Mesh Generation and Topological Data Analysis

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INRIA Saclay, France

Obvious.

Obvious.

“I could have thought of that.”

Obvious.

"I could have thought of that."

"I should have thought of that."

Outline

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1 **The Obvious.**

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   Mesh generation as a preprocess for TDA

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   The main theorems about approximating
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4 **Some things that might be true.**

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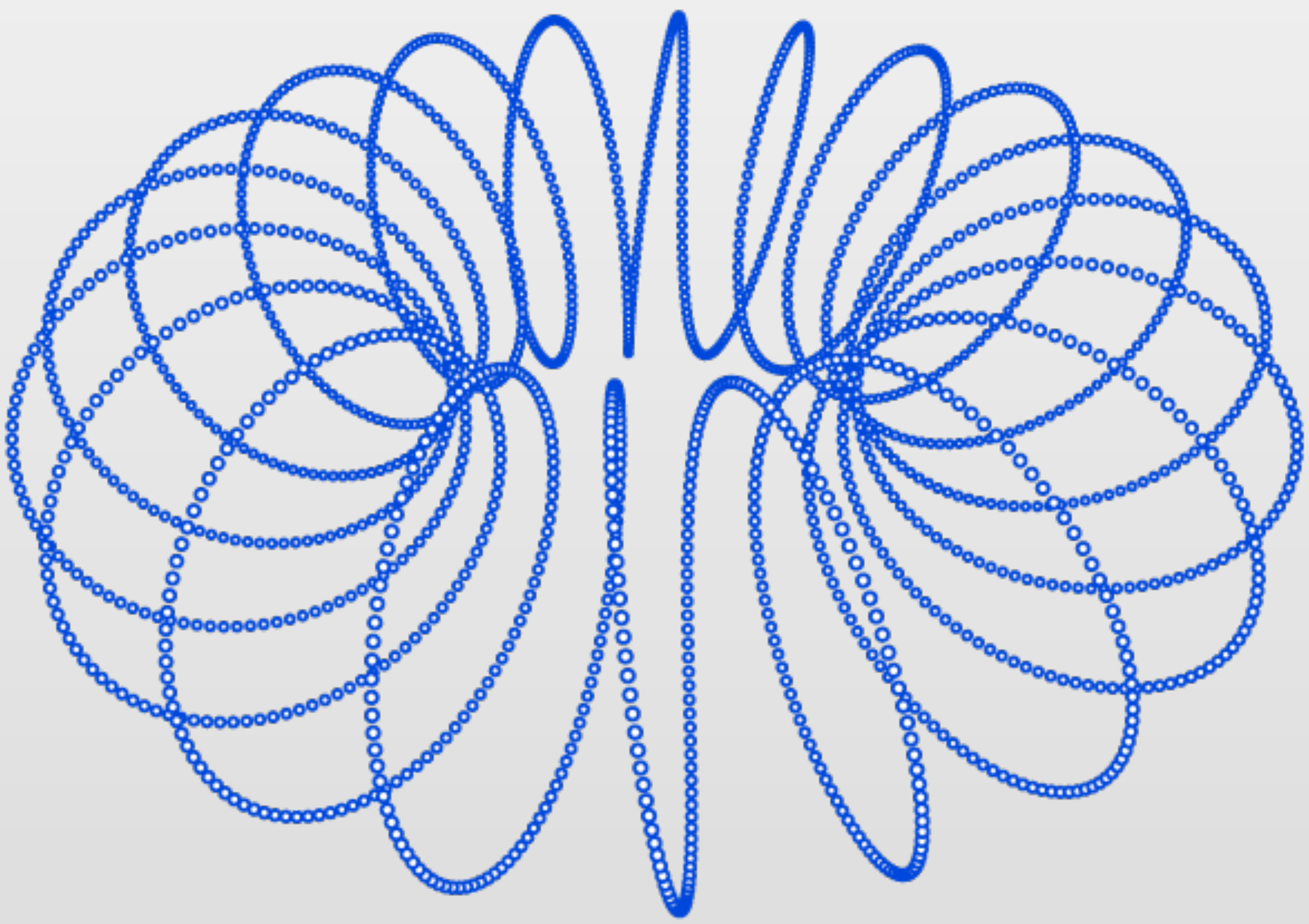
The main theorems about approximating persistence diagrams using mesh filtrations

4 **Some things that might be true.**

Wild speculation.

We consider point clouds in low-dimensional Euclidean space.



Maybe there is underlying structure, but maybe not.

A trip down memory lane...

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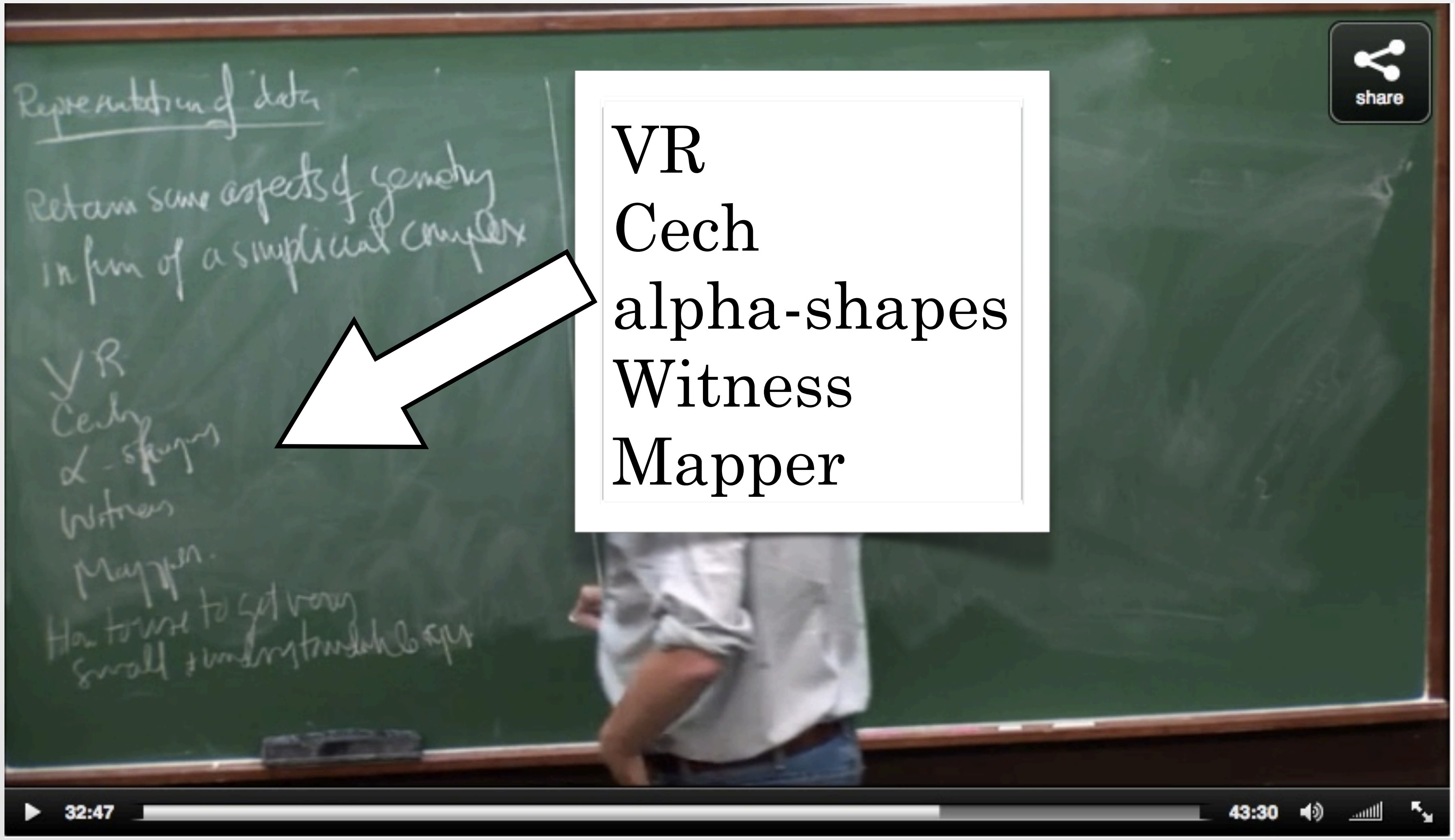

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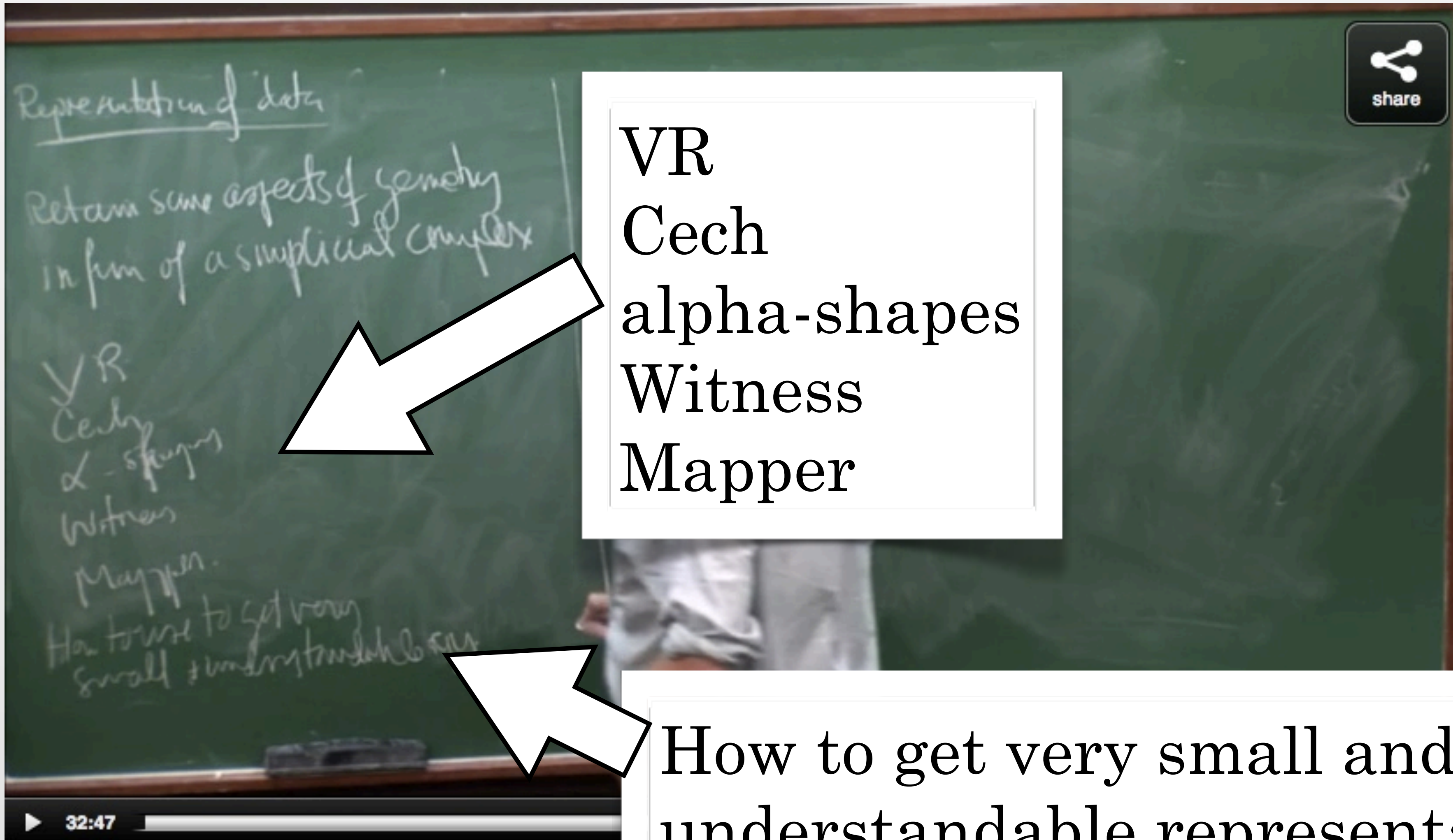

