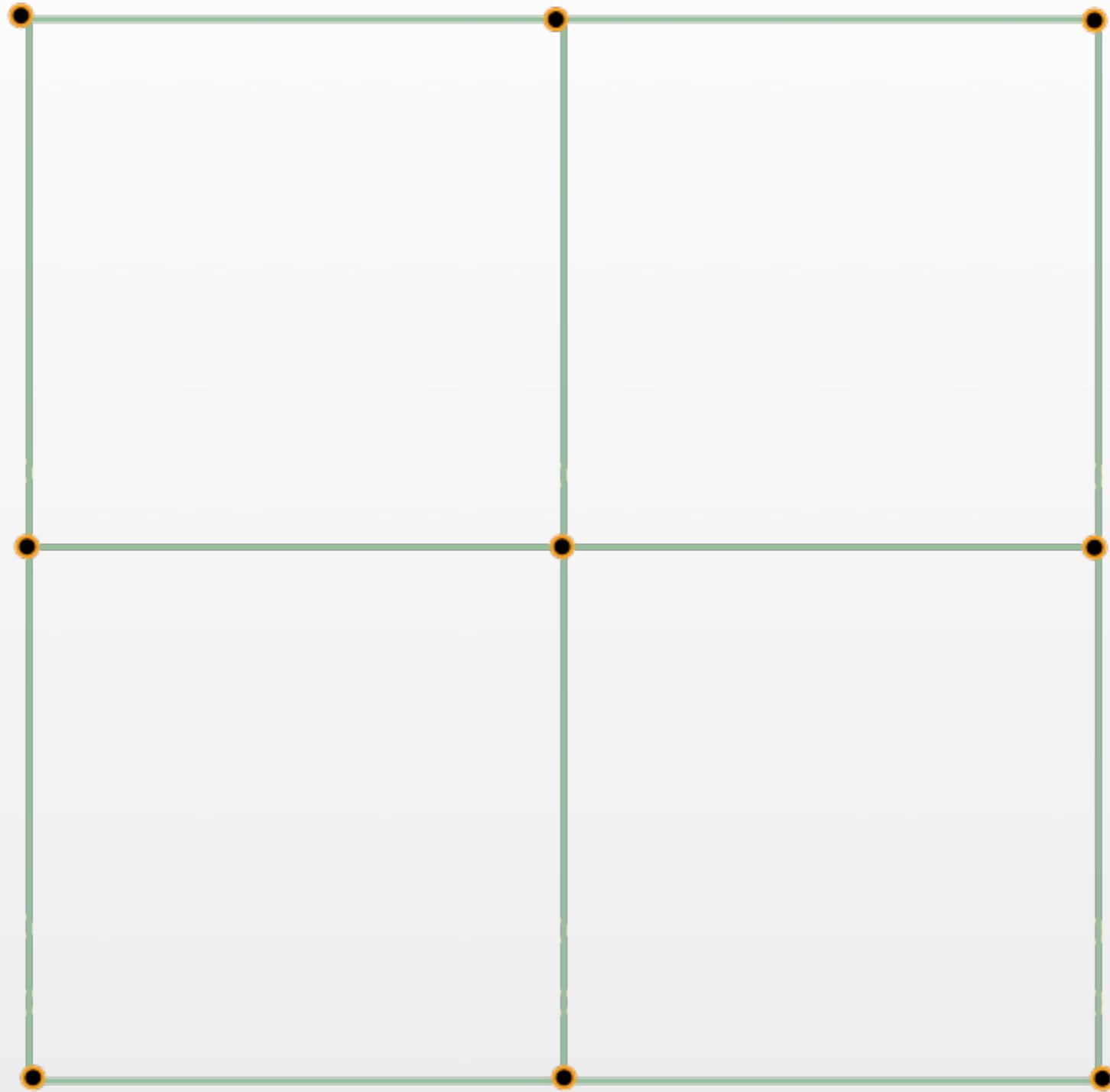
Well Paced Points

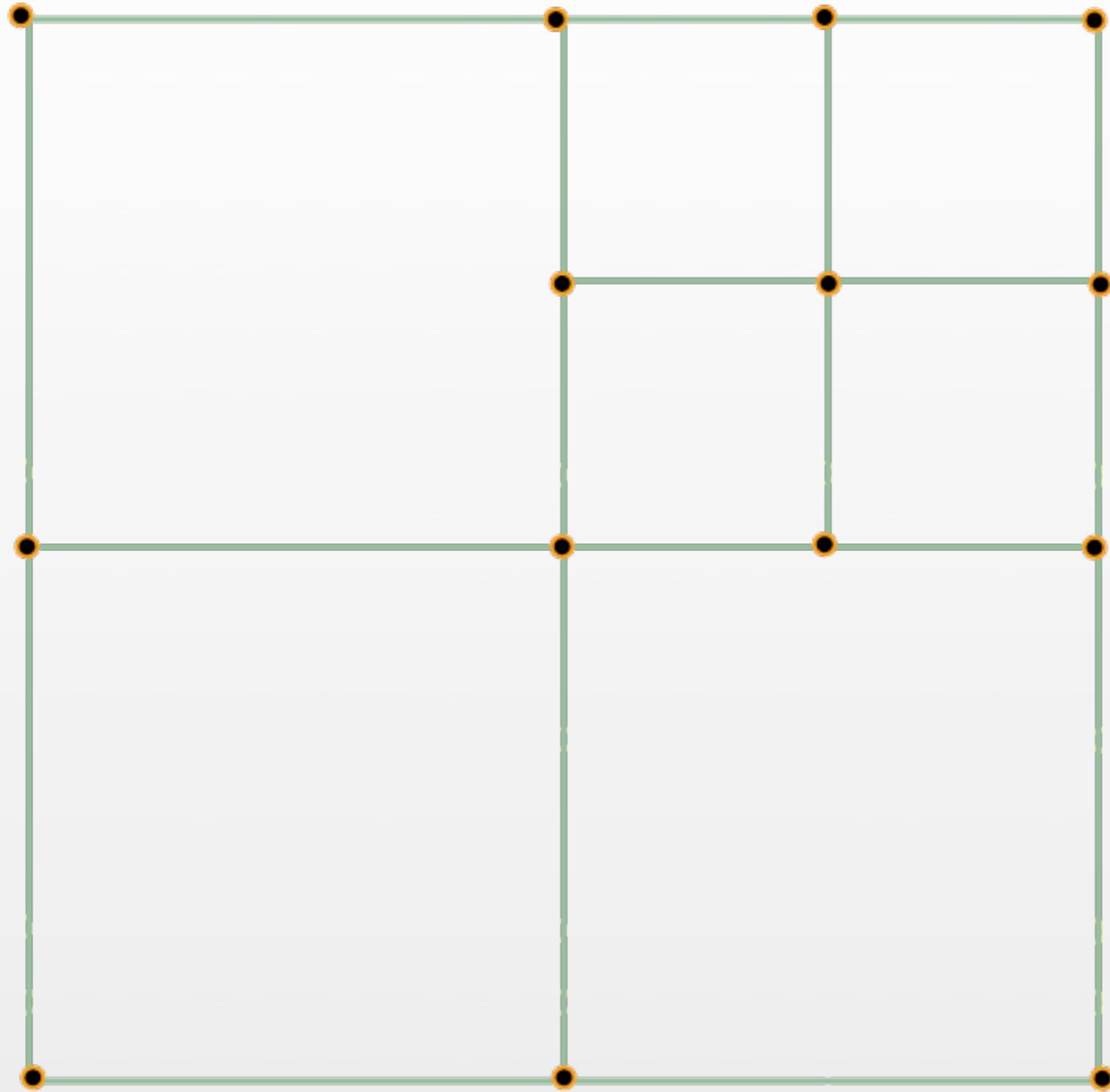
