Linear Size Meshes

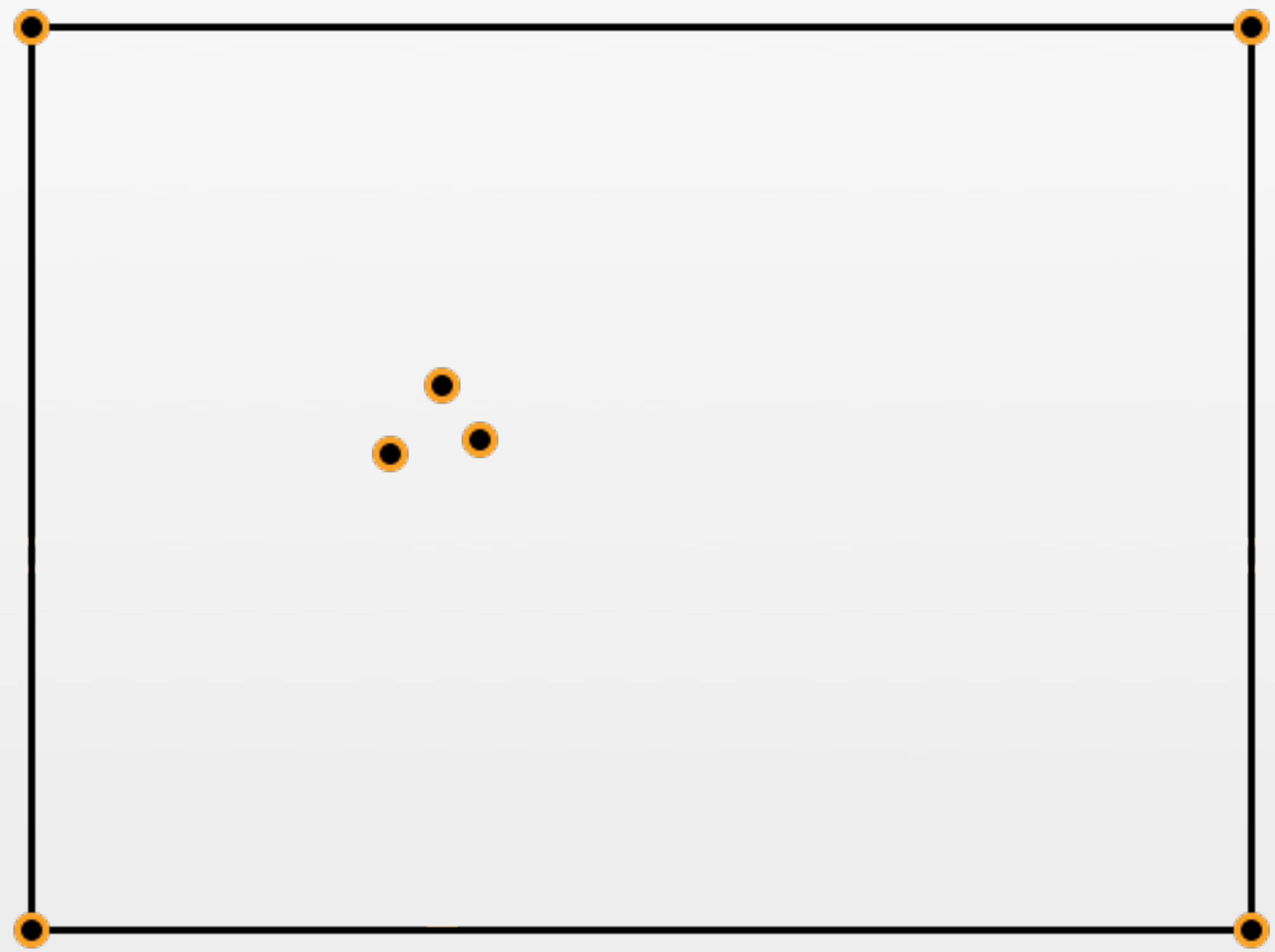
Gary Miller, Todd Phillips, Don Sheehy

The Meshing Problem

Decompose a geometric domain into simplices.

Recover features.

Guarantee Quality.



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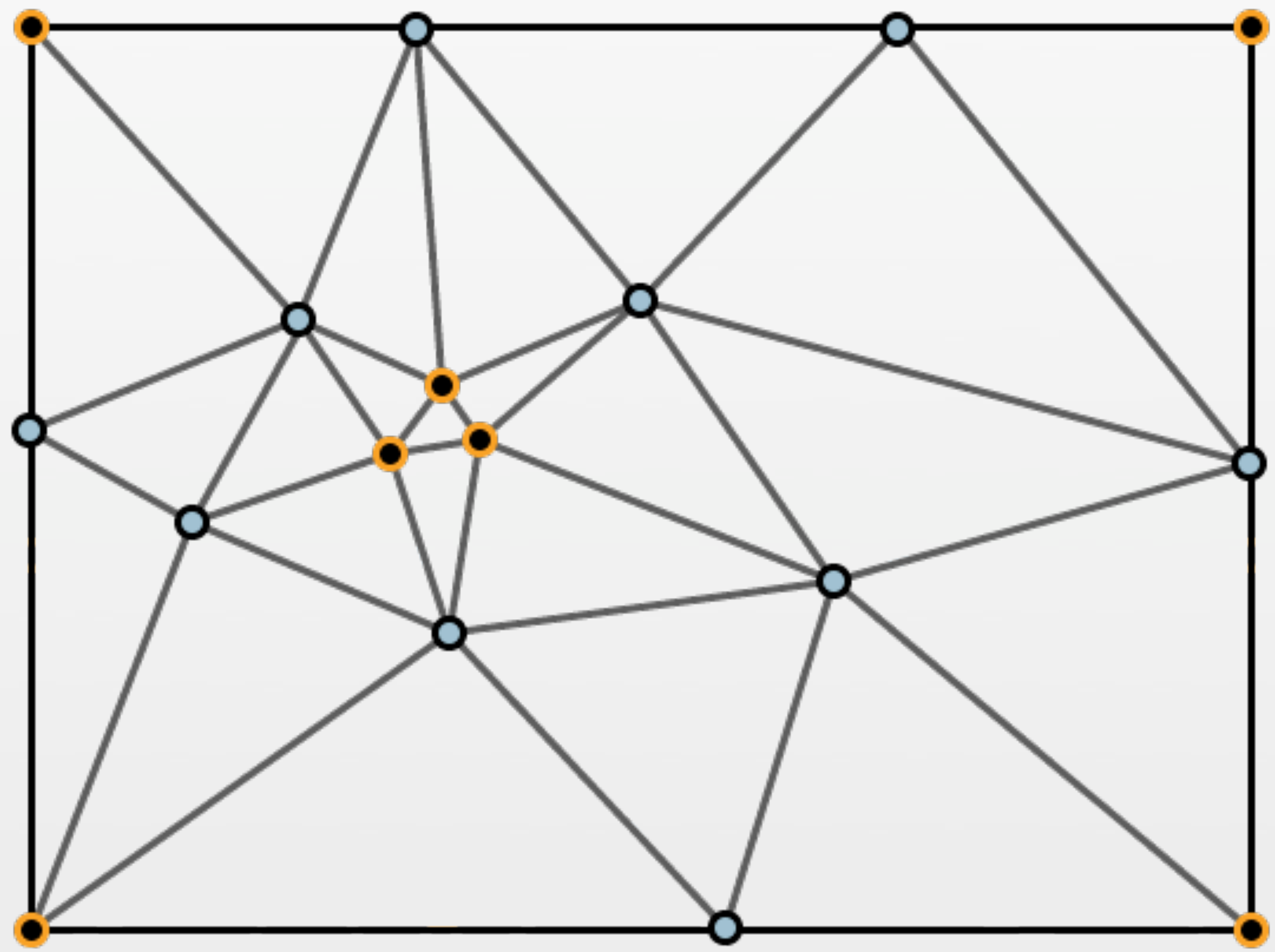
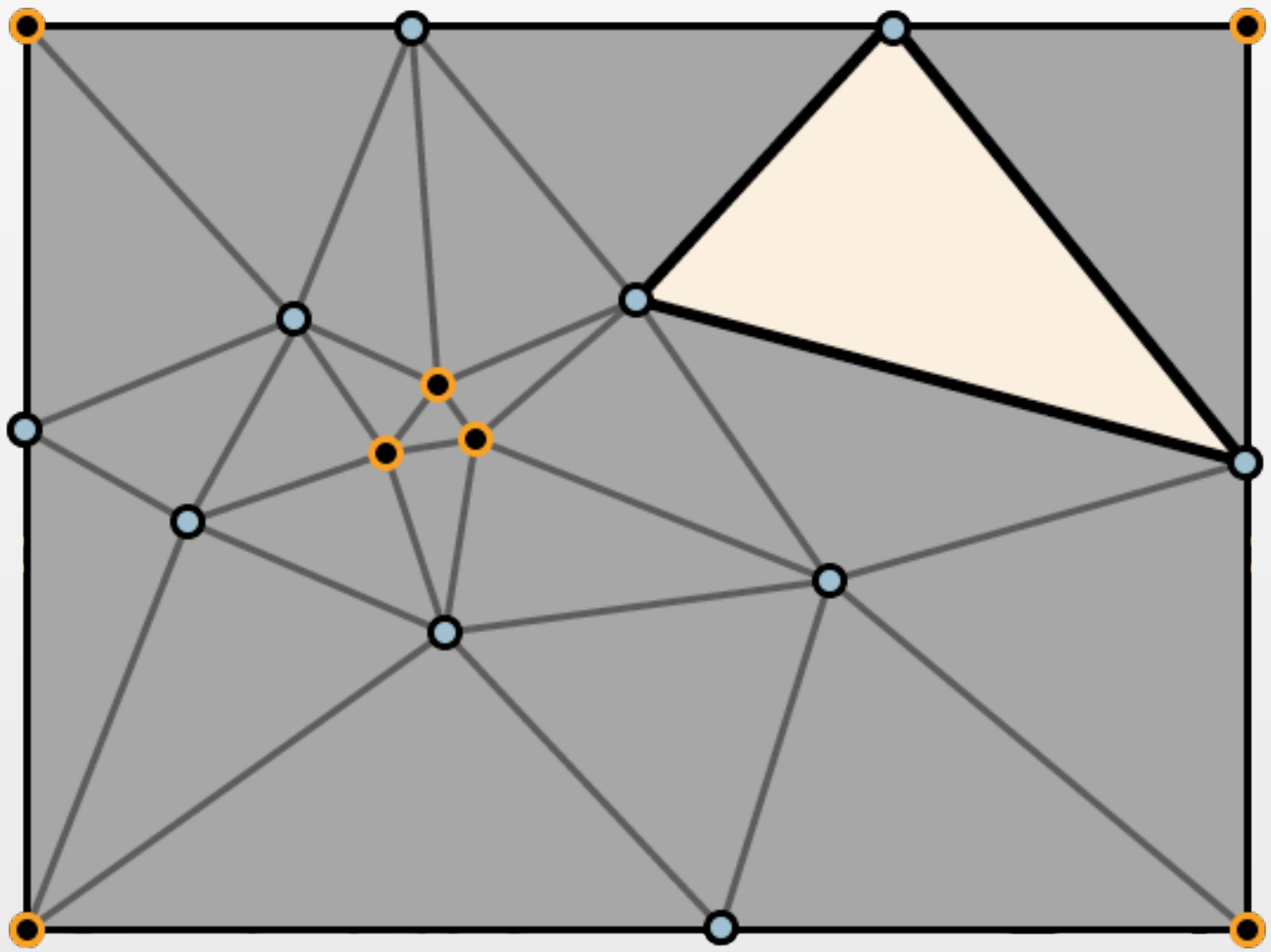


Meshes Represent Functions

Store $f(v)$ for all mesh vertices v.

Approximate $f(x)$ by interpolating from vertices in simplex containing x.