VR
Cech
alpha-shapes
Witness
Mapper

How to get very small and
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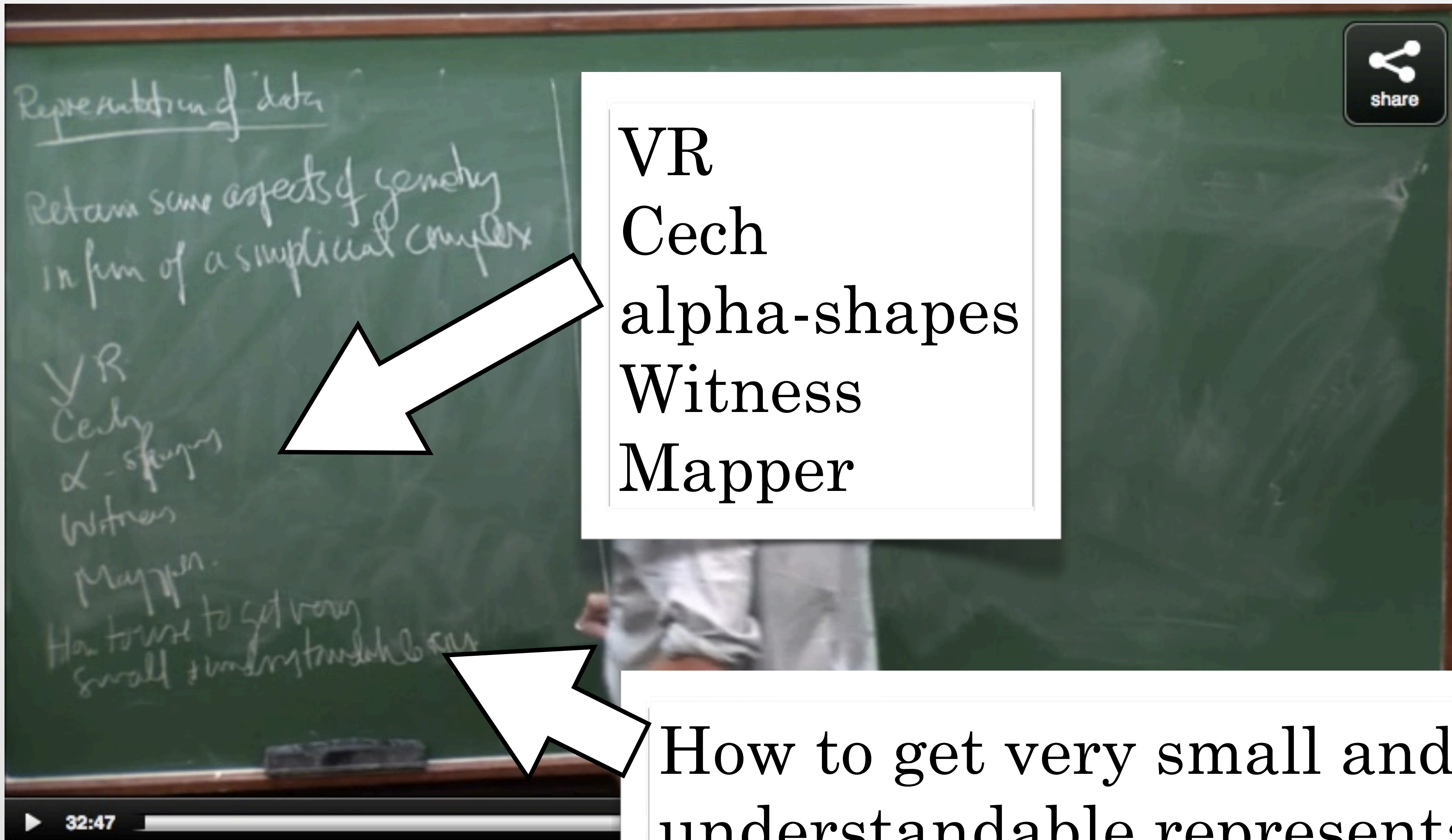

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How to get very small and understandable representations

Conventional Wisdom: *If you want a smaller complex, you need to use fewer vertices.*

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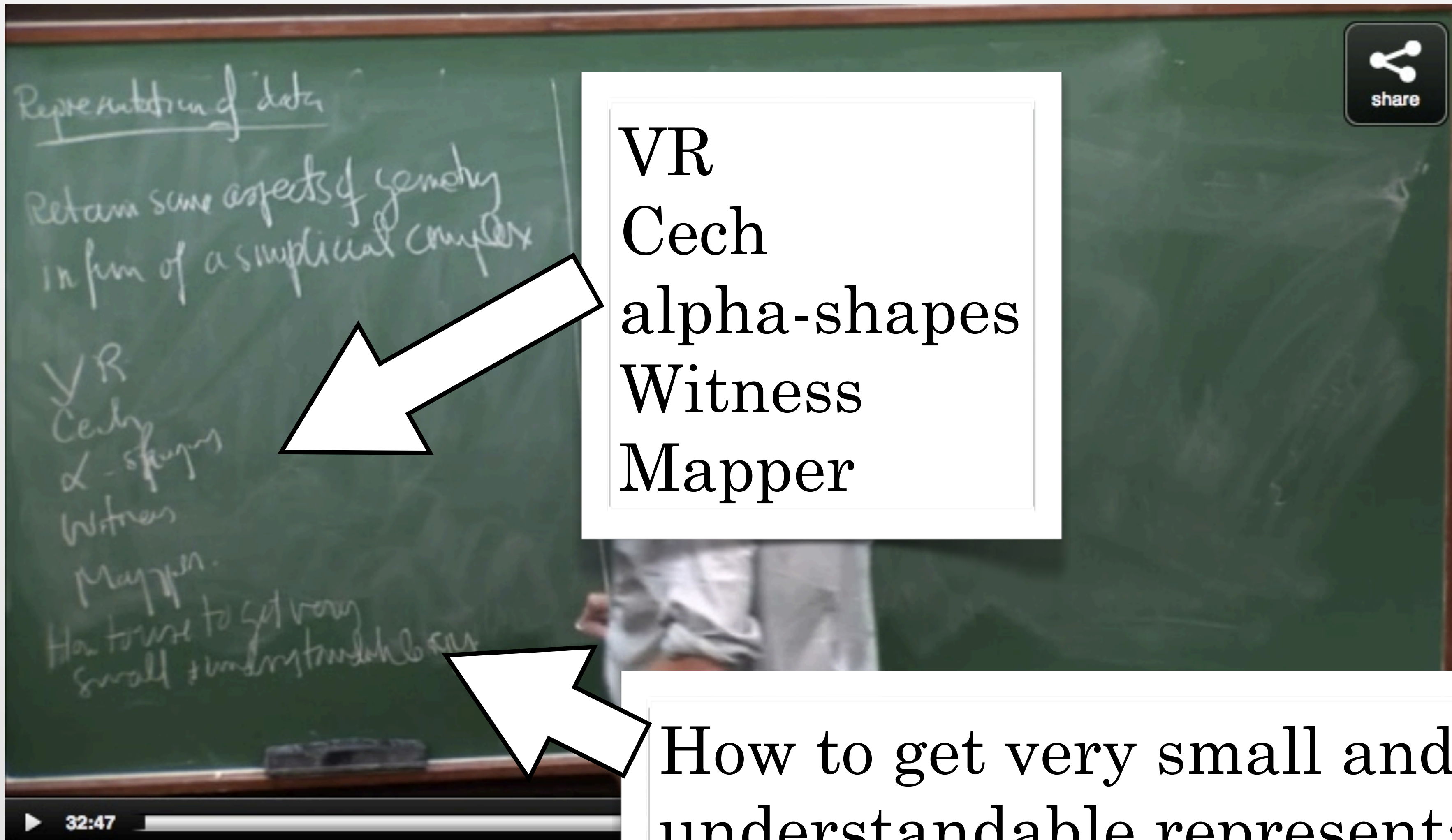

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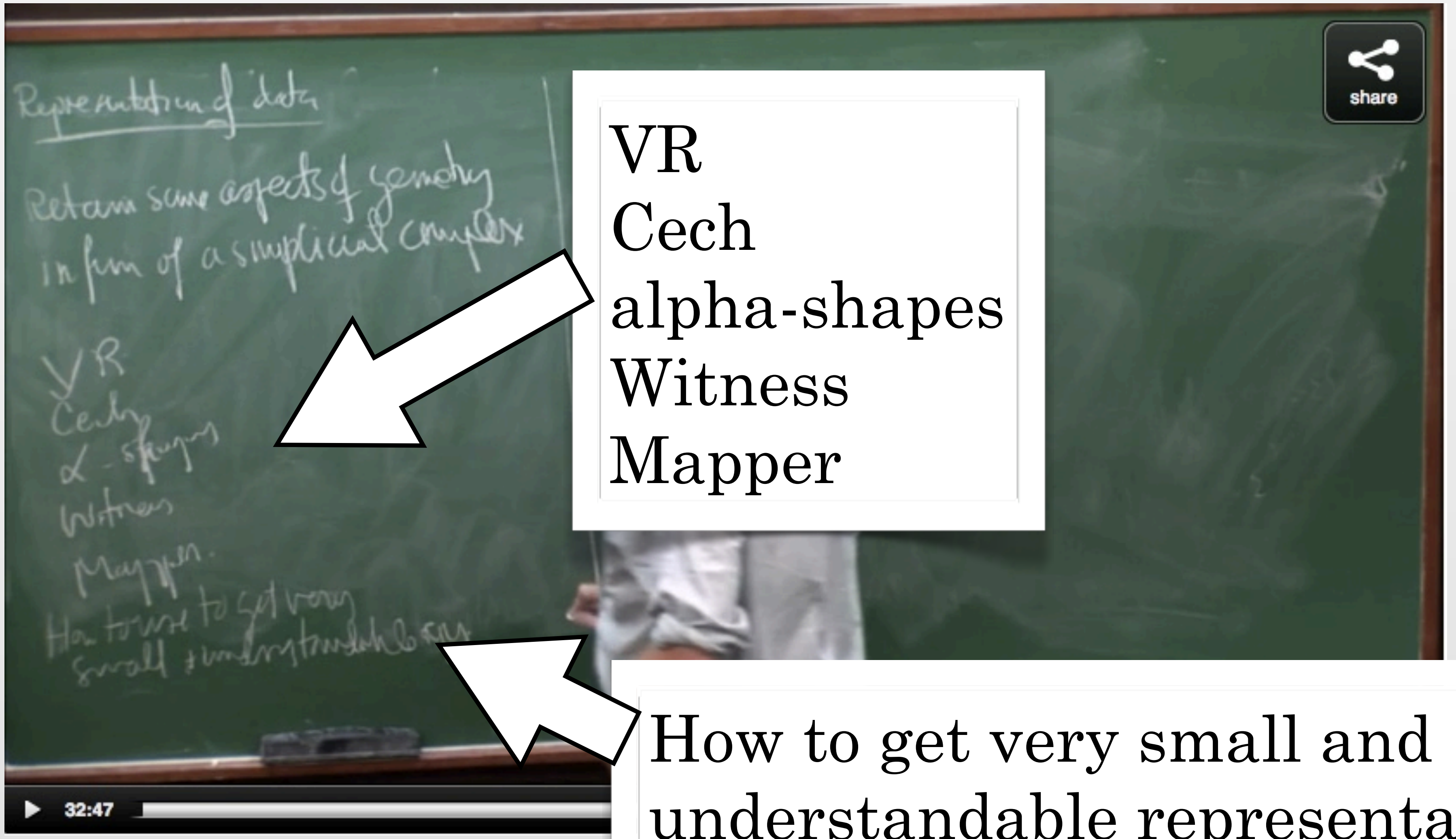