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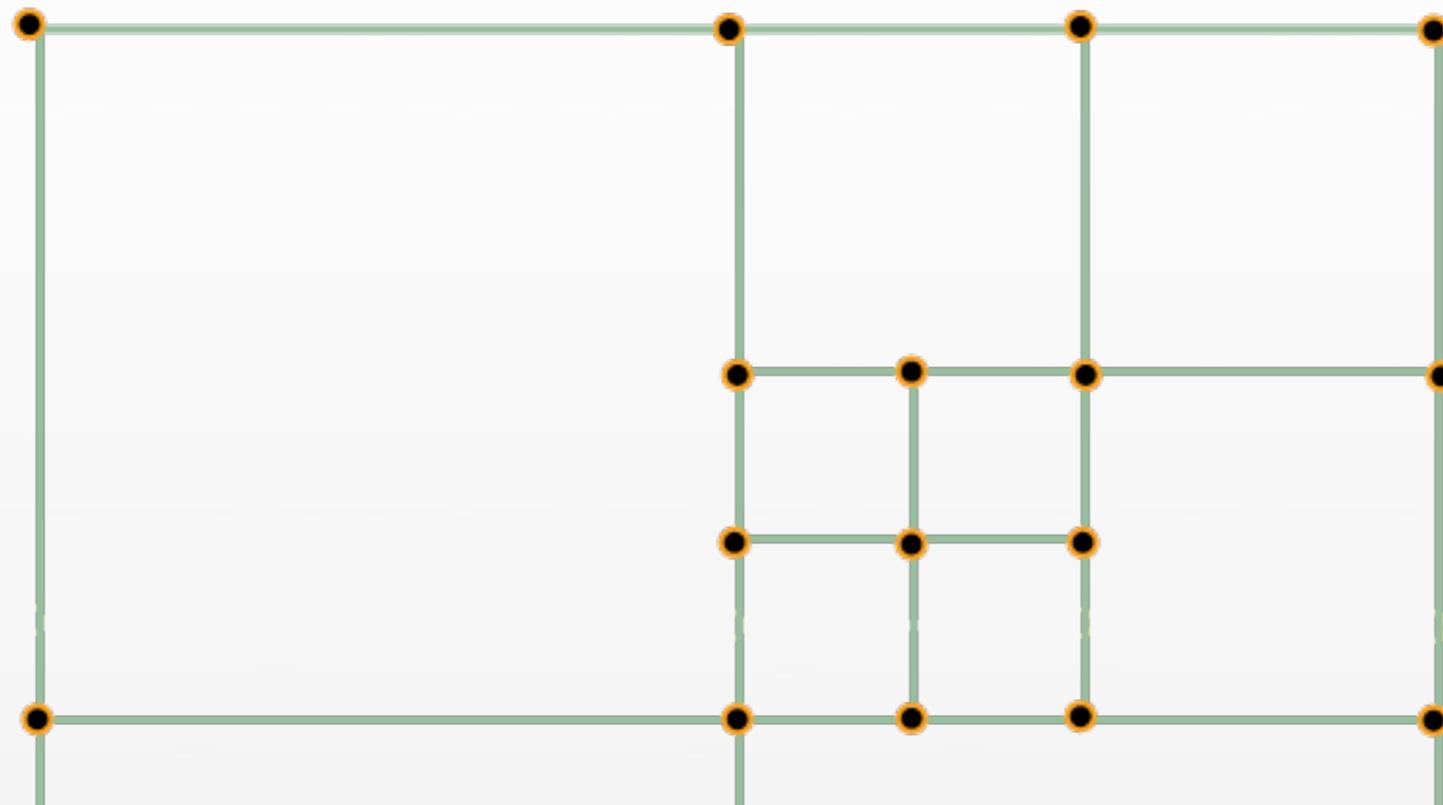