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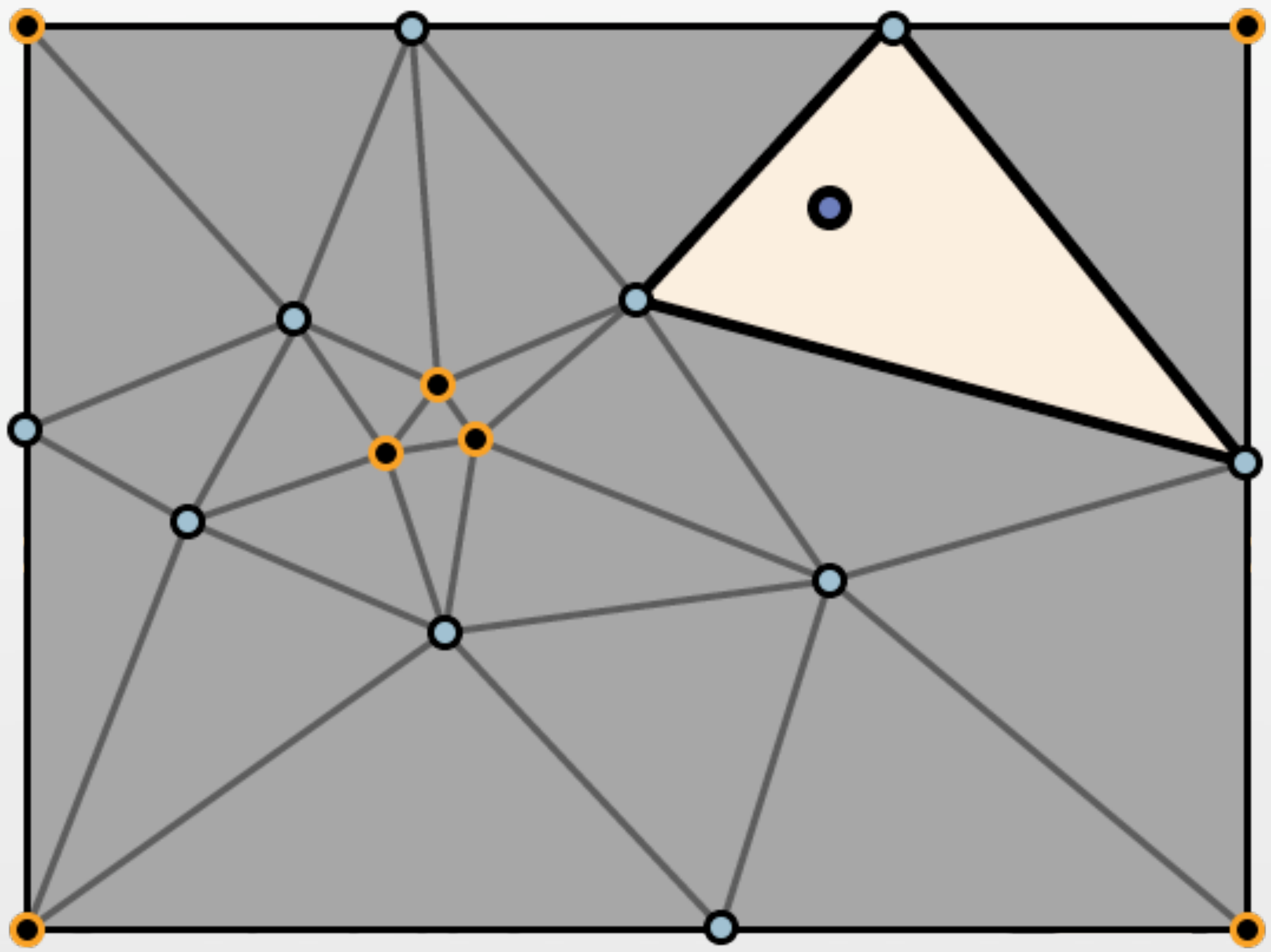

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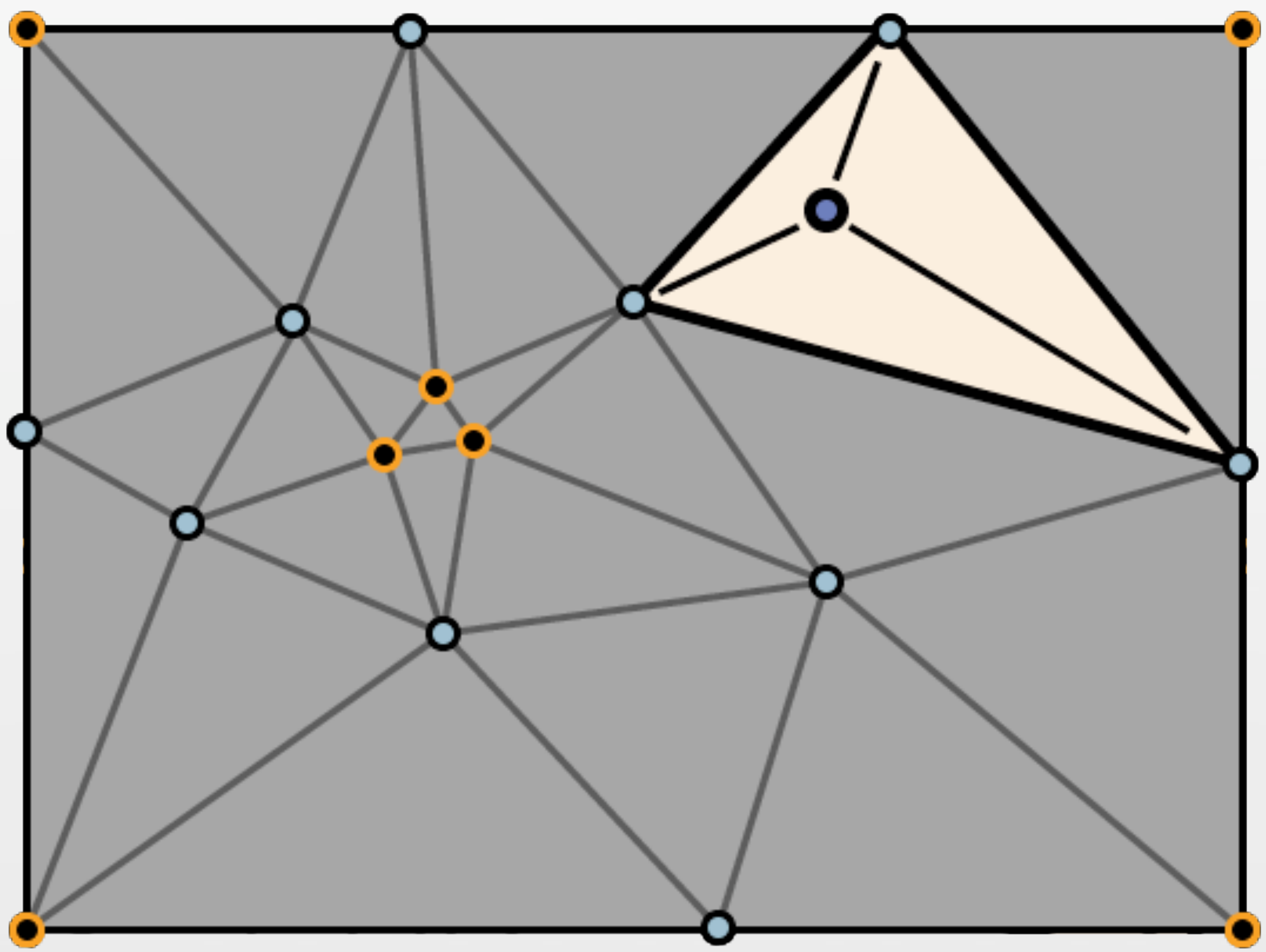
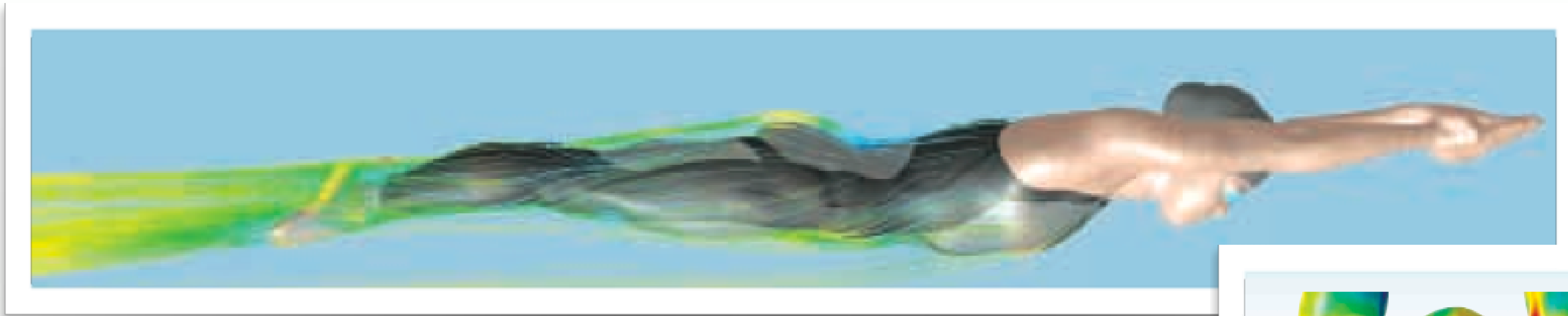
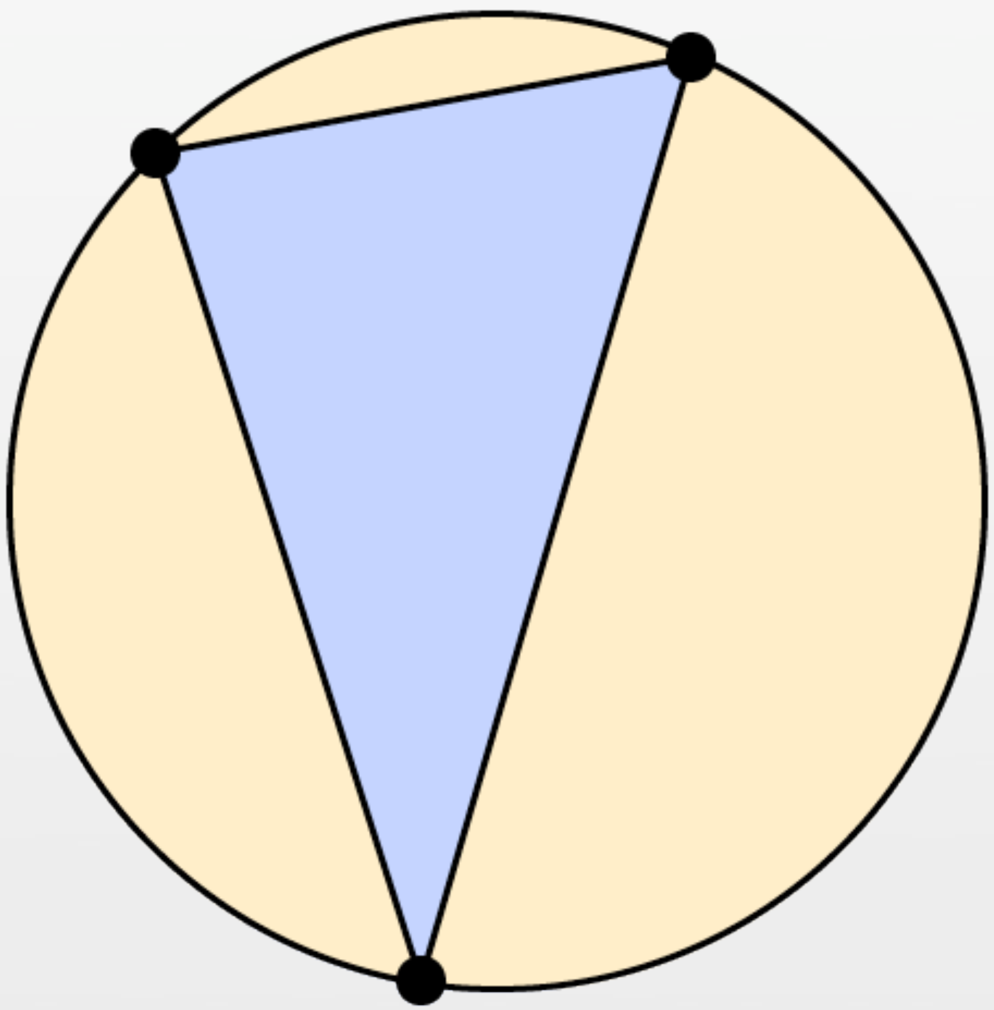




Meshes for fluid flow simulation.