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How to get very small and understandable representations

Conventional Wisdom: *If you want a smaller complex, you need to use fewer vertices.*

Obvious? True?

The complexity of simplicial complexes is not dominated by vertex counts.

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Does noise help?

A TDA Pipeline

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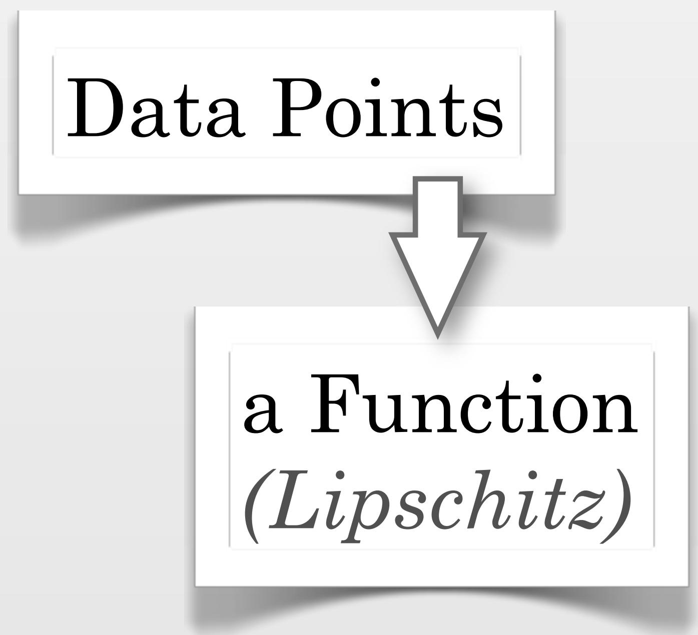
Data Points

A TDA Pipeline



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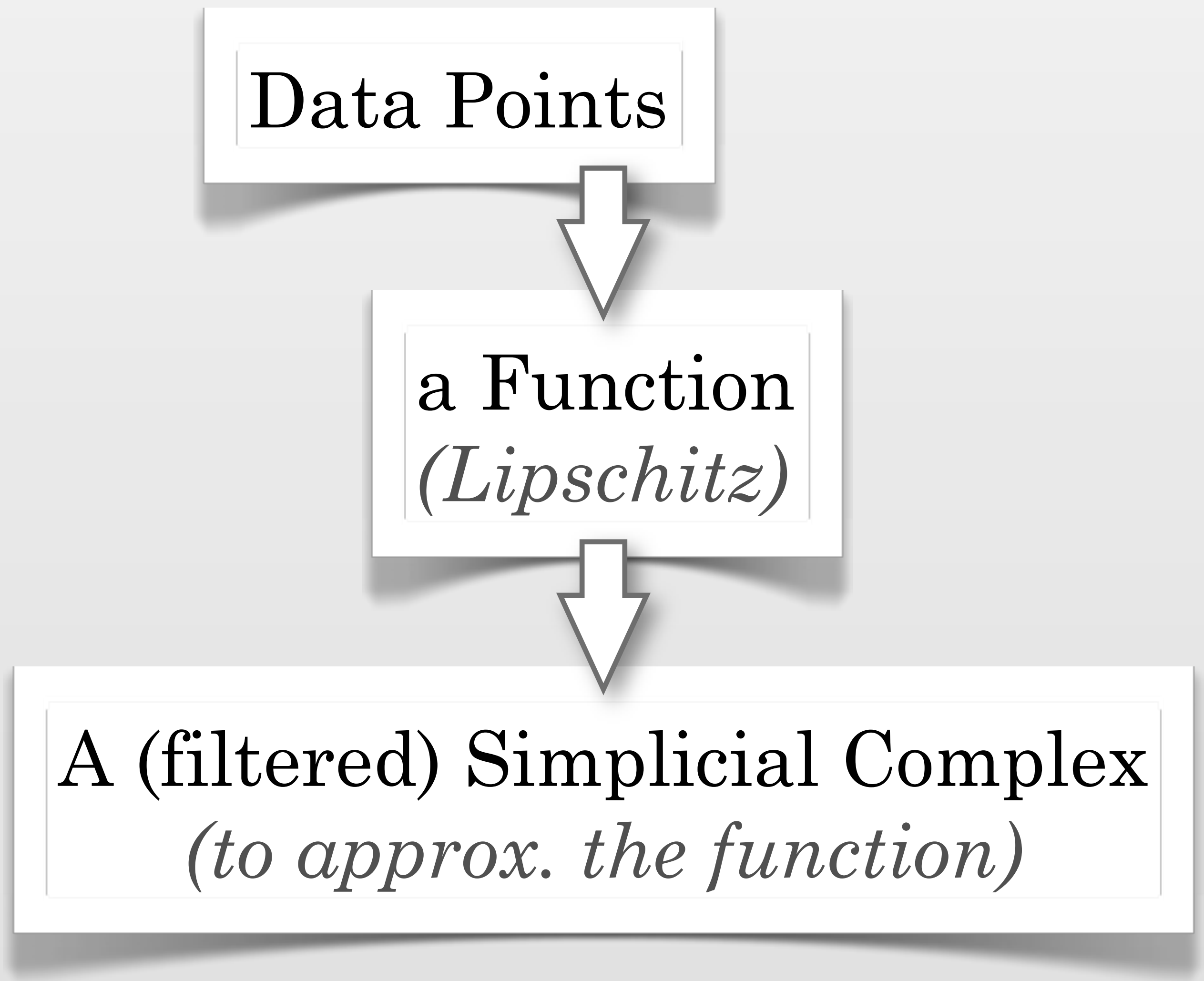

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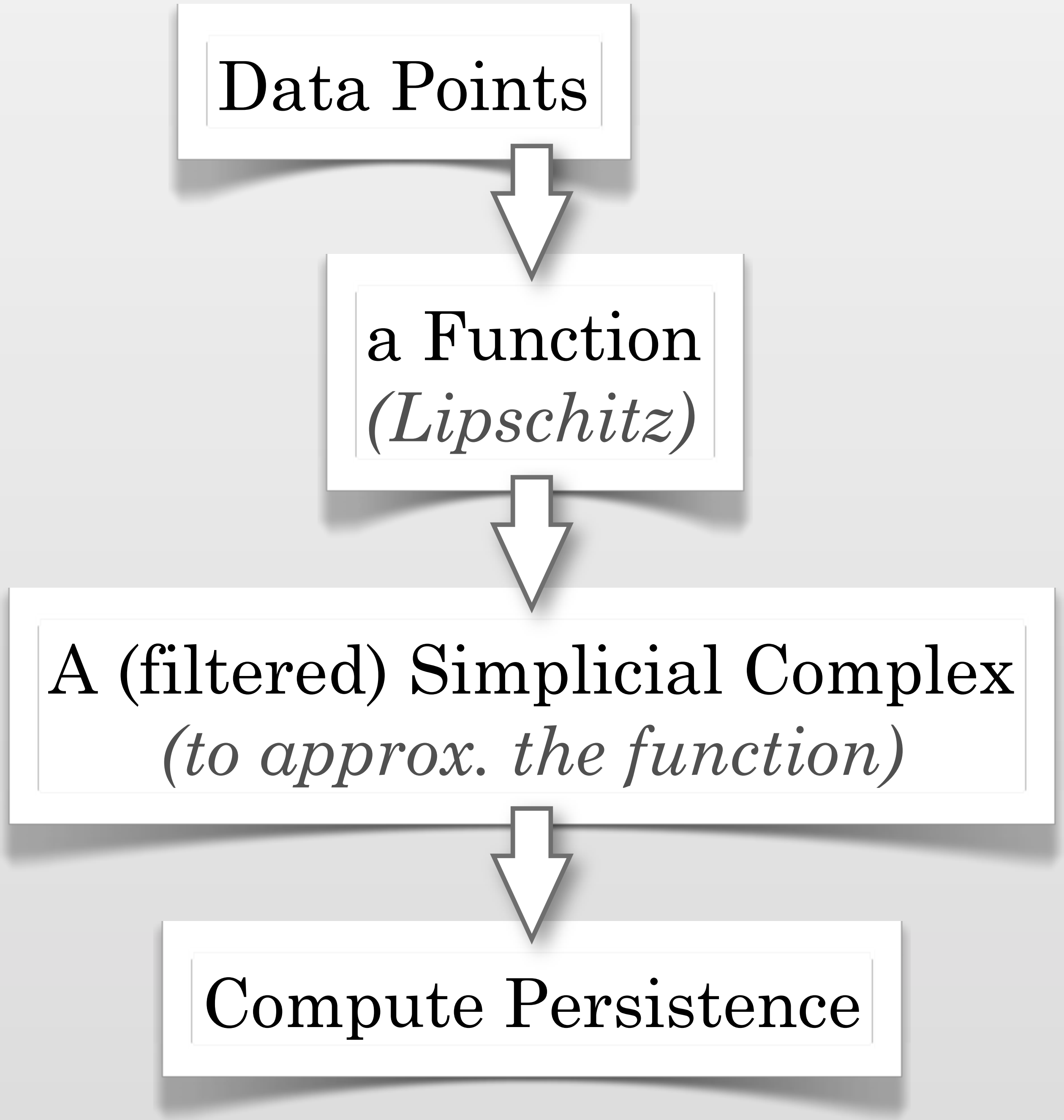