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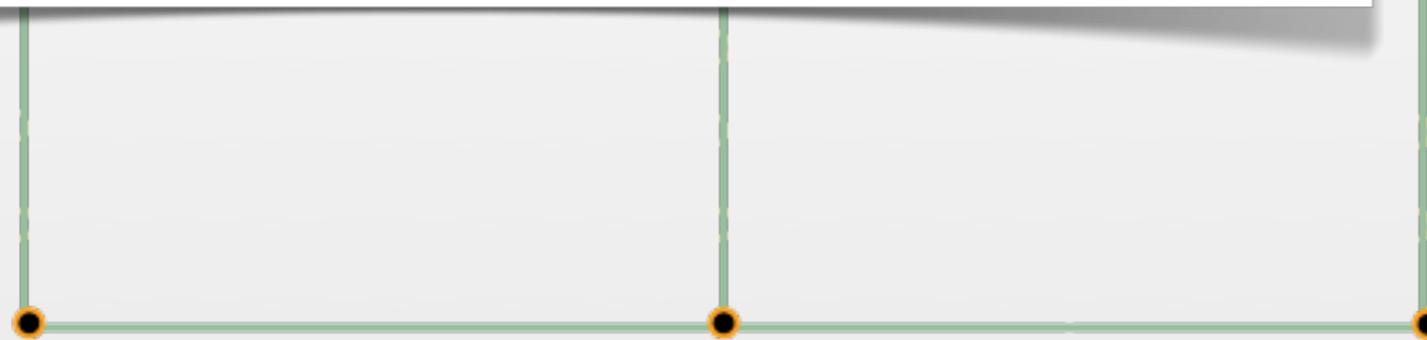