Quality

A Quality triangle is one with radius-edge ratio bounded by a constant.



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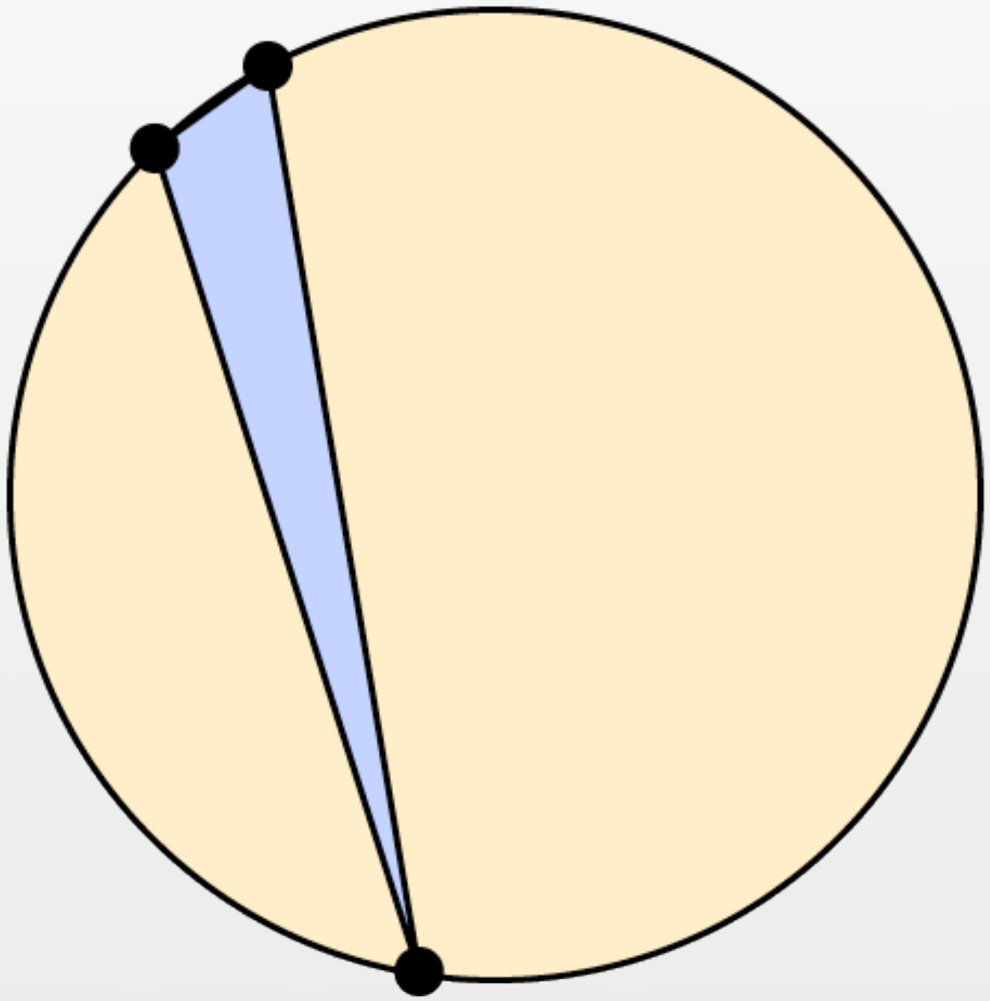

Output Size

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- Quality meshes with m vertices have O(m) simplices.

The number of points we add matters.

Size Optimality

m: The size of our mesh.

m_{OPT}: The size of the smallest possible mesh achieving similar quality guarantees.

Size Optimal: $m = O(m_{OPT})$

The Ruppert Bound

Theorem 1 *The number of number of vertices in any optimal-size quality mesh of a domain* $\Omega \subseteq \mathbb{R}^d$ *is* $\Theta(\int_{x\in\Omega} \frac{1}{\mathrm{lfs}(x)^d} dx)$.

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Note: This bound is tight.

[BernEppsteinGilbert94, Ruppert95, Ungor04, HudsonMillerPhillips06]

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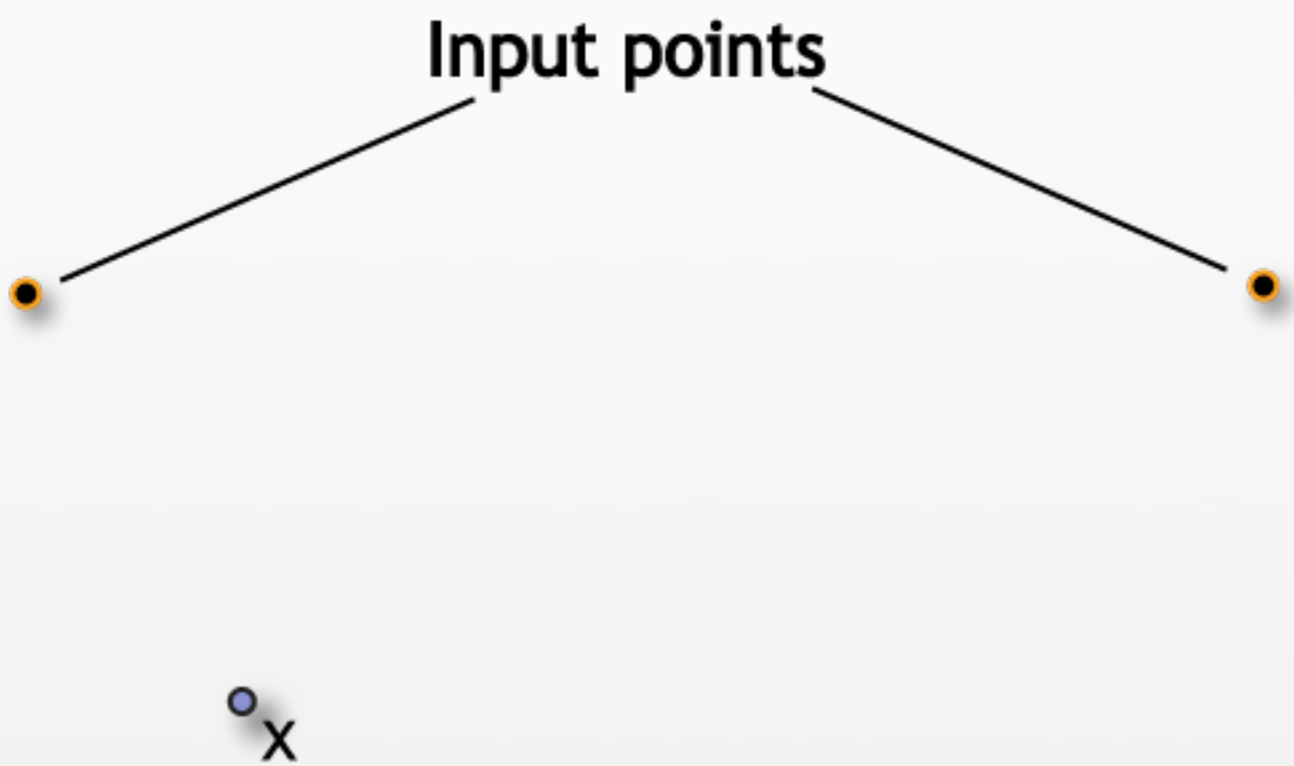
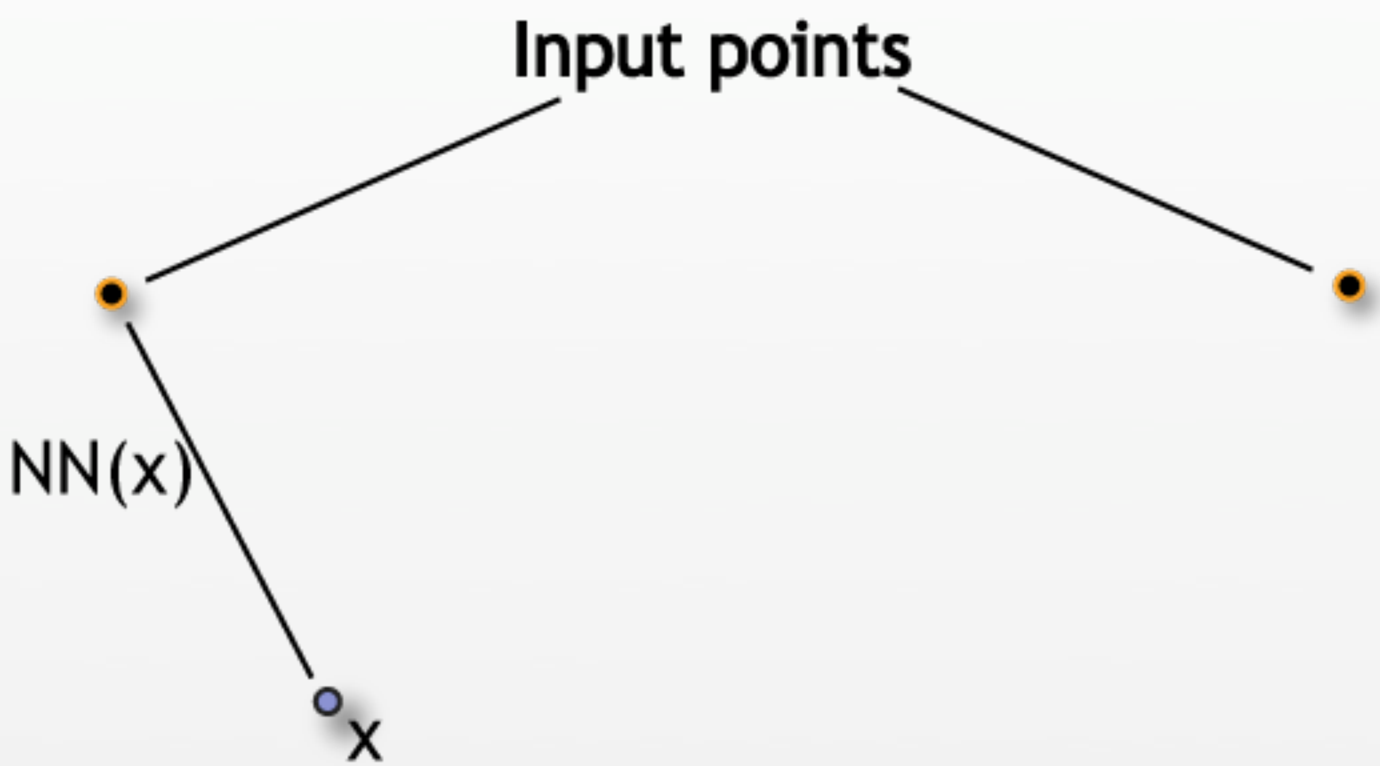
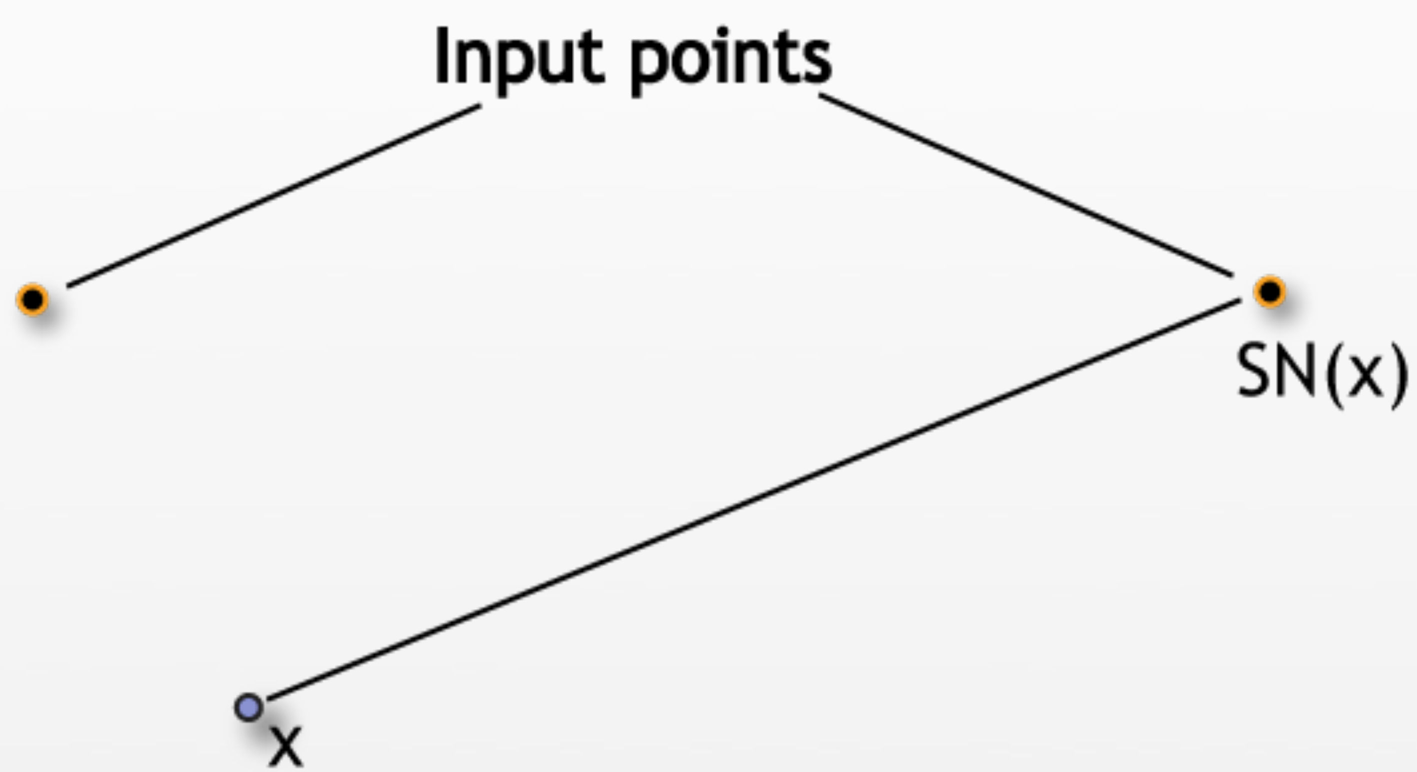
Input points