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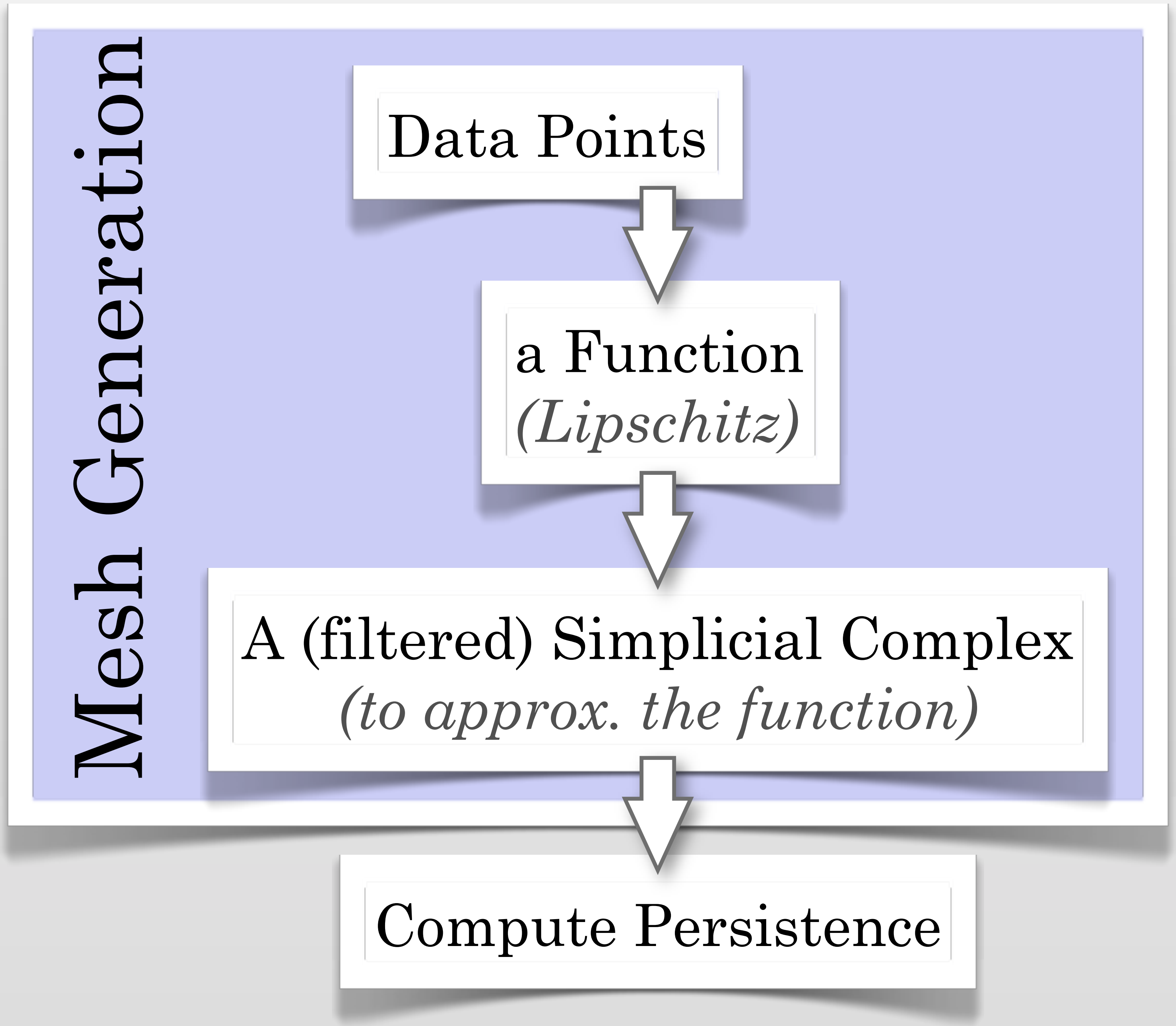

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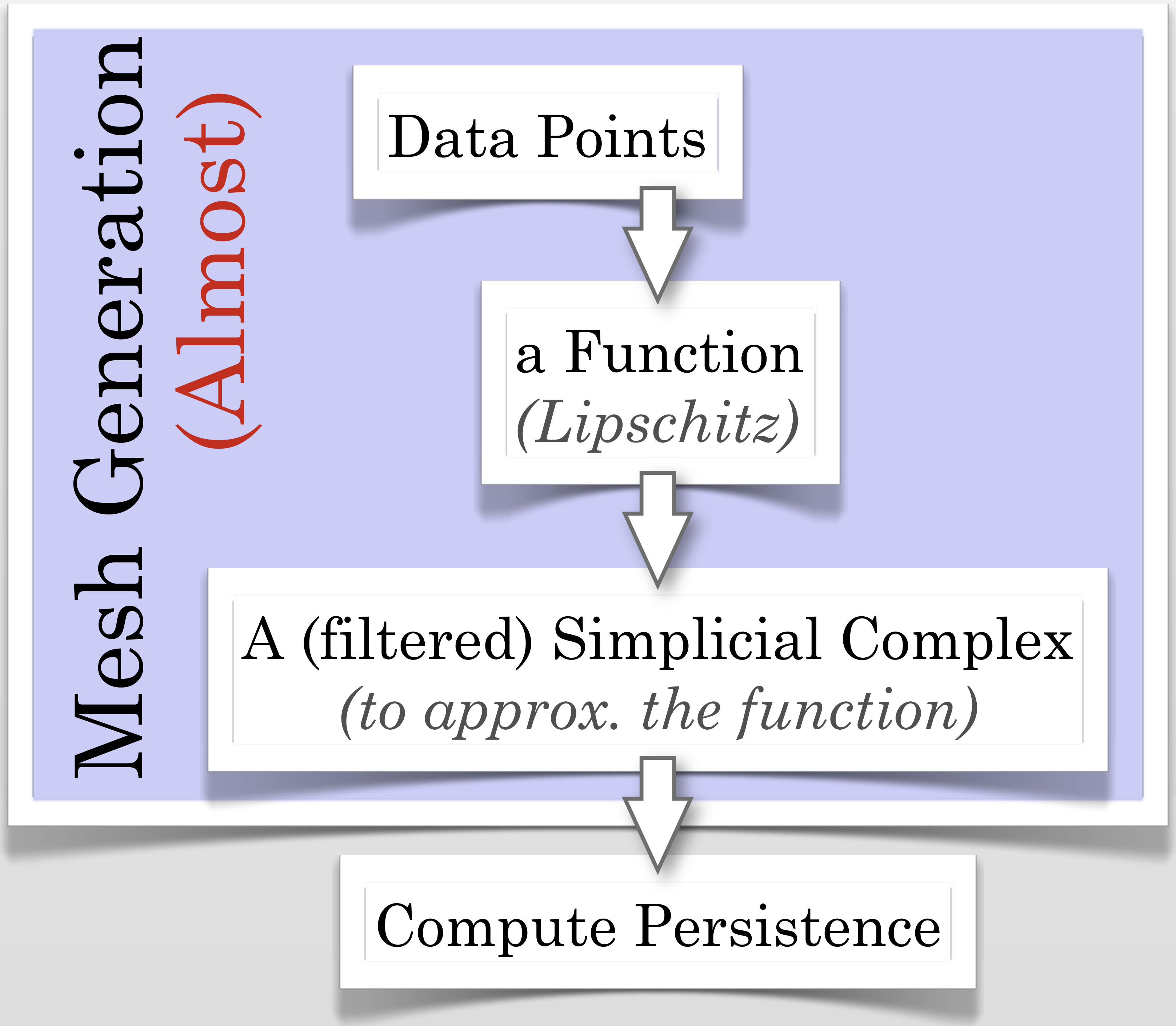

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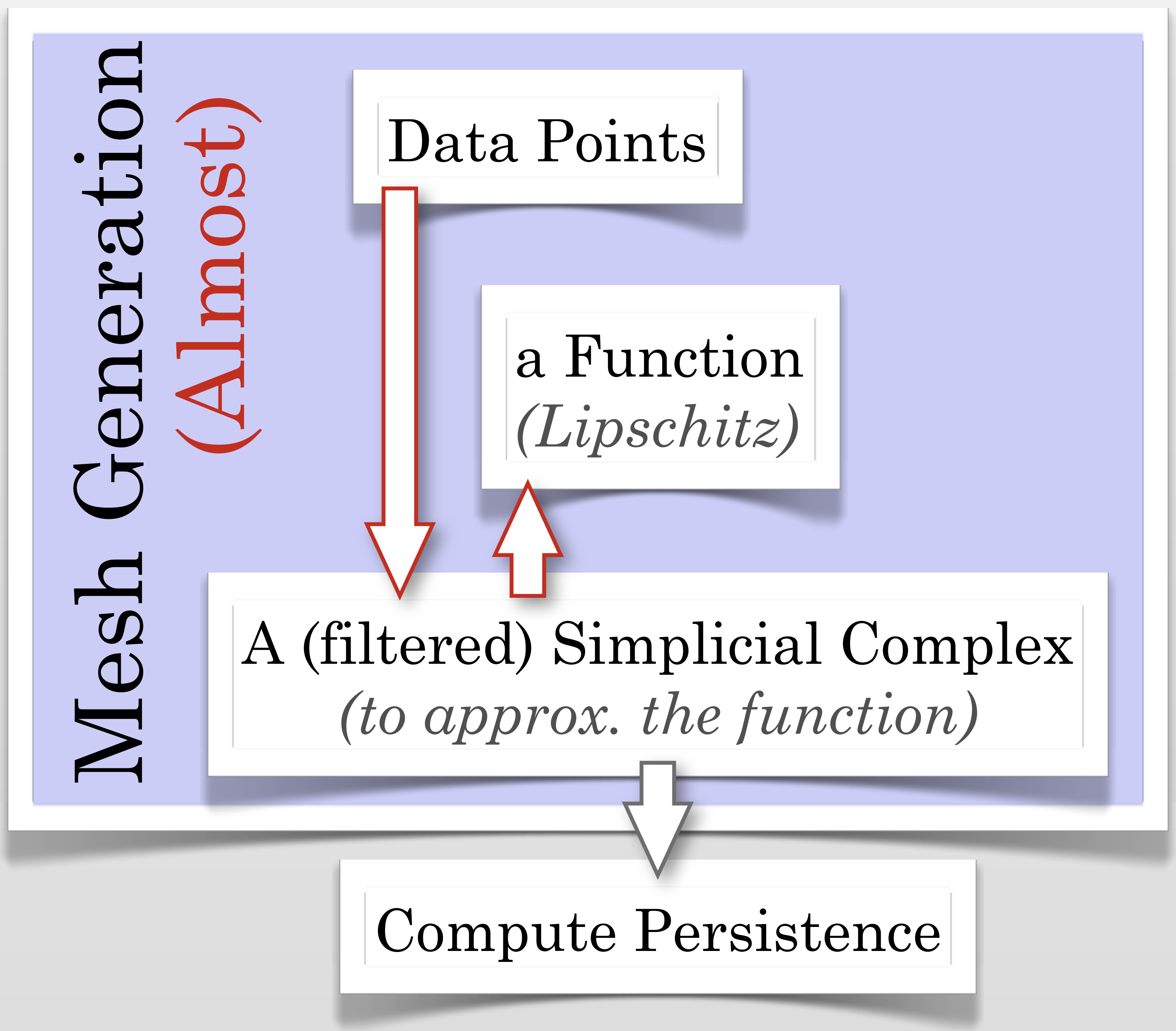