Linear Cost to balance a Quadtree.

[Moore95, Weiser81]



The Main Result

Theorem 2 *If P is a θ -well-paced extension of Q , then $Cost(Q \cup P) = O(Cost(Q) + |P|)$.*

Proof Idea

We want to prove that the amortized change in m_{opt} as we add a θ -medial point is **constant**.

$$\int_{x \in \Omega} \frac{1}{\text{lfs}'(x)^d} - \frac{1}{\text{lfs}(x)^d} dx$$

lfs' is the new local feature size after adding one point.

Proof Idea

$$\int_{x \in \Omega} \frac{1}{|\text{fs}'(x)|^d} - \frac{1}{|\text{fs}(x)|^d} dx$$

Key ideas to bound this integral.

- Integrate over the entire space using polar coordinates.
- Split the integral into two parts: the region near the the new point and the region far from the new point.
- Use every trick you learned in high school.

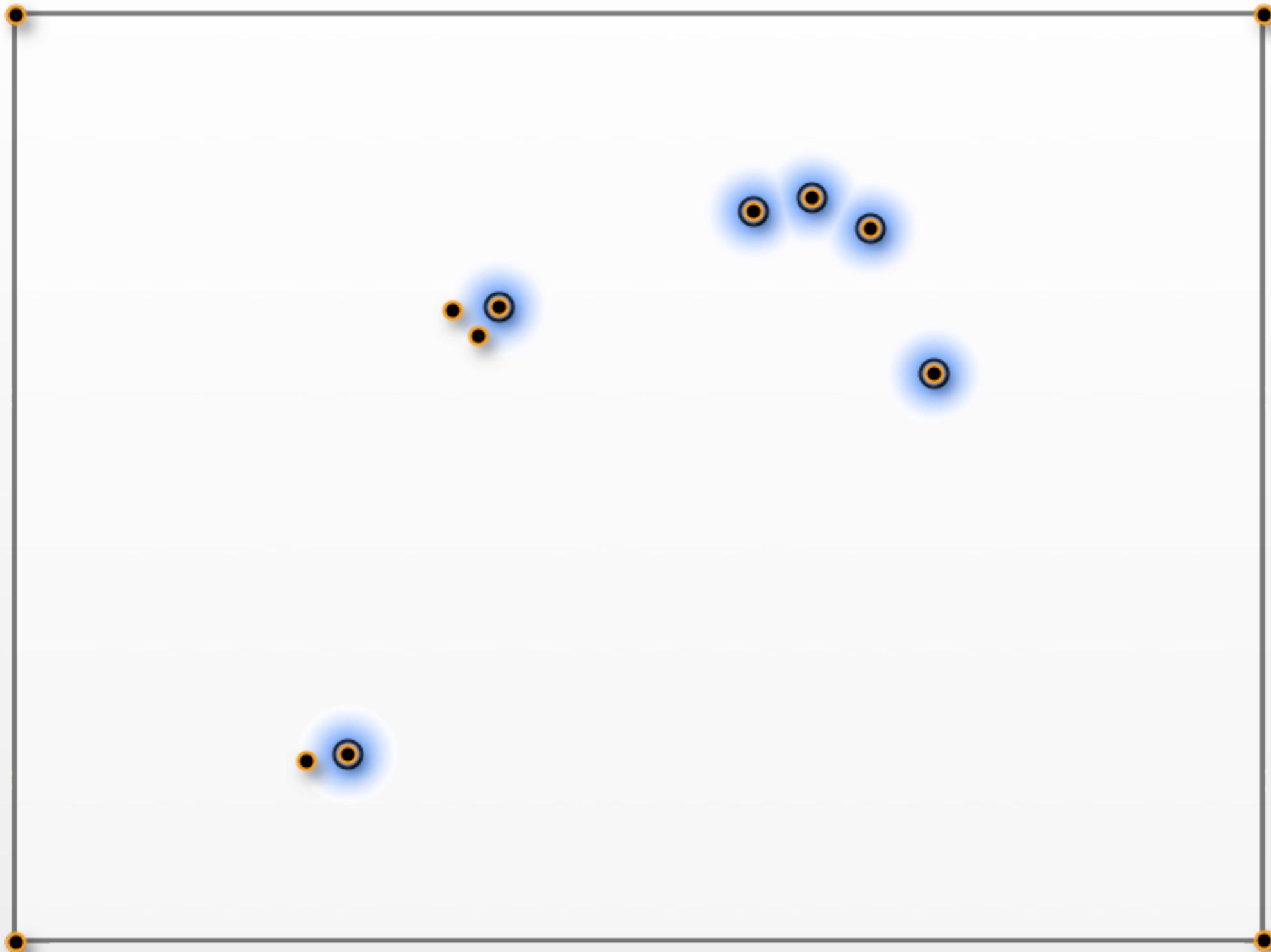
How do we get around the
Ruppert lower bound?