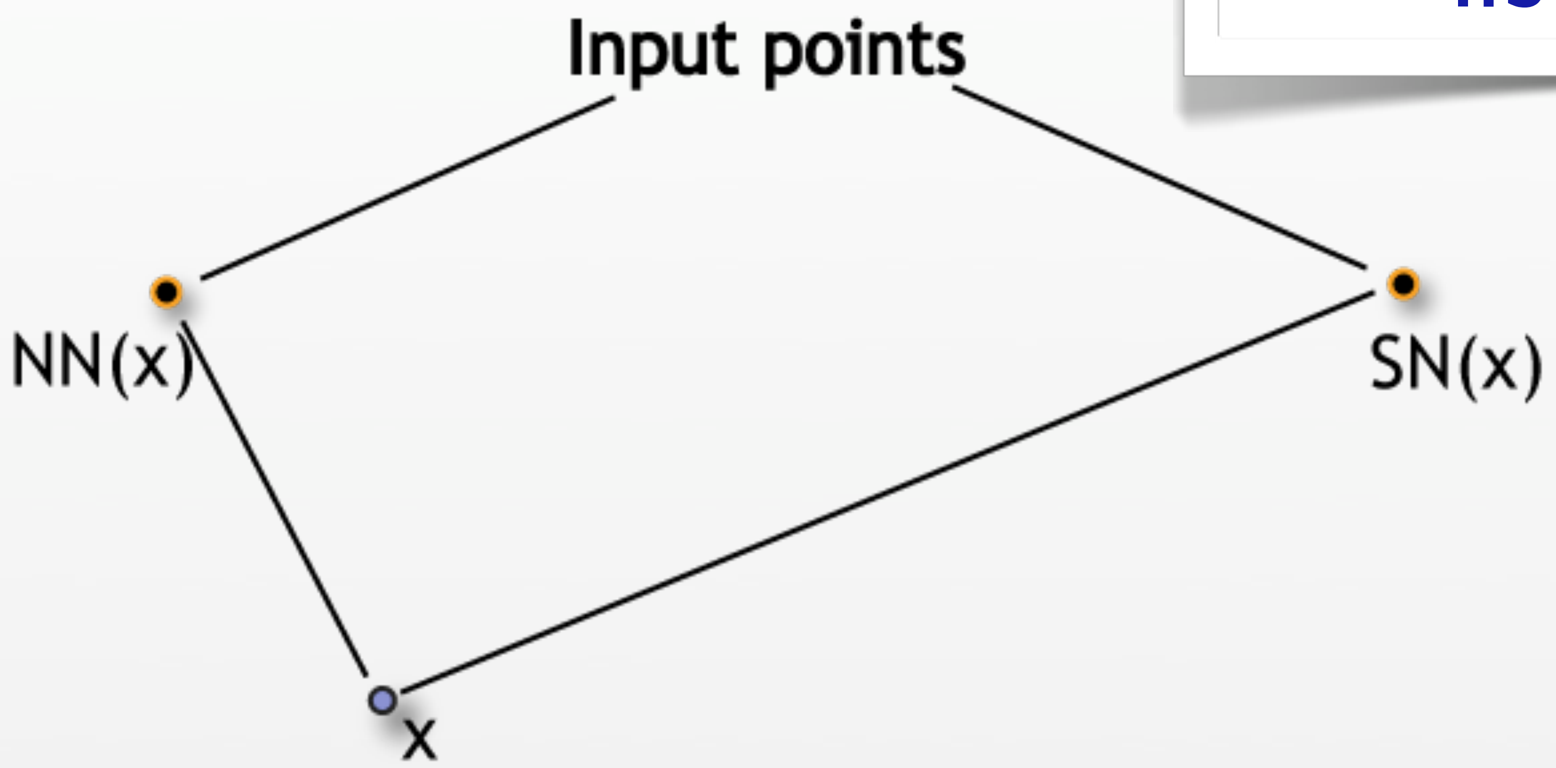

Input points
x

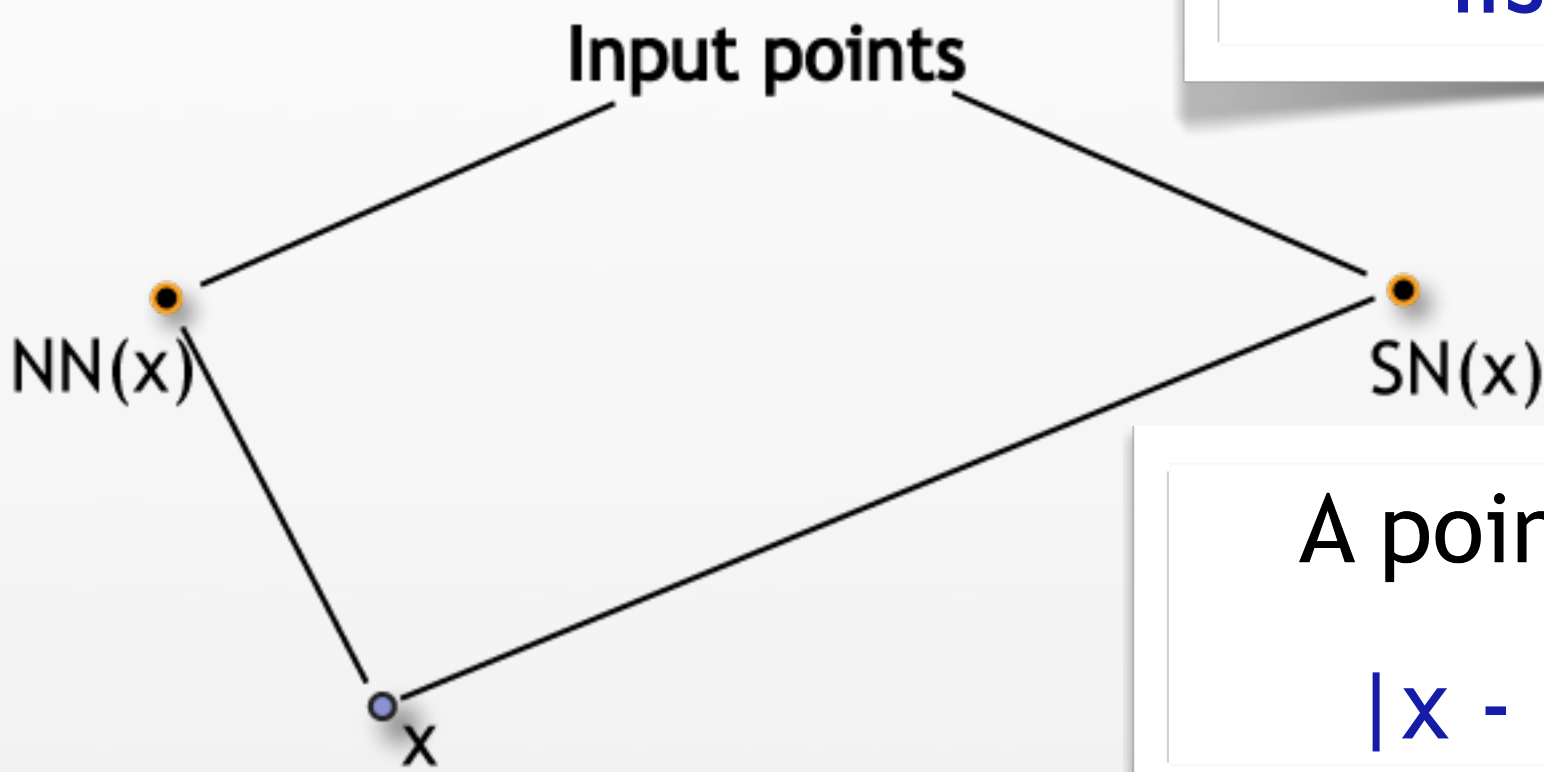

Input points
NN(x)
x


Input points
SN(x)
x

The local feature size at x is

$lfs(x) = |x - SN(x)|$.





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A point x is θ-medial w.r.t S if

$|x - NN(x)| \geq \theta\, |x - SN(x)|$.