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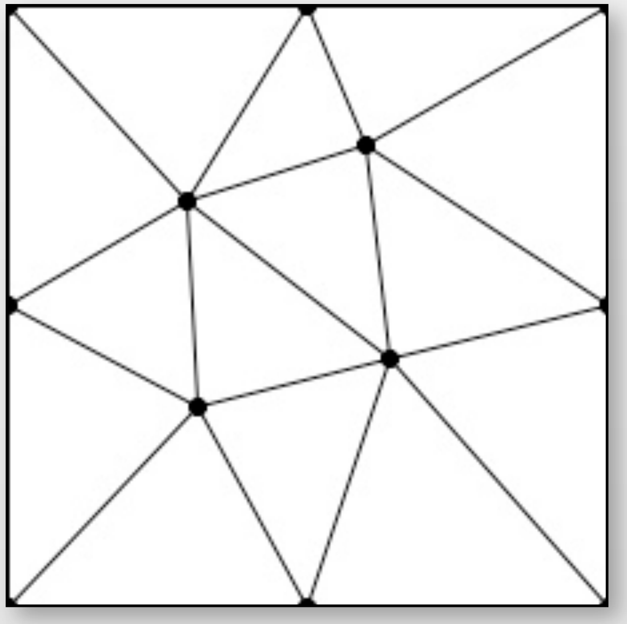

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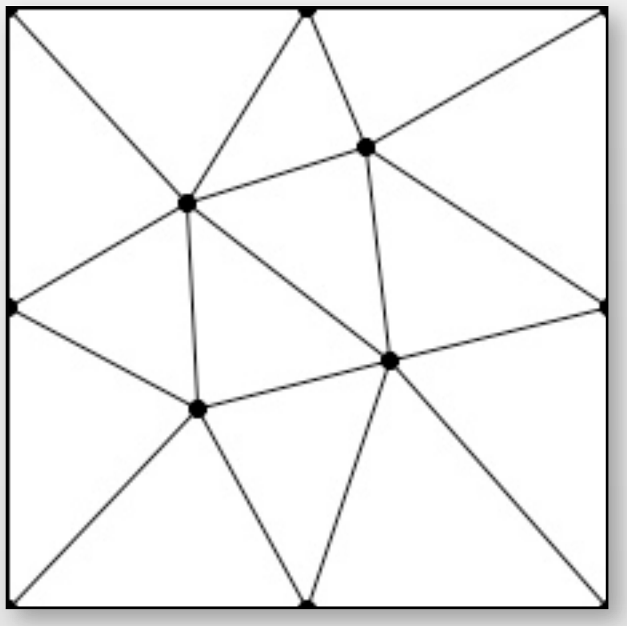

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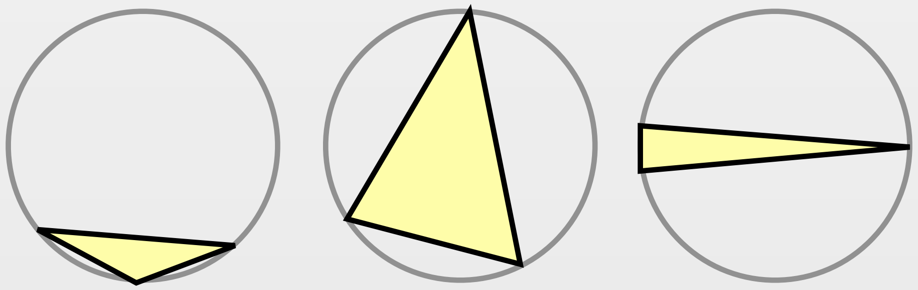

Radius/Edge < const

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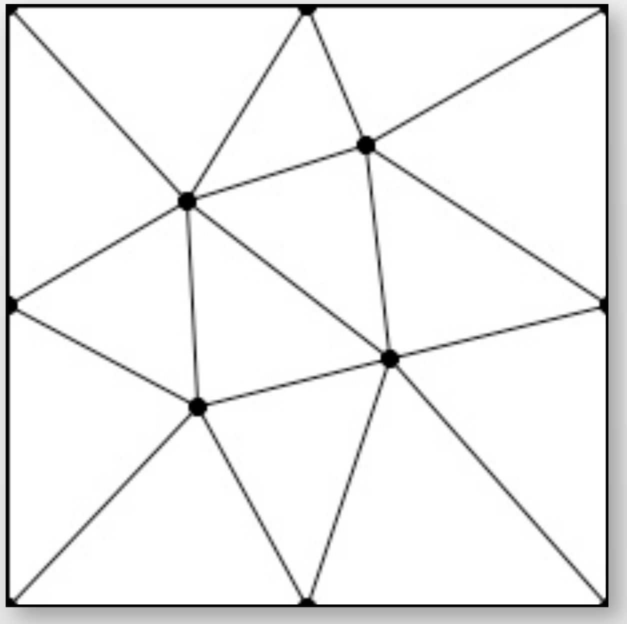

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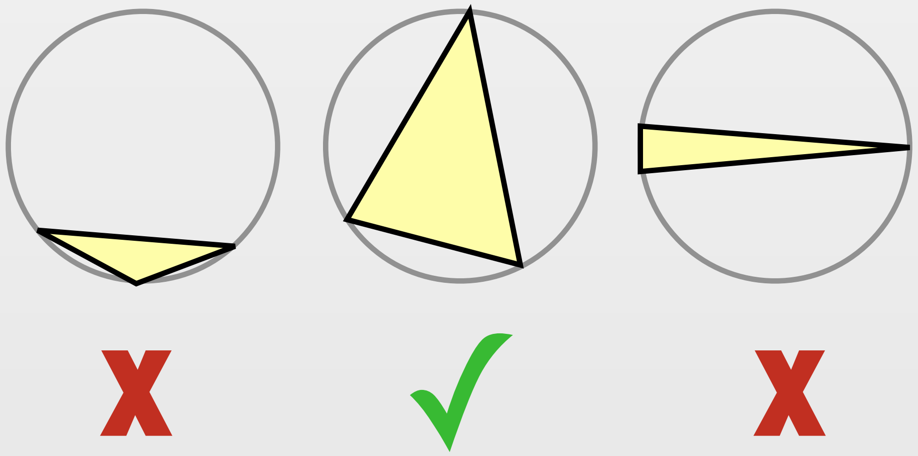

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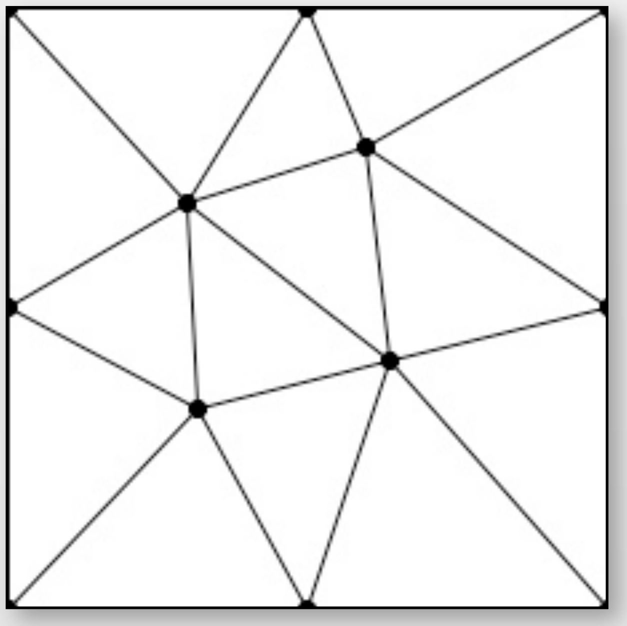

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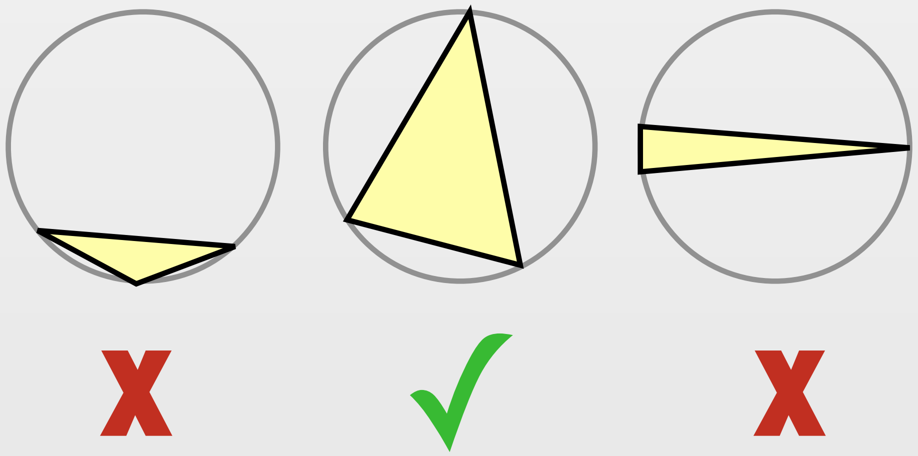

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Conforming to Input

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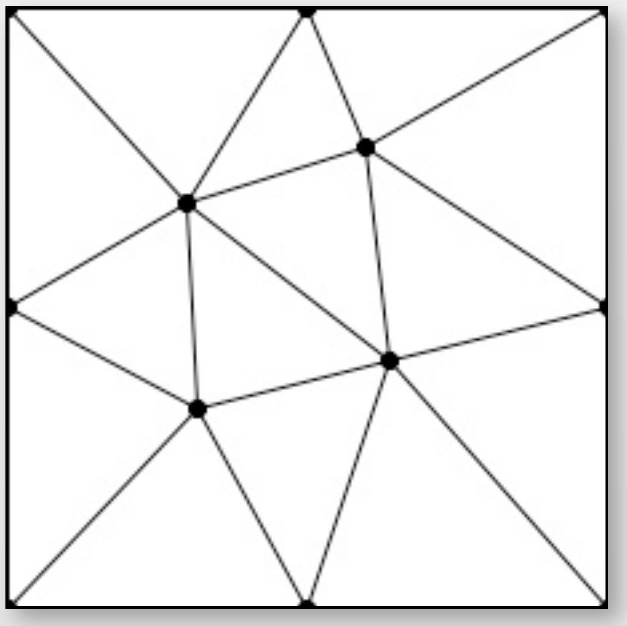