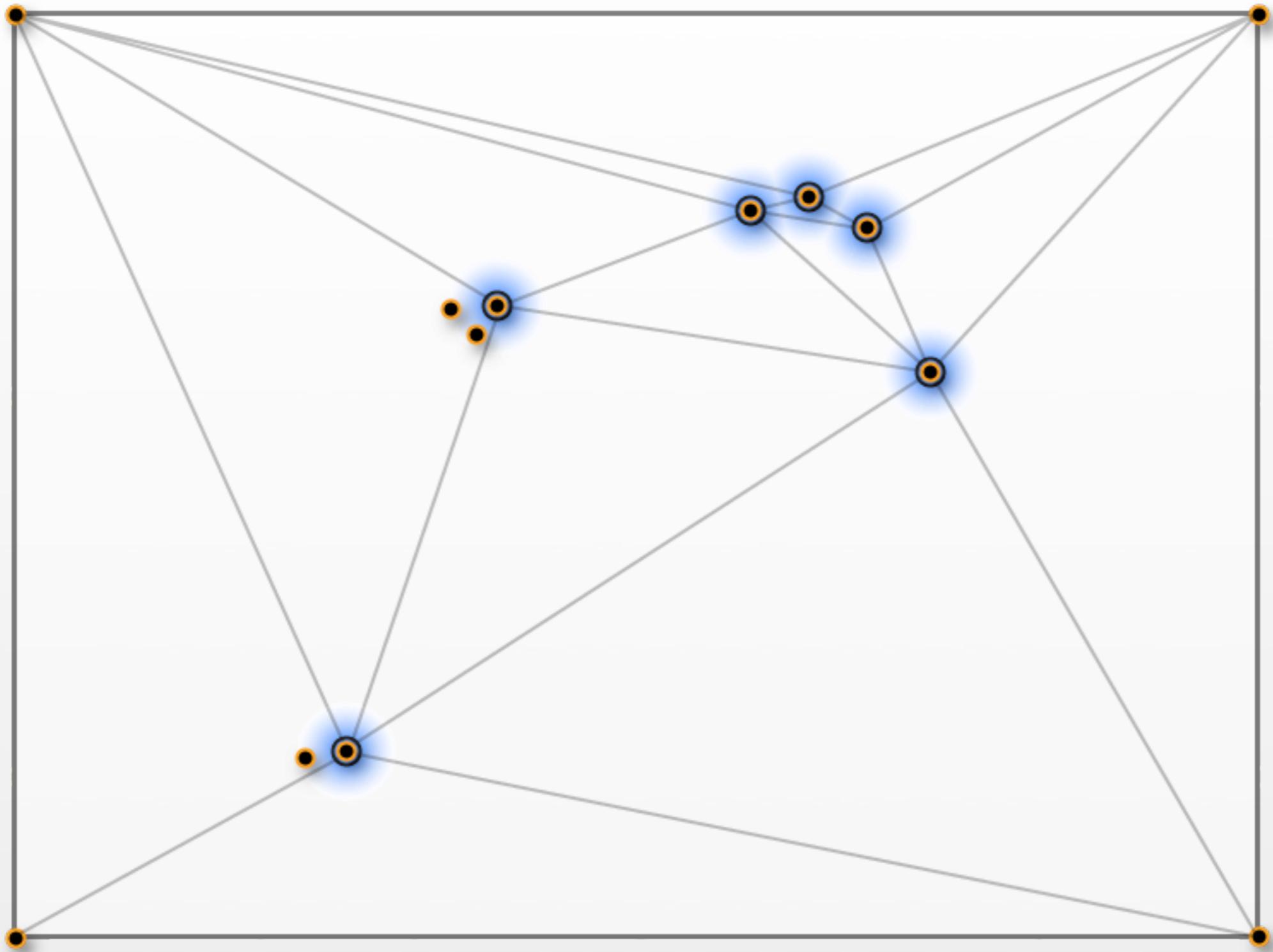
Linear Size Delaunay Meshes

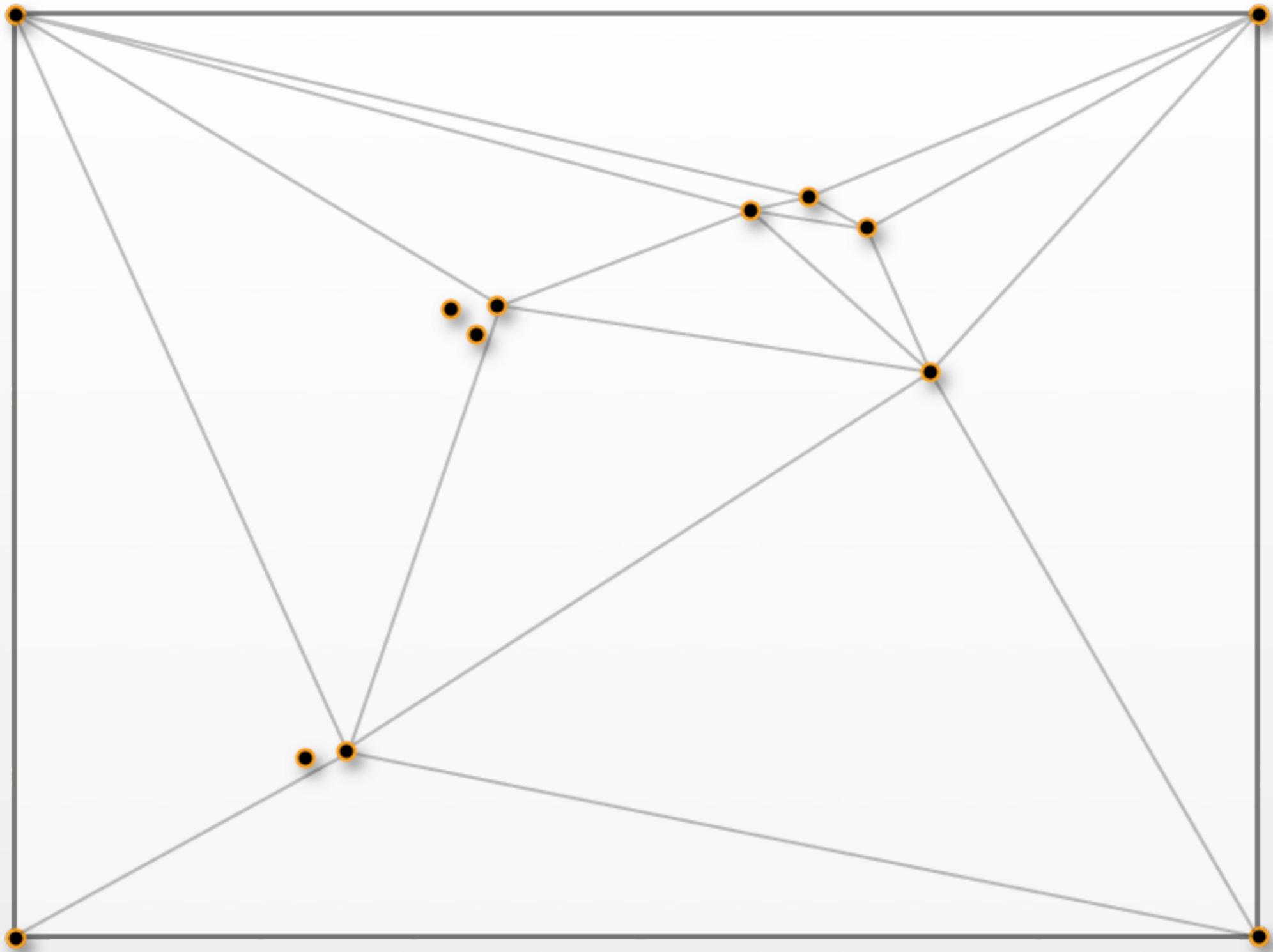
The Algorithm:

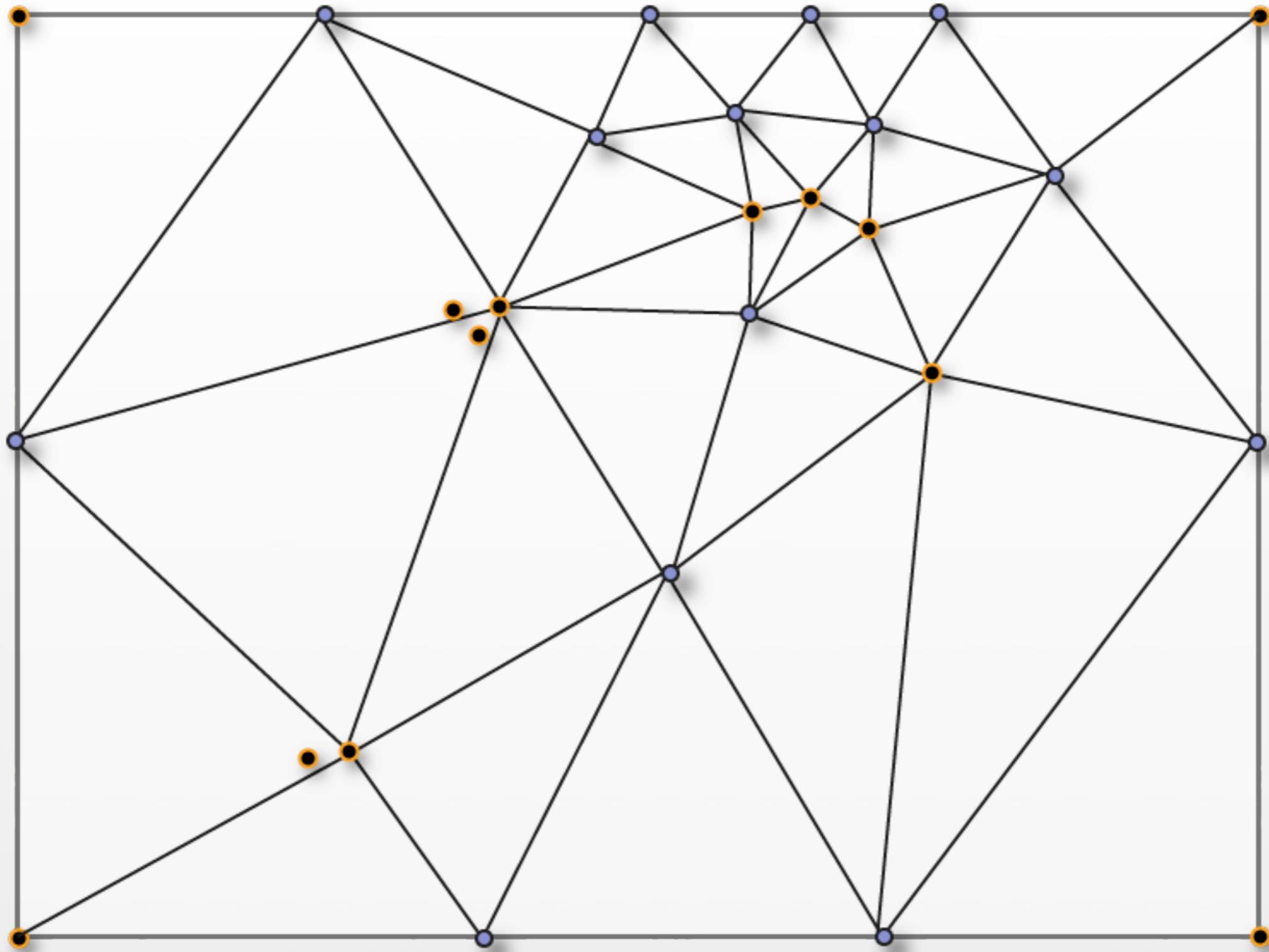
- Pick a maximal well-paced extension of the bounding box.
- Refine to a quality mesh.
- Remaining points form small clusters. Surround them with smaller bounding boxes.
- Recursively mesh the smaller bounding boxes.
- Return the Delaunay triangulation of the entire point set.