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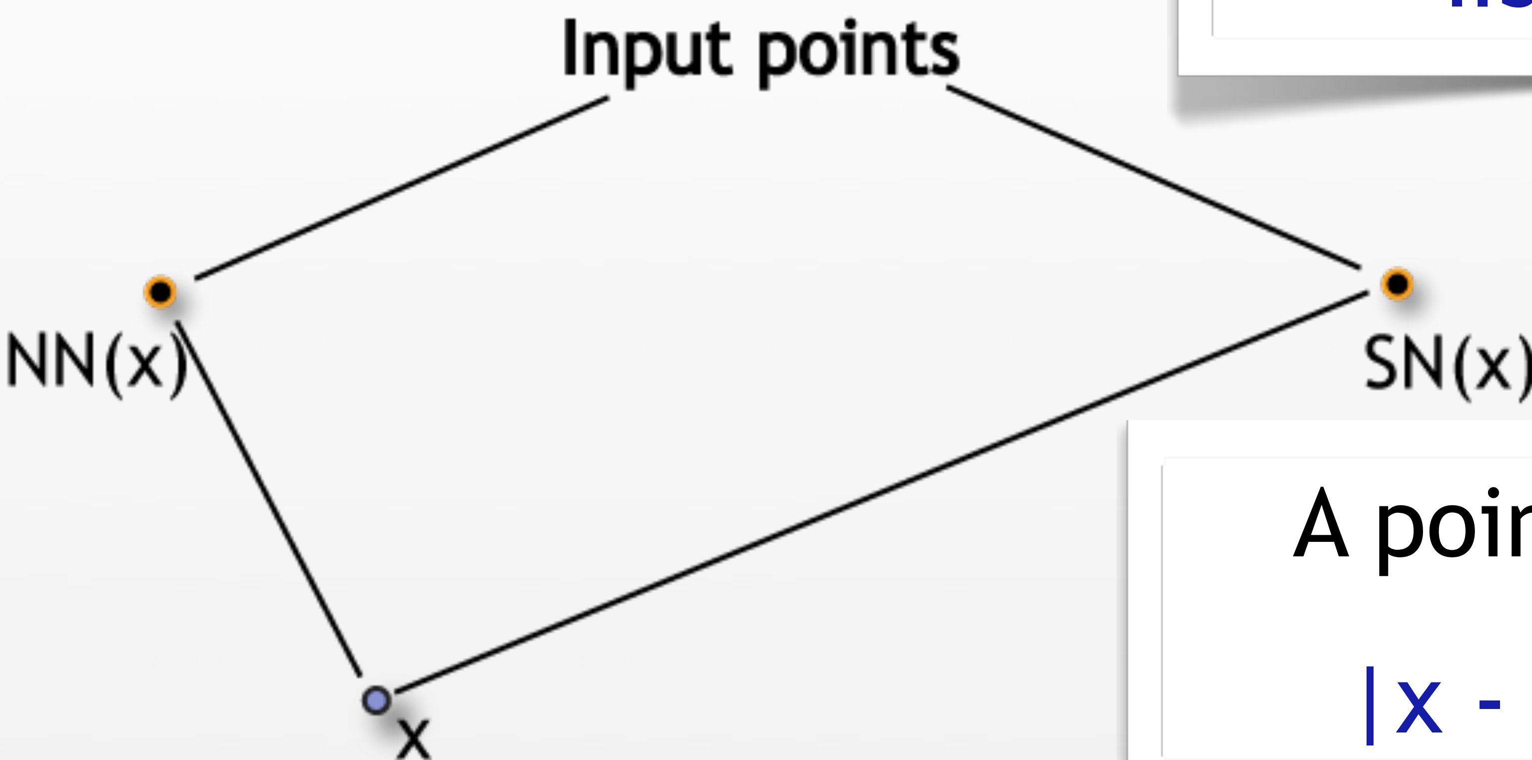

A point x is θ-medial w.r.t S if

$|x - NN(x)| \geq \theta \, |x - SN(x)|$.

An ordered set of points P is a well-paced extension of a point set Q if p_i is θ-medial w.r.t. $Q \cup \{p_1, \ldots, p_{i-1}\}$.

Well Paced Points

A point x is θ-medial w.r.t S if

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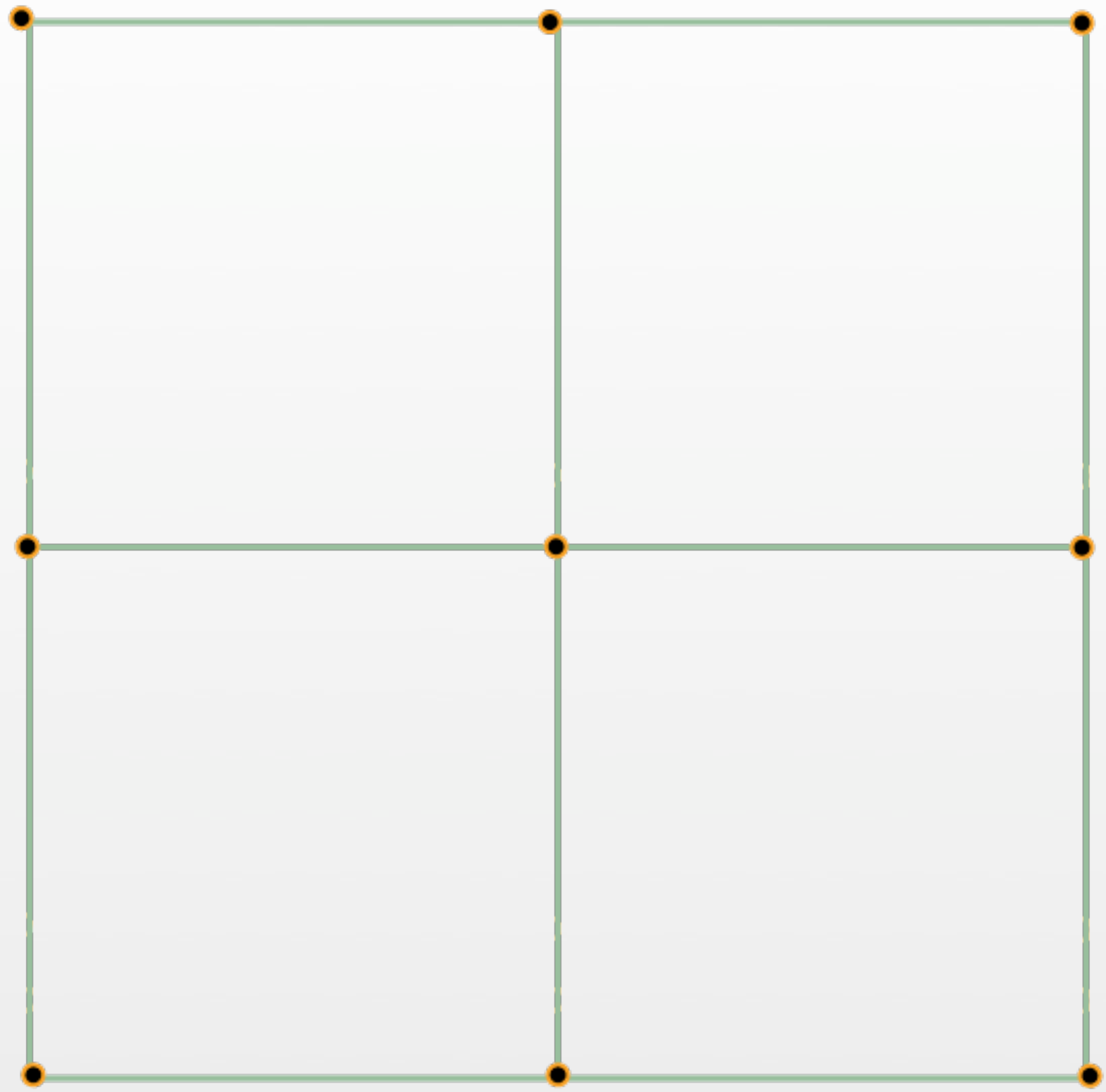
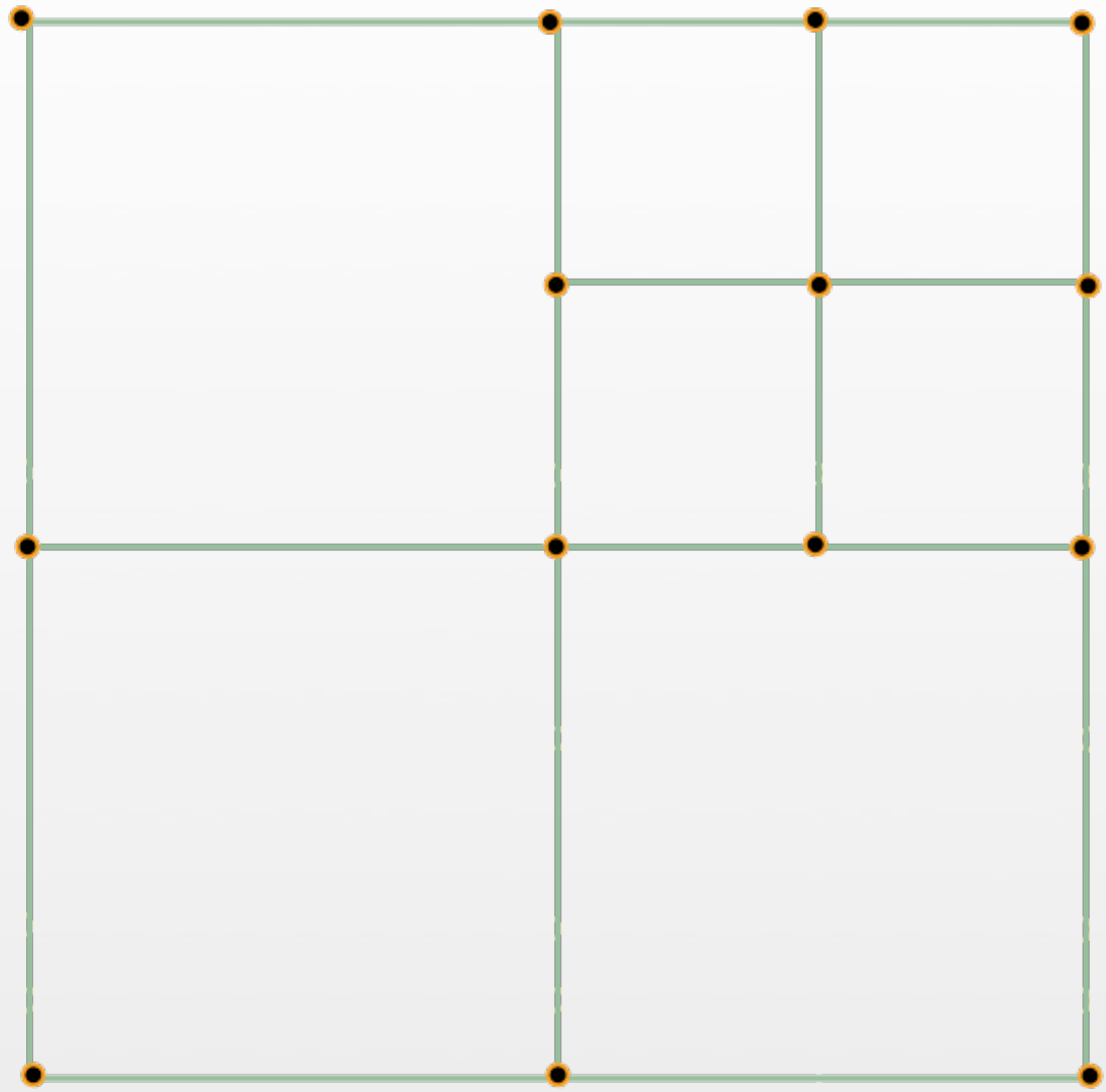
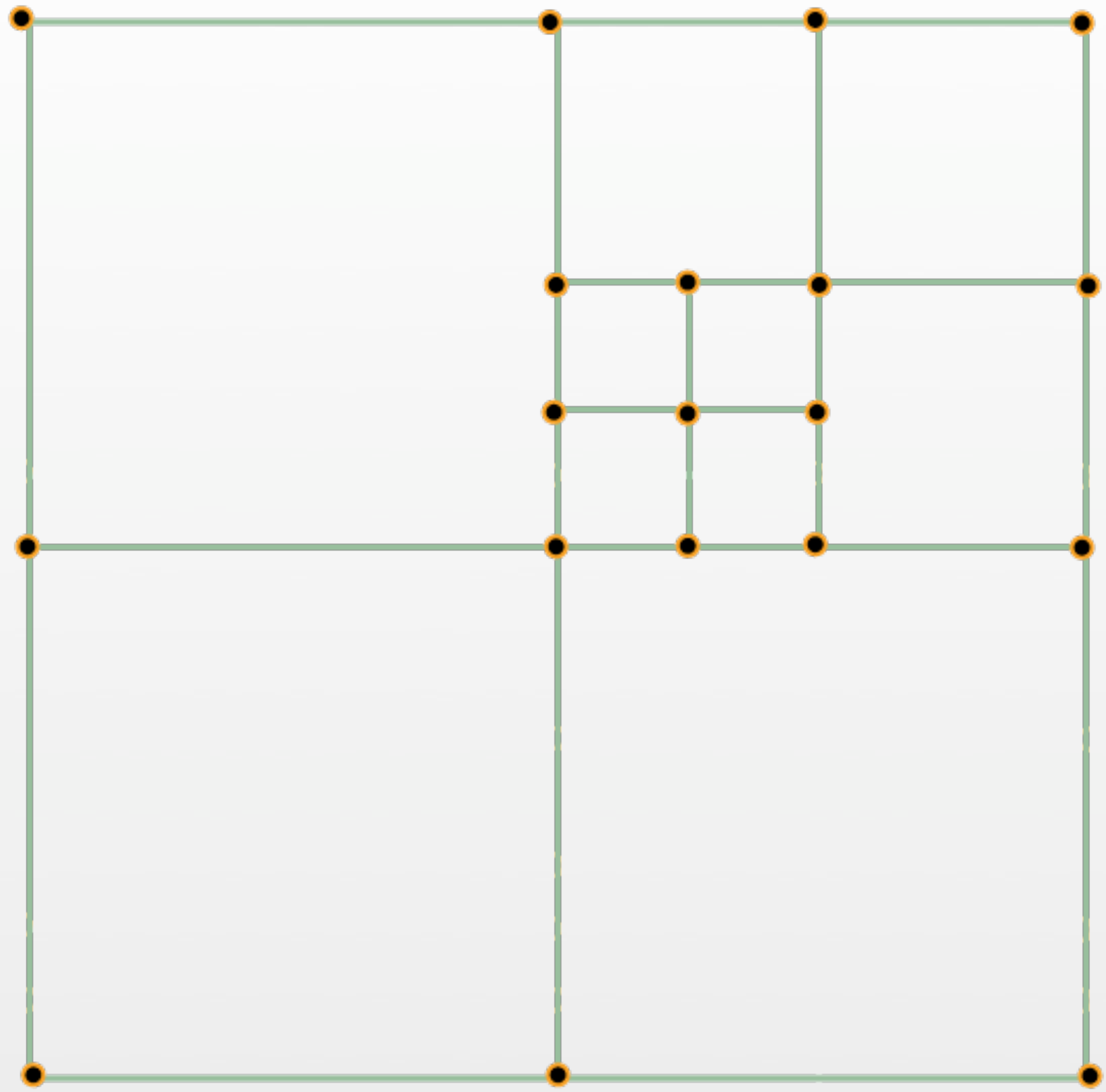
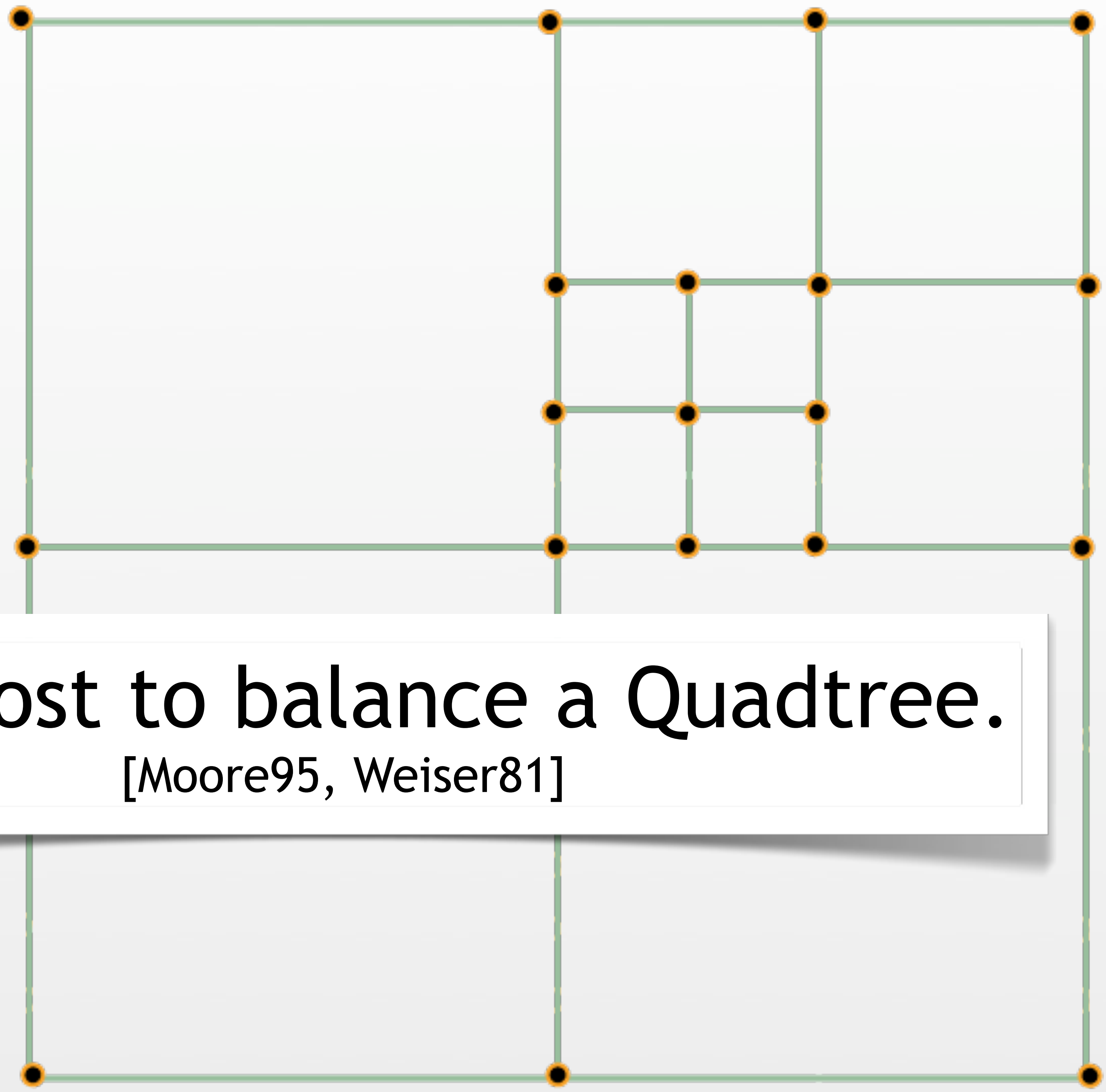