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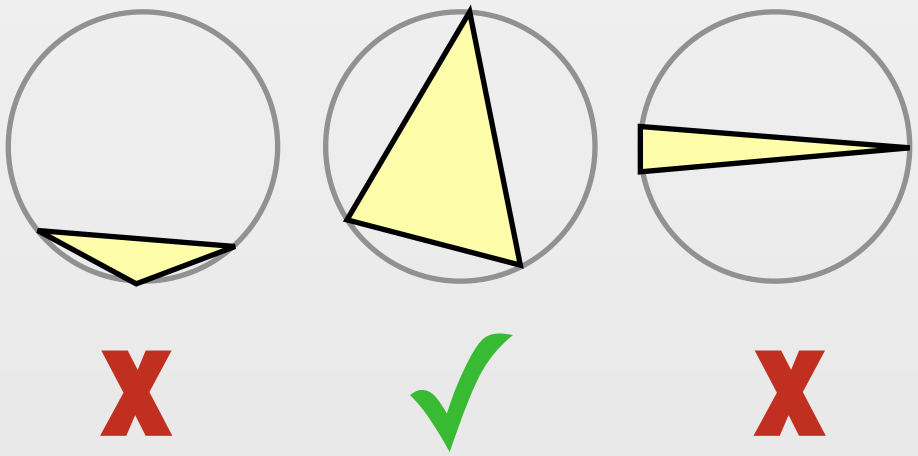

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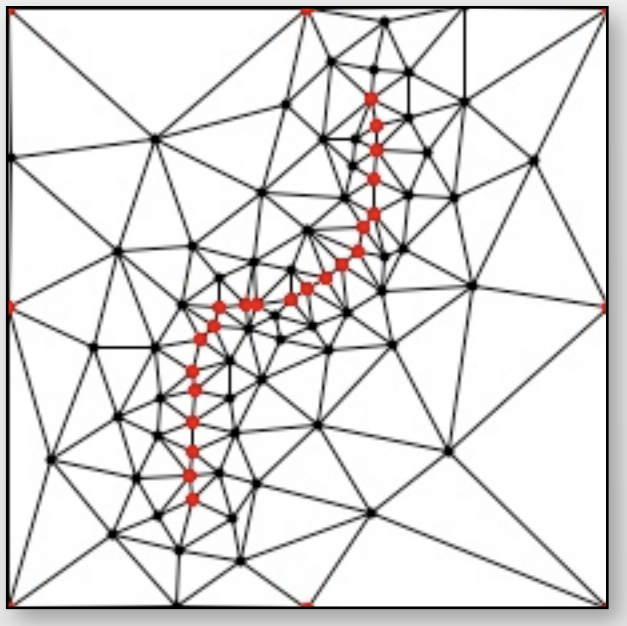

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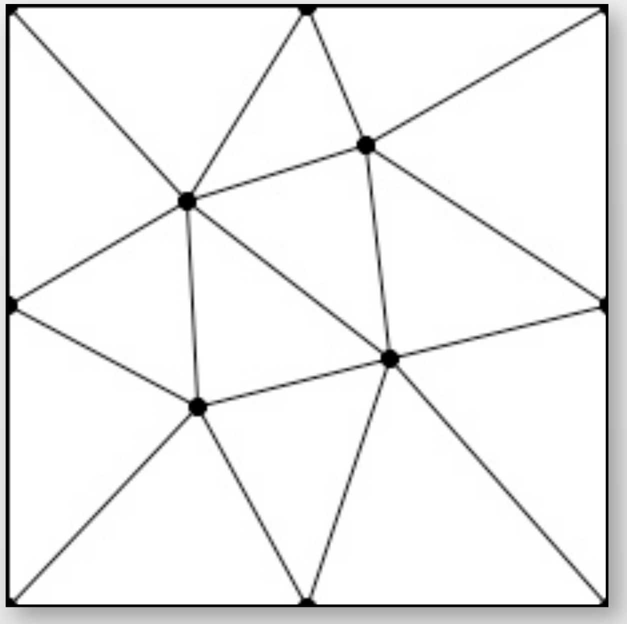

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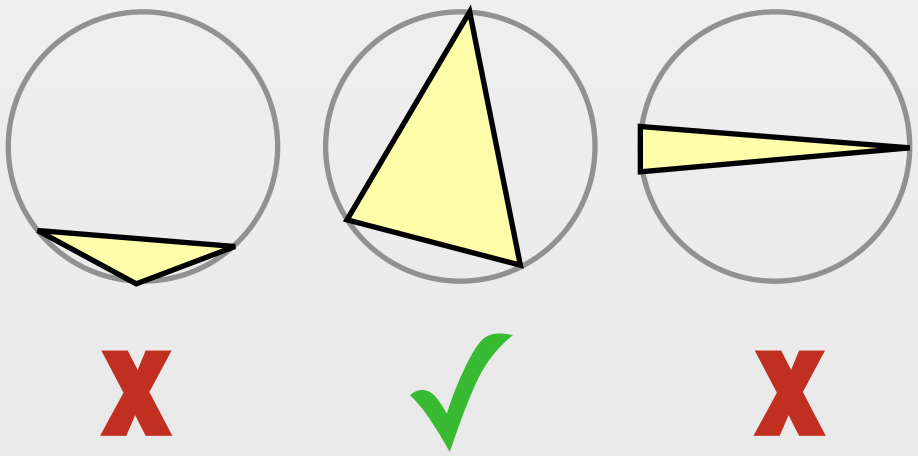

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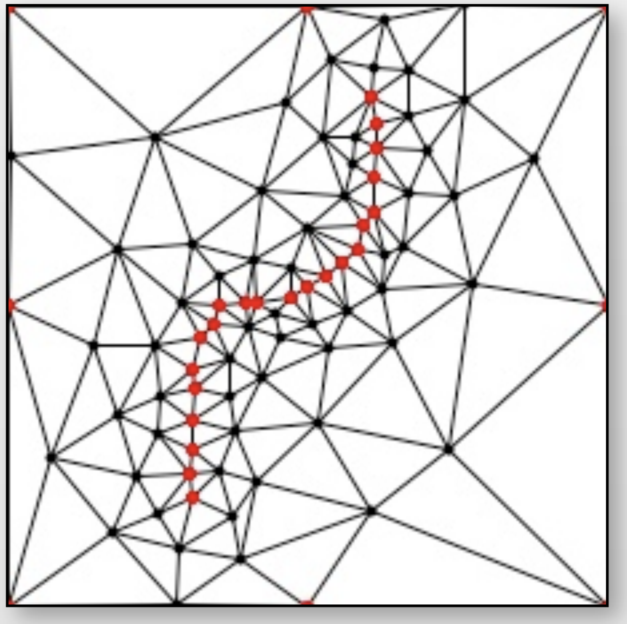

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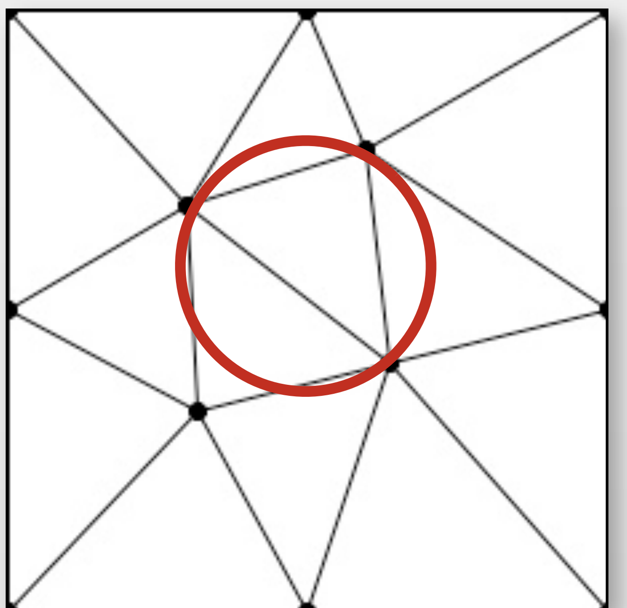

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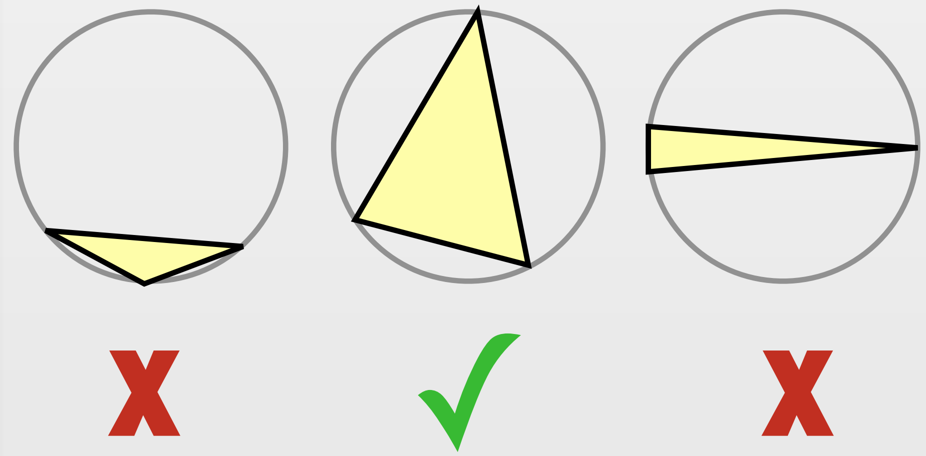

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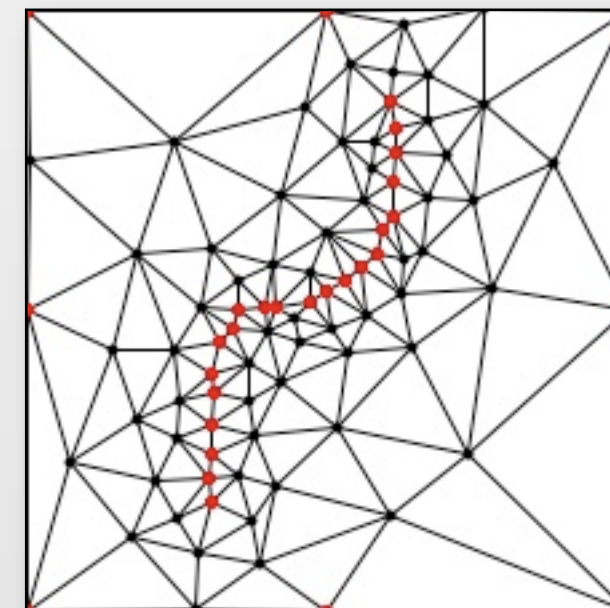

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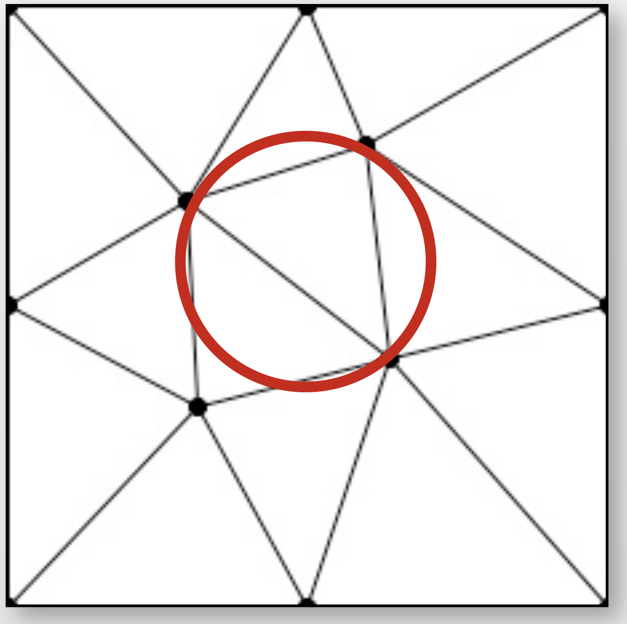