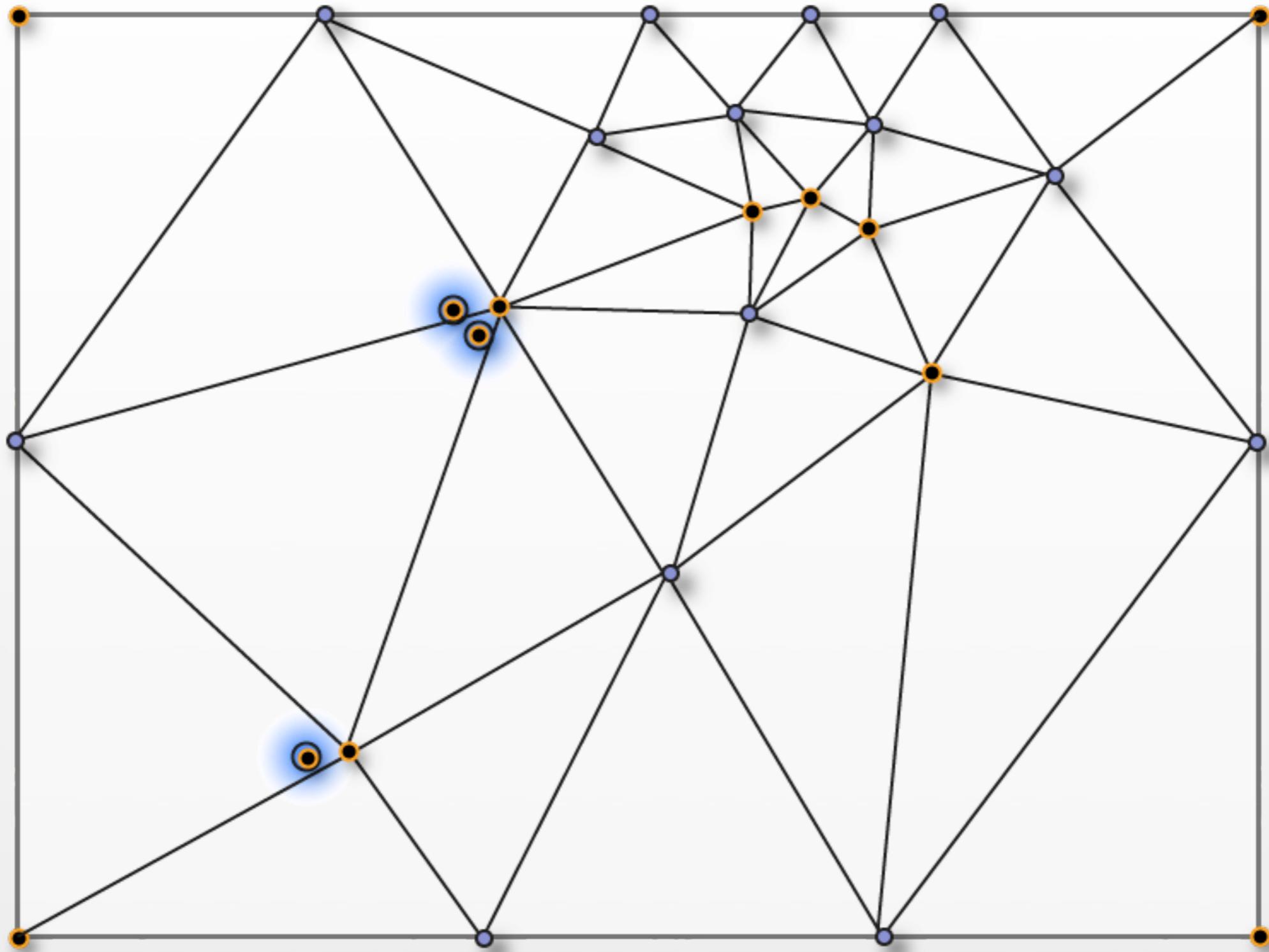


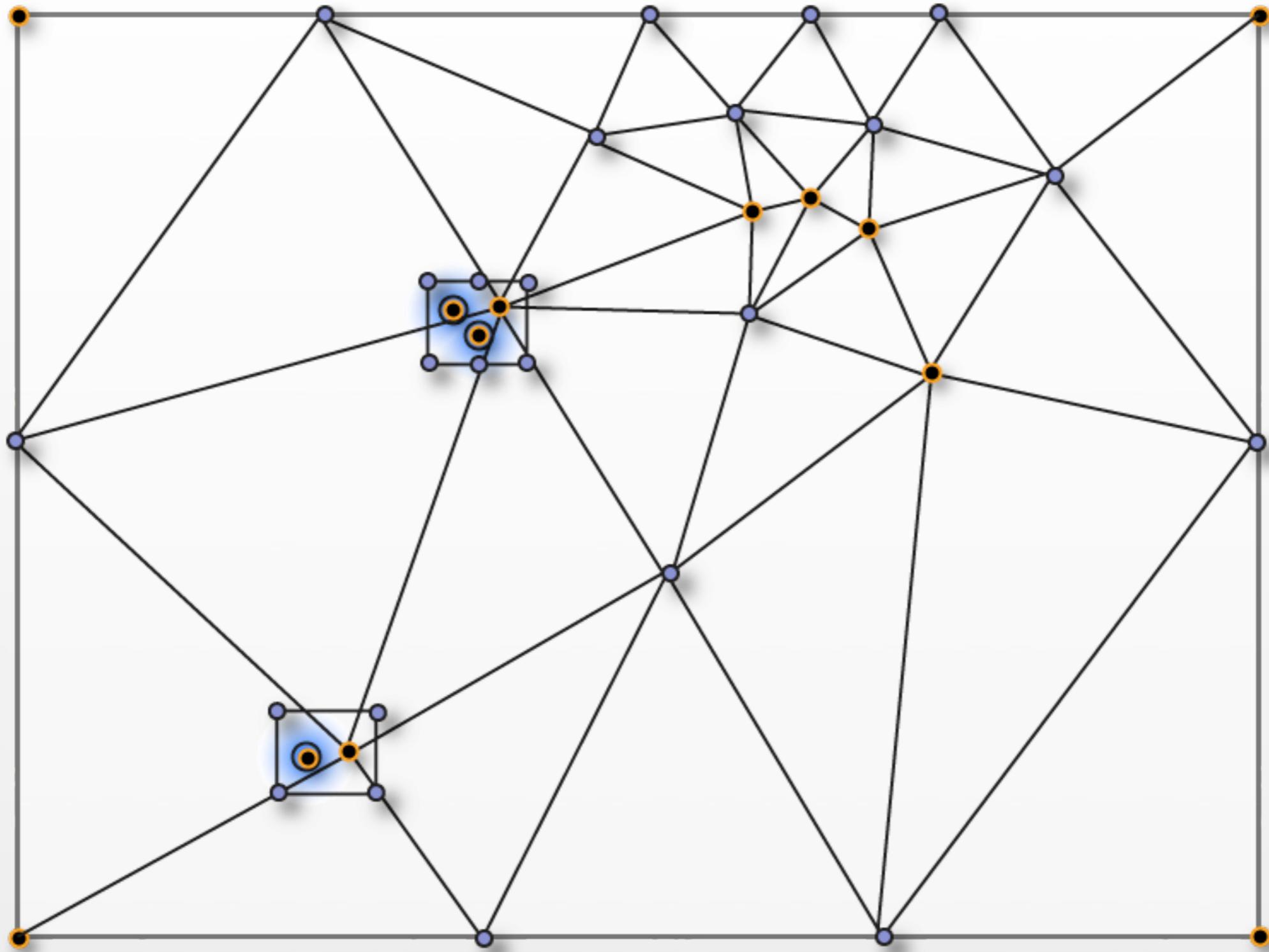


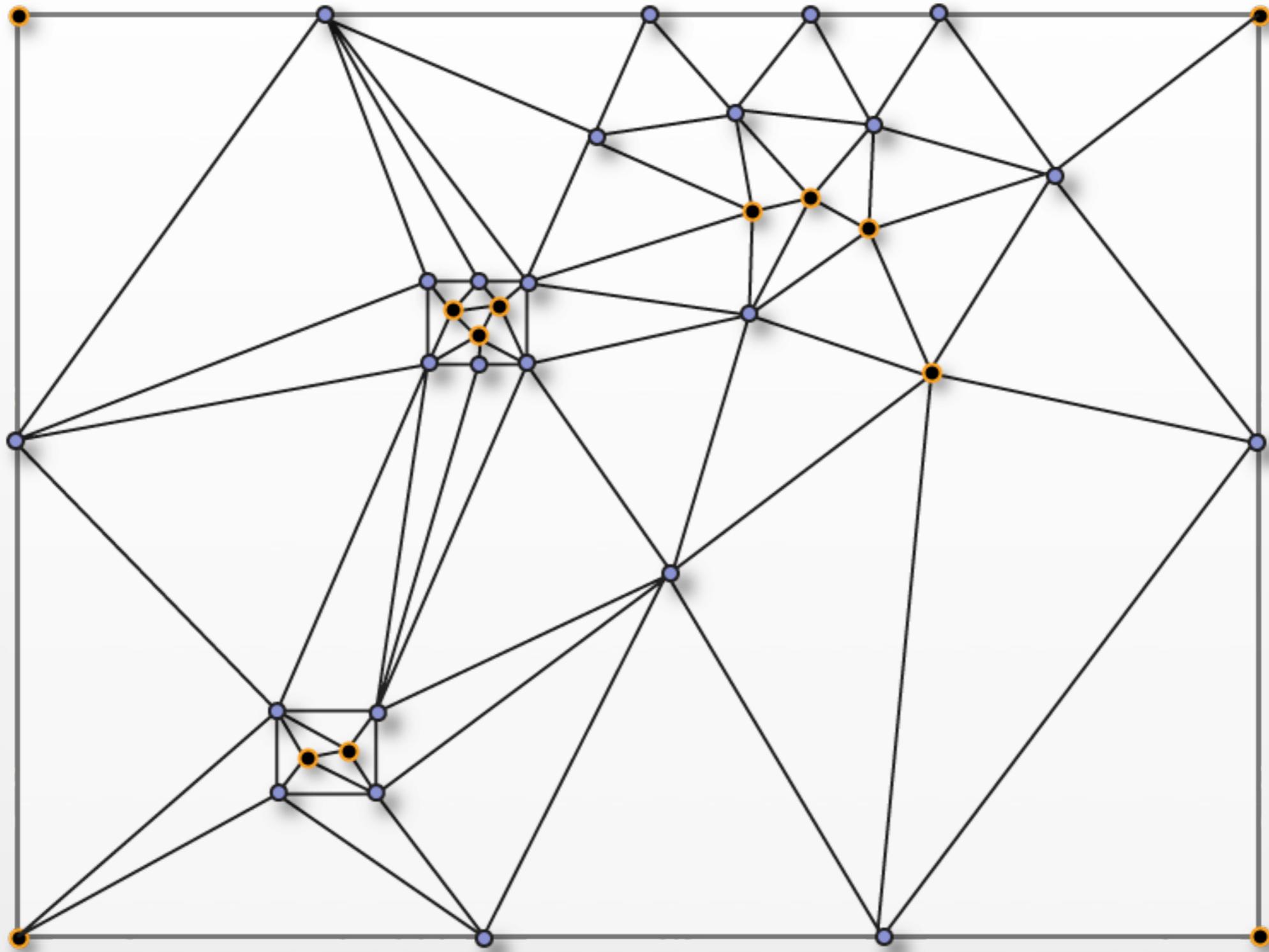












Guarantees

- Output size is $O(n)$.
- Bound on radius/longest edge. (No-large angles in R^2)

Summary

- An algorithm for producing linear size delaunay meshes of point sets in \mathbf{R}^d .
- A new method for analyzing the size of quality meshes in terms of input size.

Sneak Preview

In a follow-up paper, we show how the theory of well-paced points leads to a proof of a classic **folk conjecture**.

- Suppose we are meshing a domain with an odd shaped boundary.
- We enclose the domain in a bounding box and mesh the whole thing.
- We throw away the “extra.”
- The amount we throw away is at most a constant fraction.

Thank you.

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Questions?