Linear Cost to balance a Quadtree.

[Moore95, Weiser81]

The Main Result

Theorem 2 *If P is a θ-well-paced extension of Q, then* $Cost(Q \cup P) = O(Cost(Q) + |P|)$.

Proof Idea

We want to prove that the amortized change in m_{opt} as we add a θ-medial point is constant.

$$\int_{x\in\Omega} \frac{1}{\mathrm{lfs'}(x)^d} - \frac{1}{\mathrm{lfs}(x)^d}\, dx$$

lfs' is the new local feature size after adding one point.

Proof Idea

$$\int_{x\in\Omega} \frac{1}{\mathrm{lfs'}(x)^d} - \frac{1}{\mathrm{lfs}(x)^d}\, dx$$

Key ideas to bound this integral.

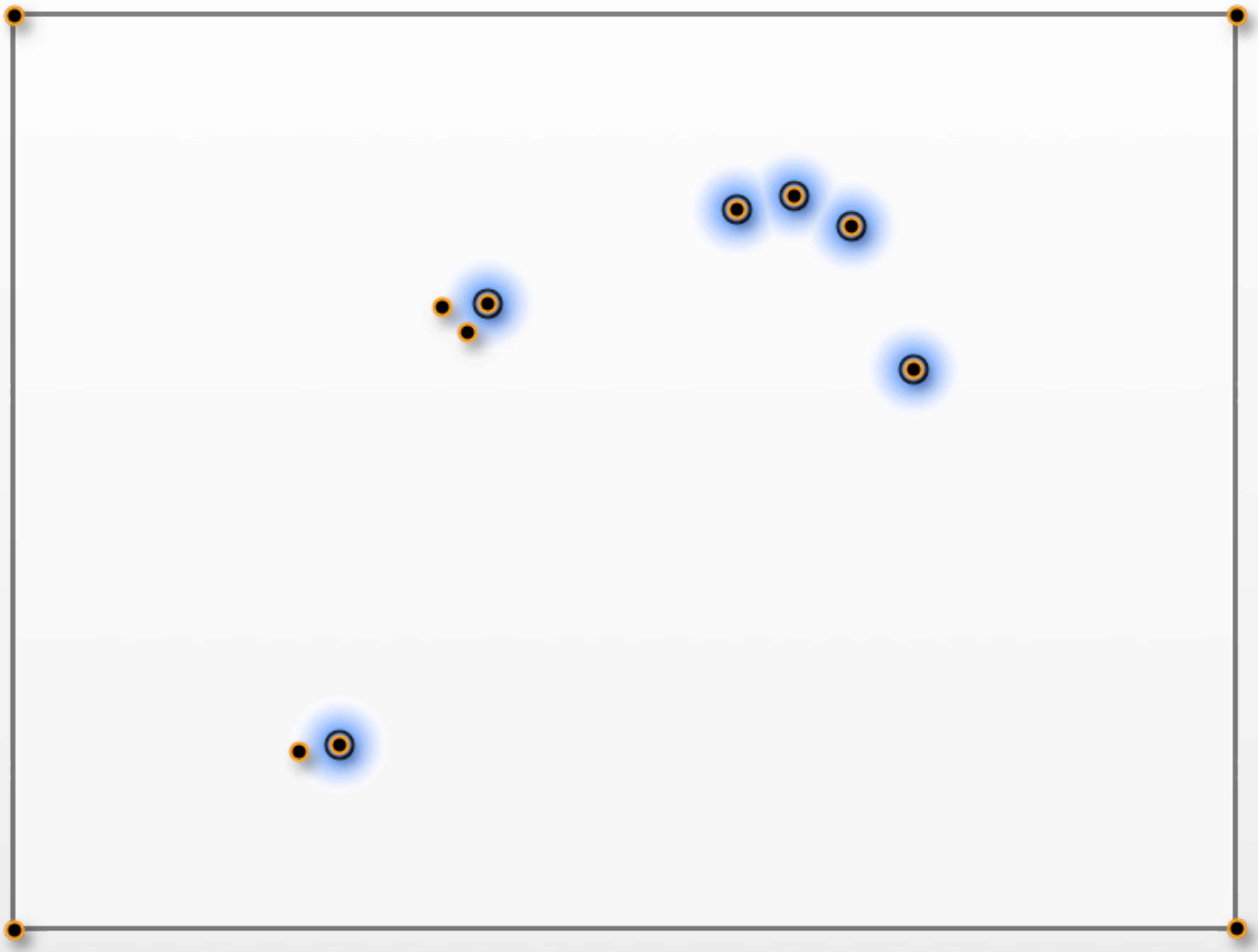
- Integrate over the entire space using polar coordinates.
- Split the integral into two parts: the region near the the new point and the region far from the new point.
- Use every trick you learned in high school.