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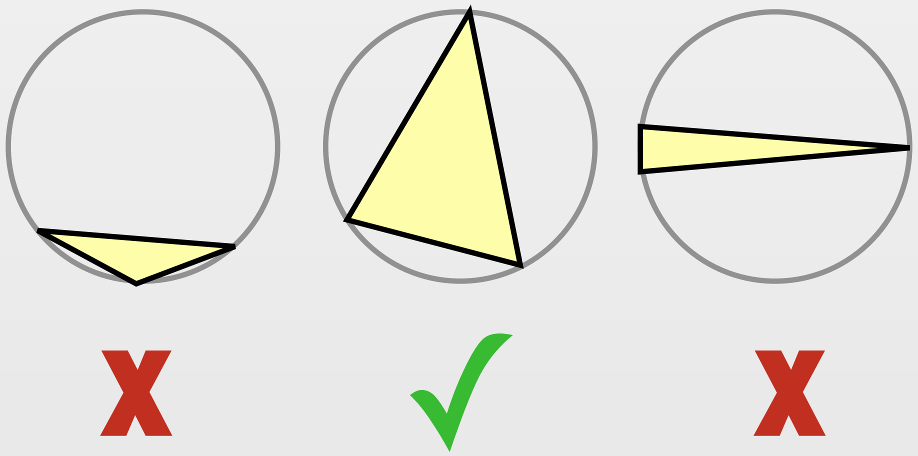

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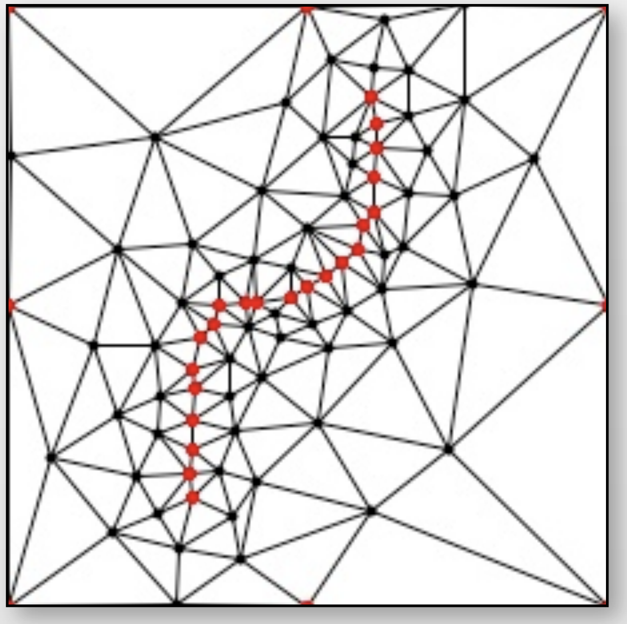

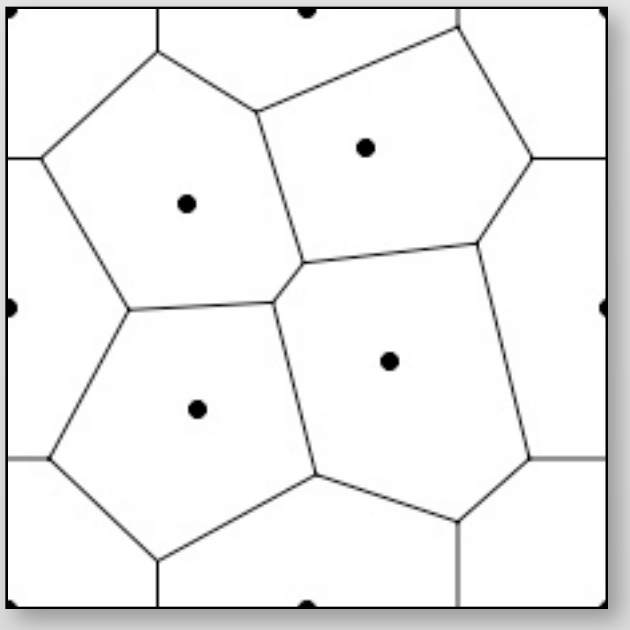

Voronoi Diagram

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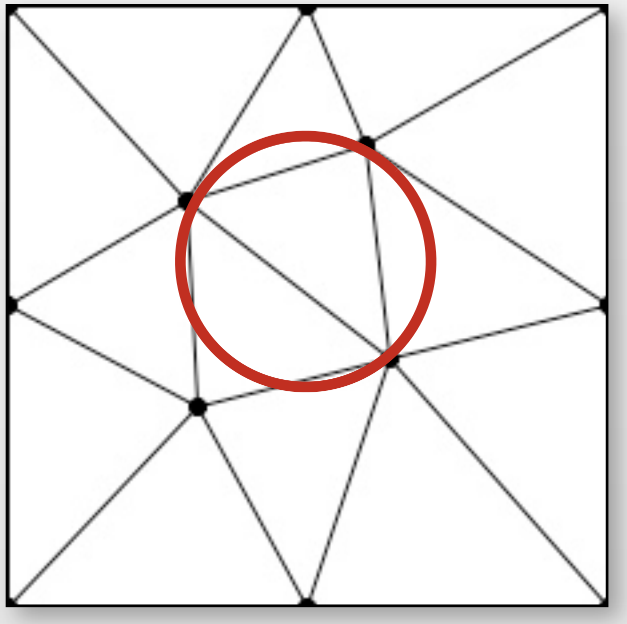

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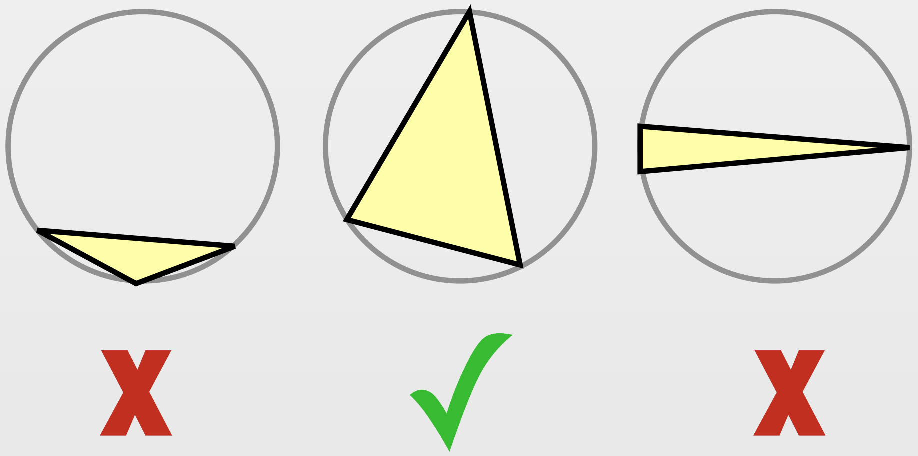

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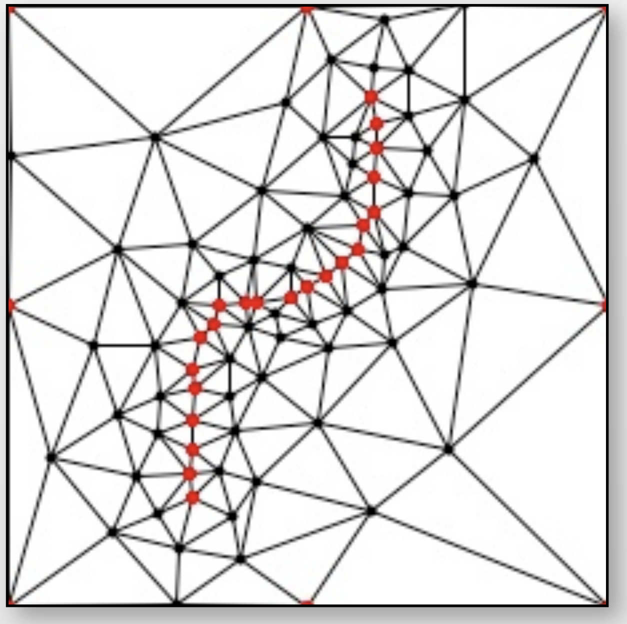

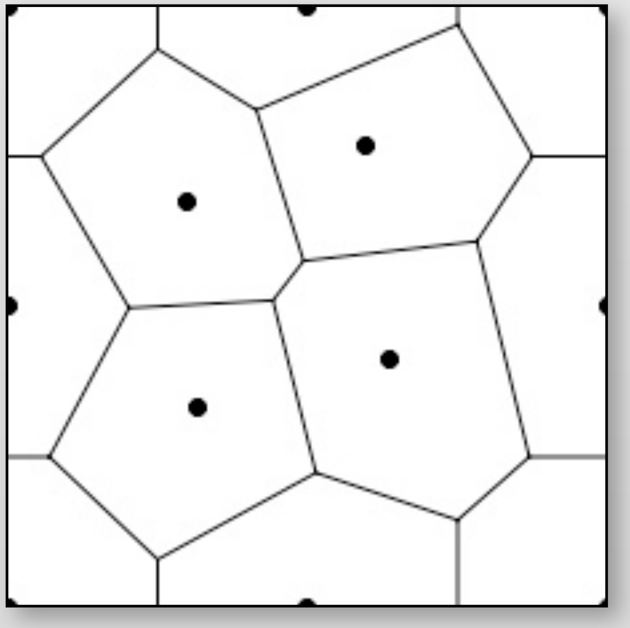

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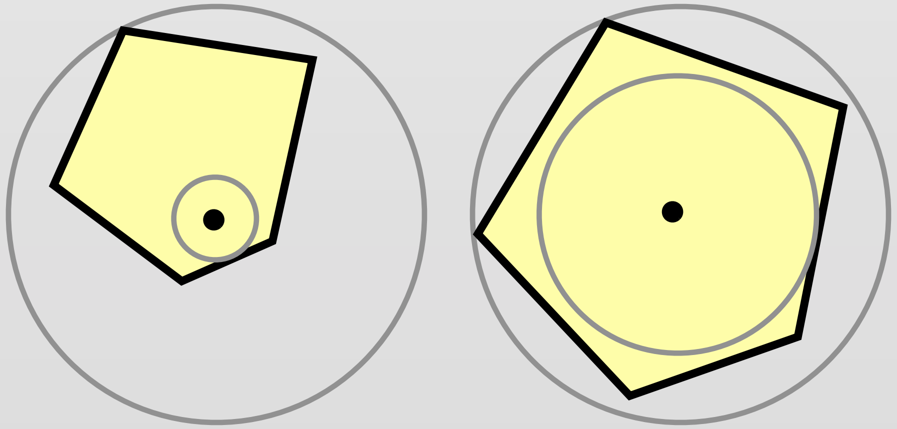