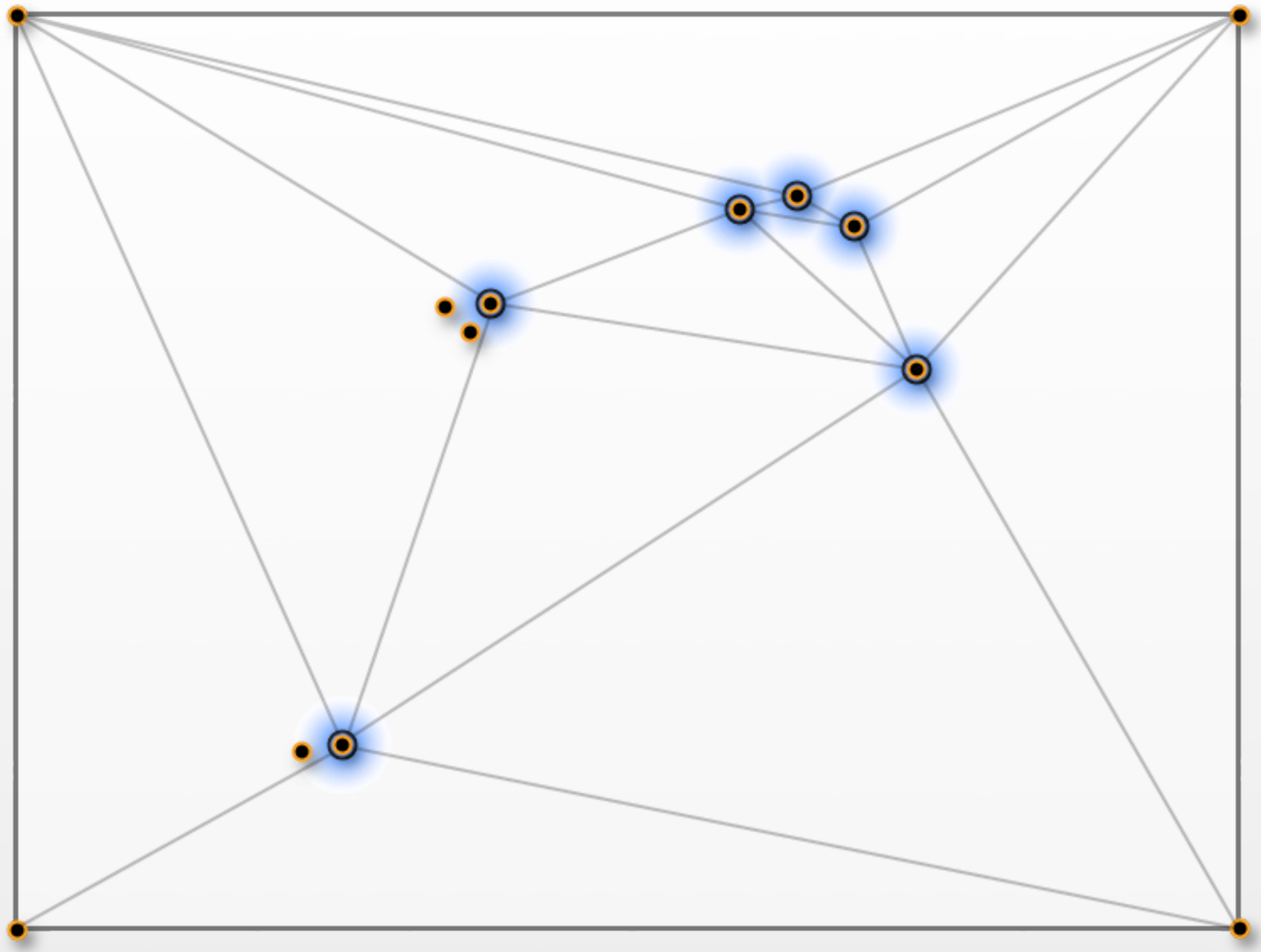
How do we get around the Ruppert lower bound?

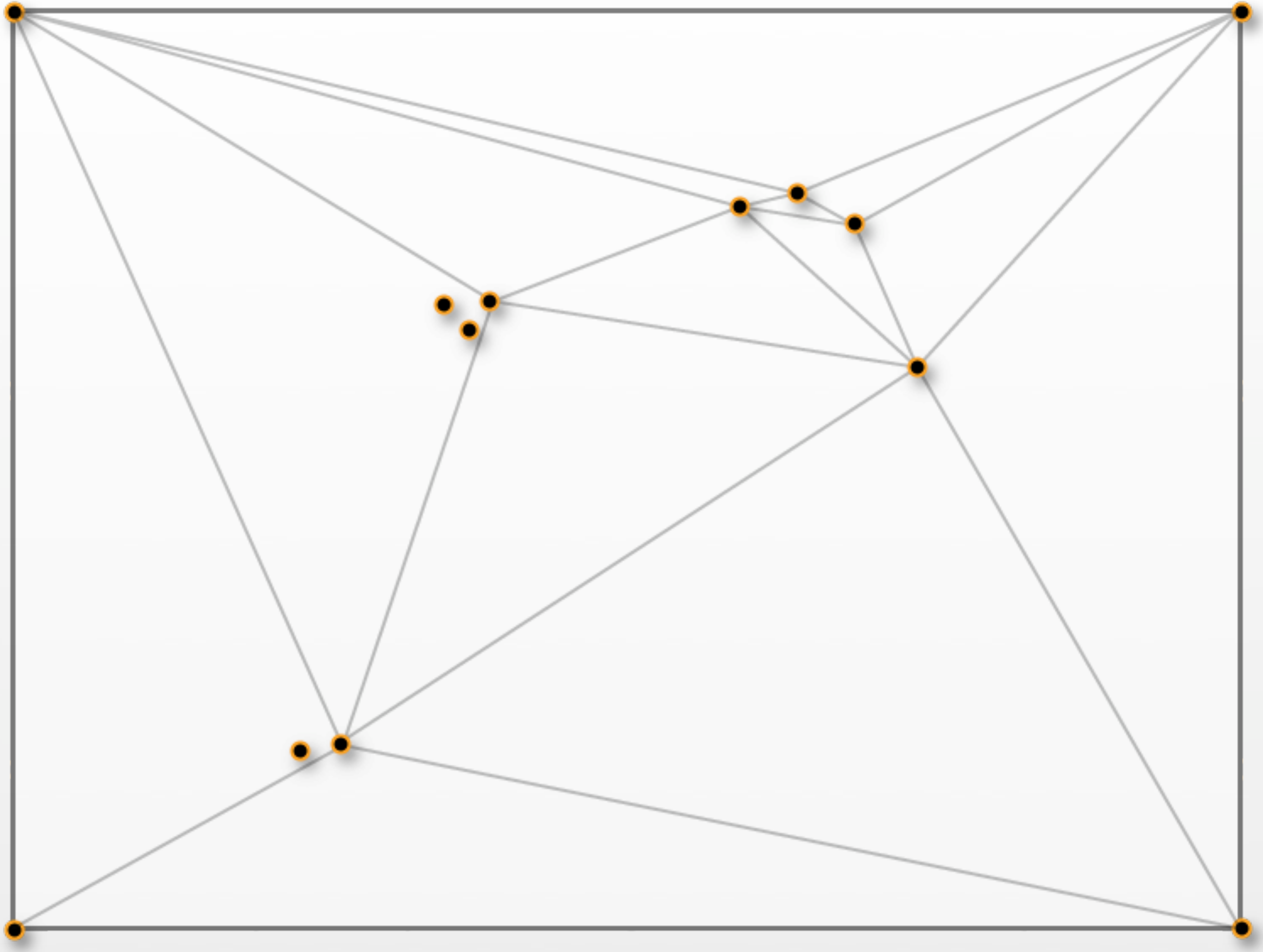
Linear Size Delaunay Meshes

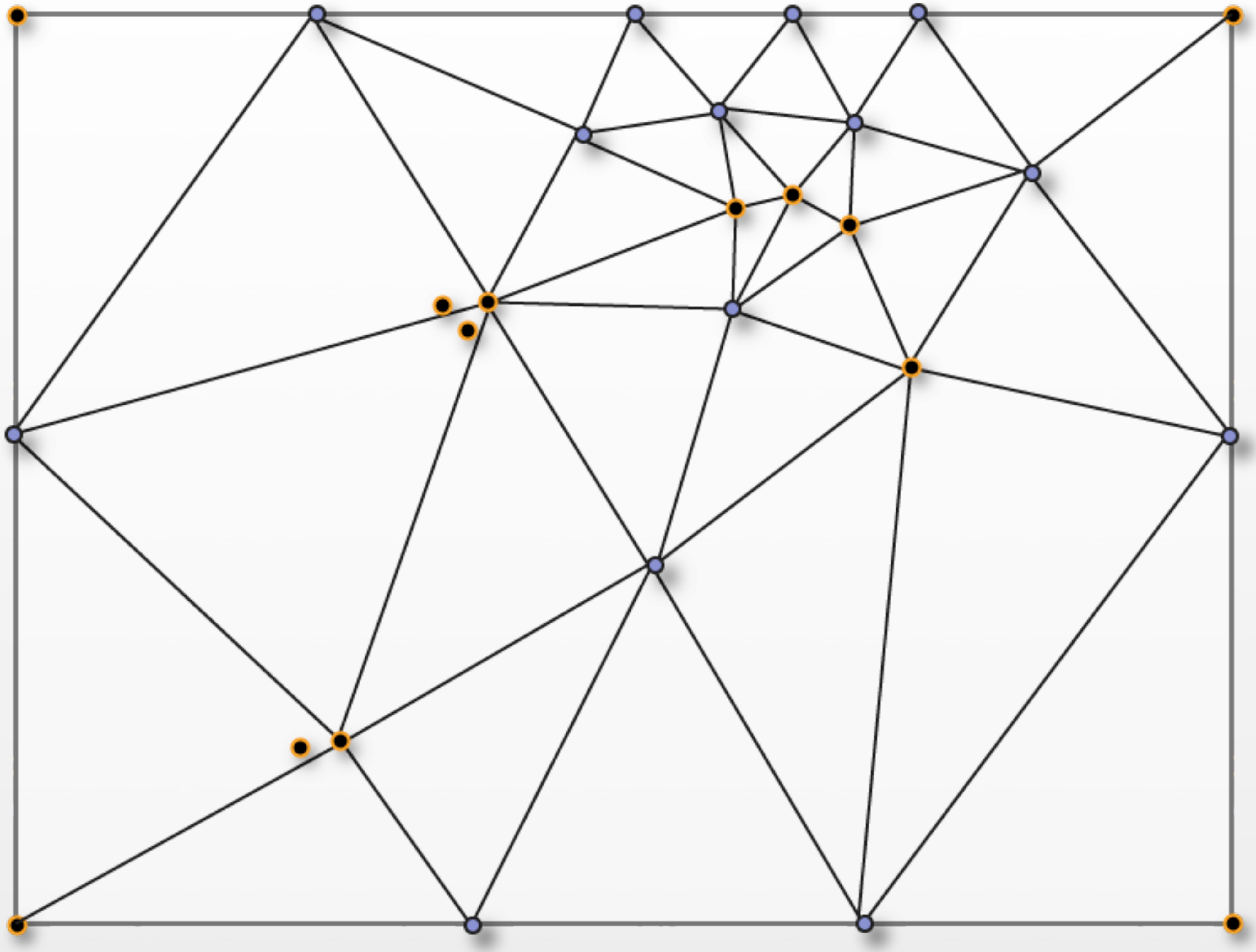
The Algorithm:

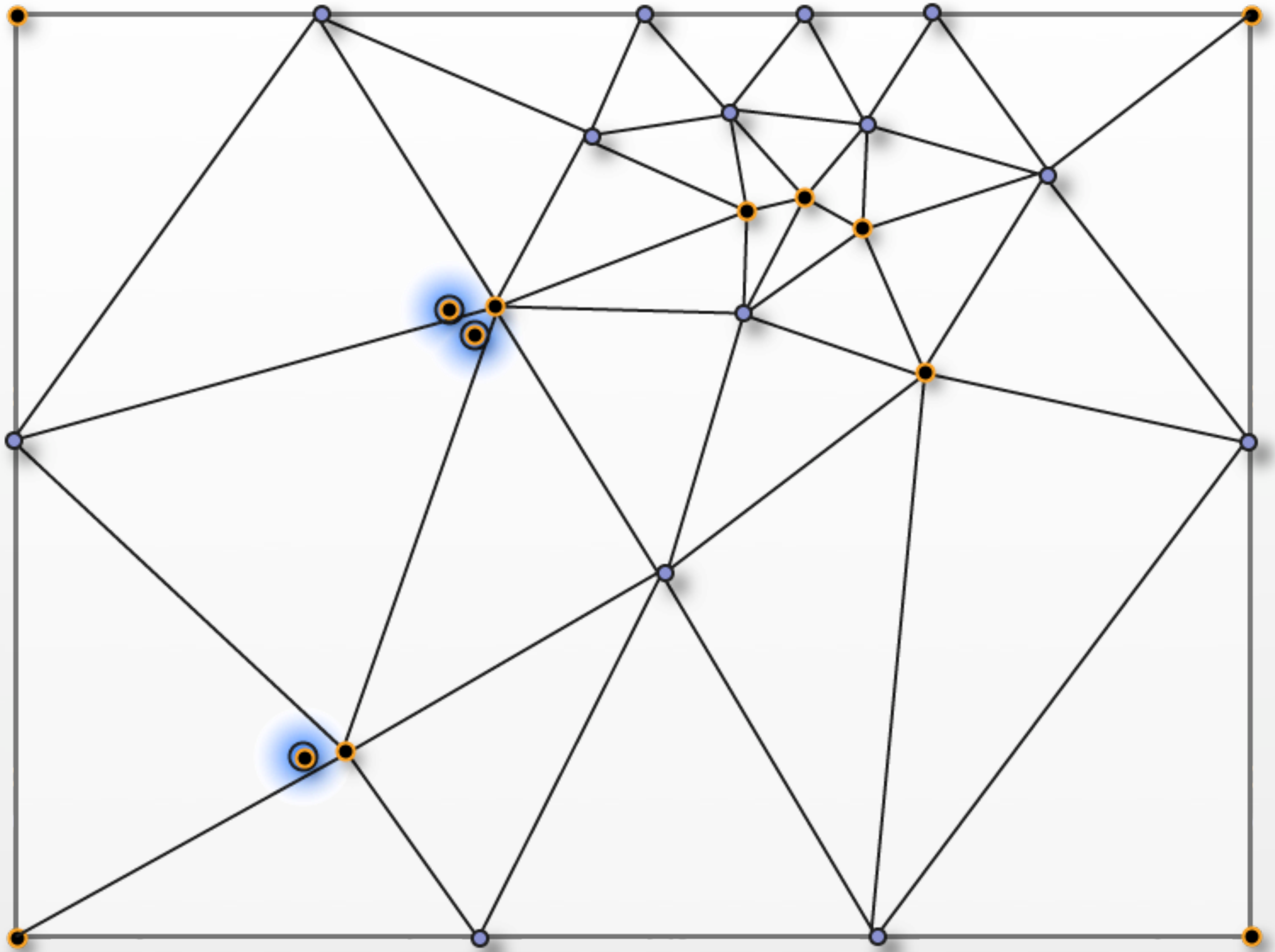
- Pick a maximal well-paced extension of the bounding box.
- Refine to a quality mesh.
- Remaining points form small clusters. Surround them with smaller bounding boxes.
- Recursively mesh the smaller bounding boxes.
- Return the Delaunay triangulation of the entire point set.

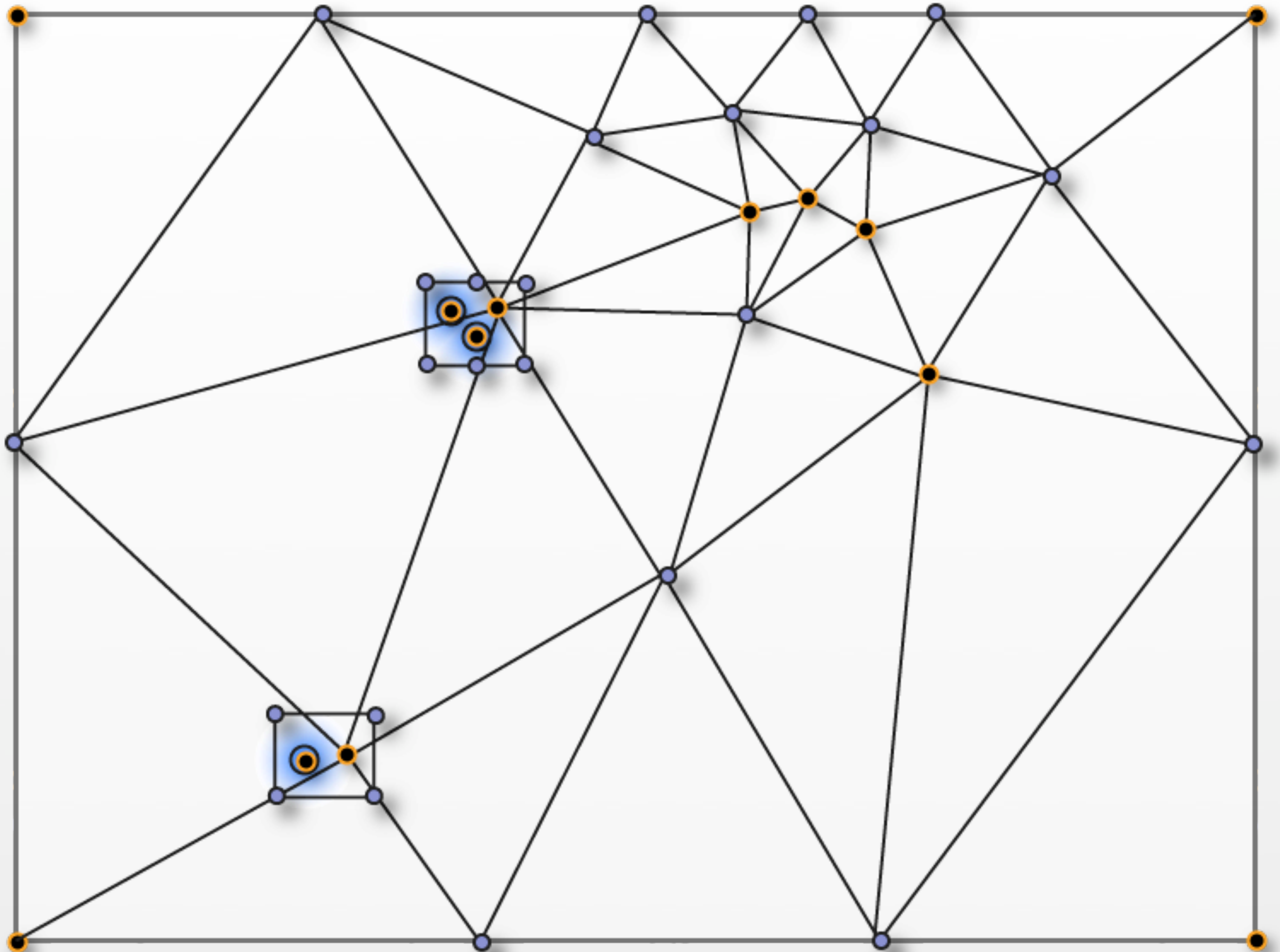


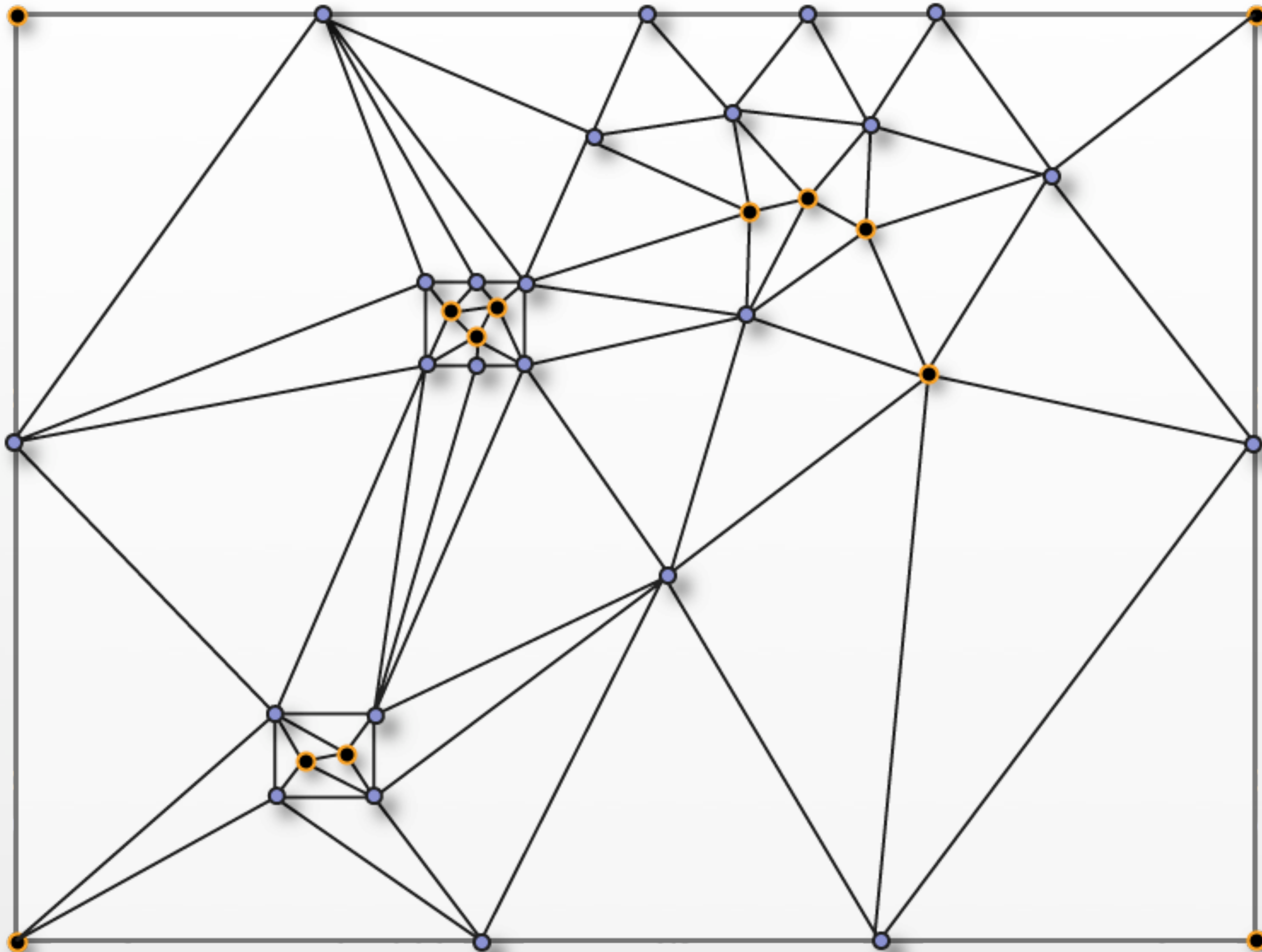












Guarantees

- Output size is O(n).
- Bound on radius/longest edge. (No-large angles in R^2)

Summary

- An algorithm for producing linear size delaunay meshes of point sets in $\mathbf{R}^d$.
- A new method for analyzing the size of quality meshes in terms of input size.

Sneak Preview

In a follow-up paper, we show how the theory of well-paced points leads to a proof of a classic folk conjecture.

- Suppose we are meshing a domain with an odd shaped boundary.
- We enclose the domain in a bounding box and mesh the whole thing.
- We throw away the "extra."
- The amount we throw away is at most a constant fraction.

Thank you.

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Questions?