OutRadius/InRadius < const

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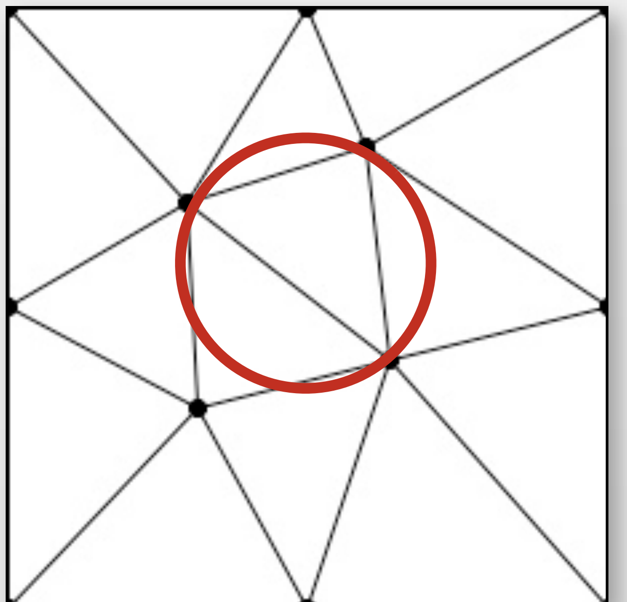

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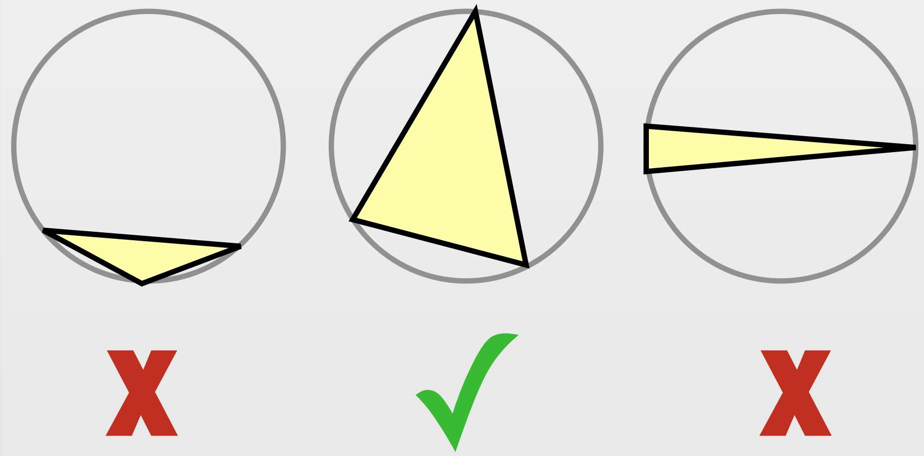

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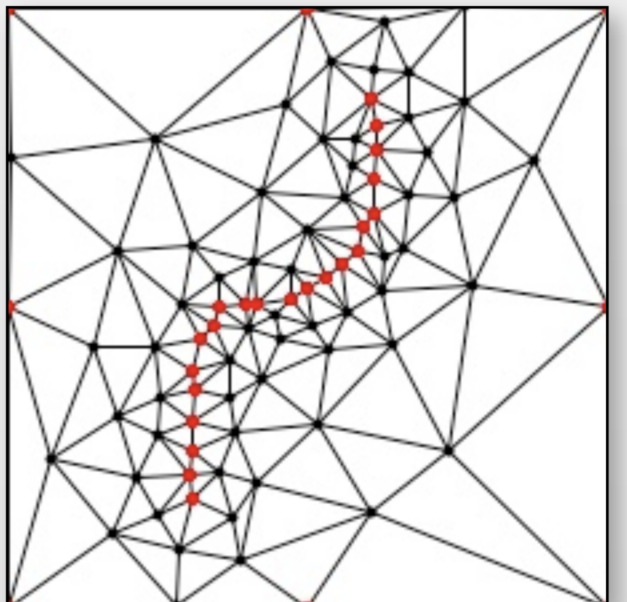

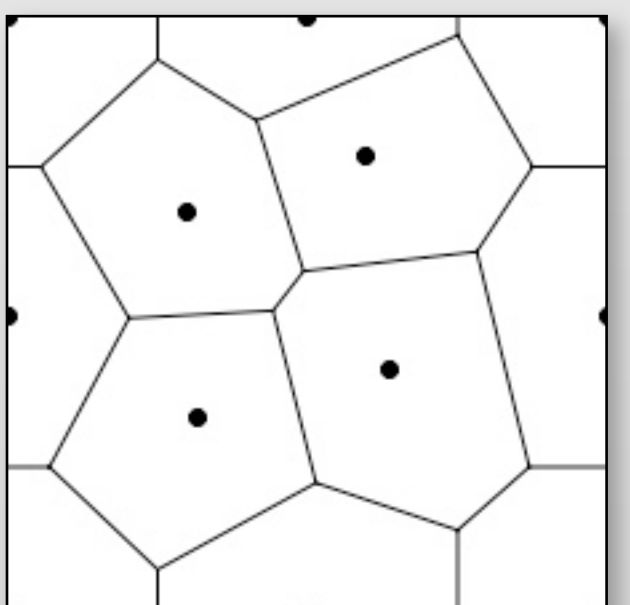

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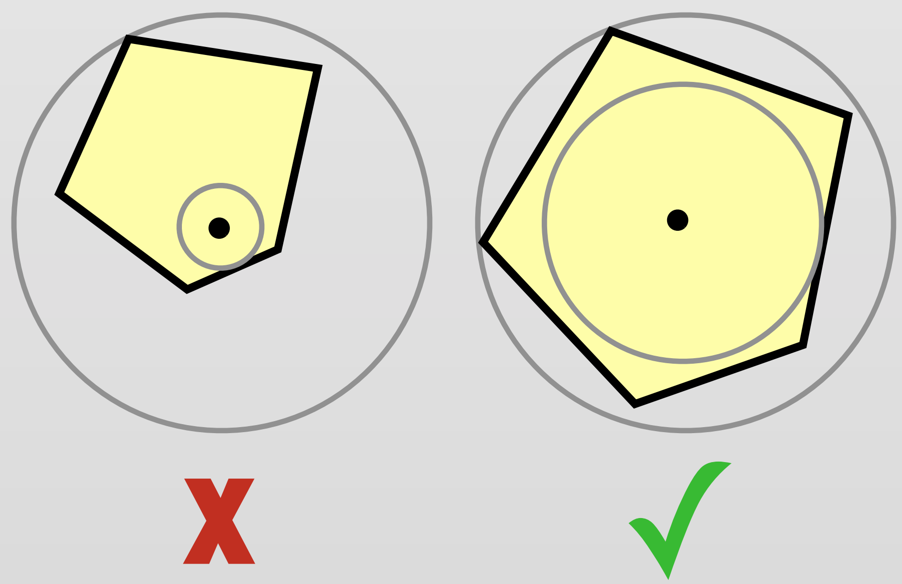

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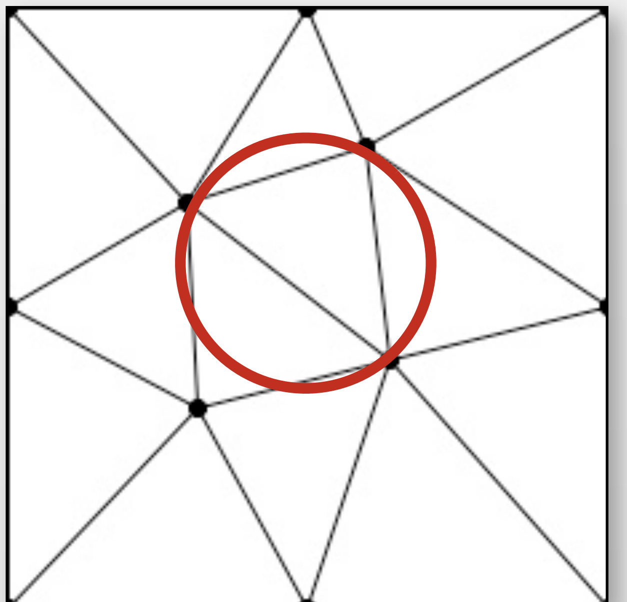

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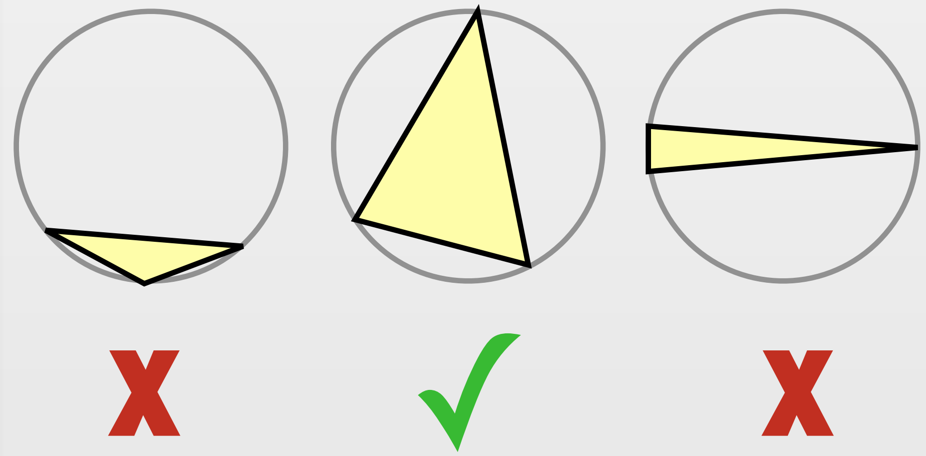

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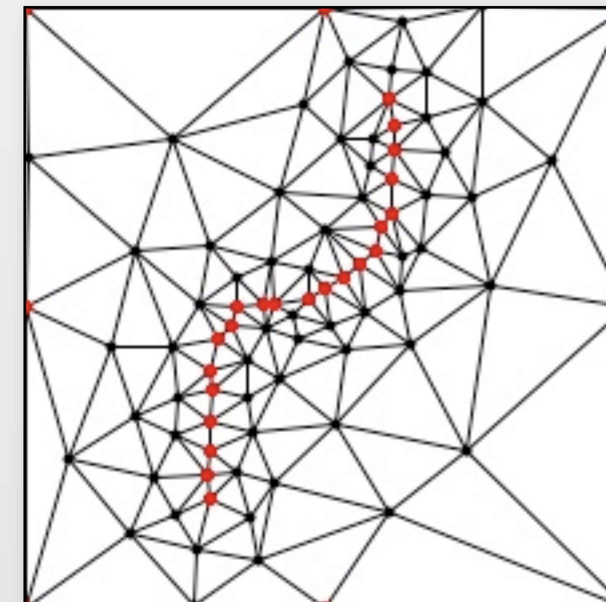

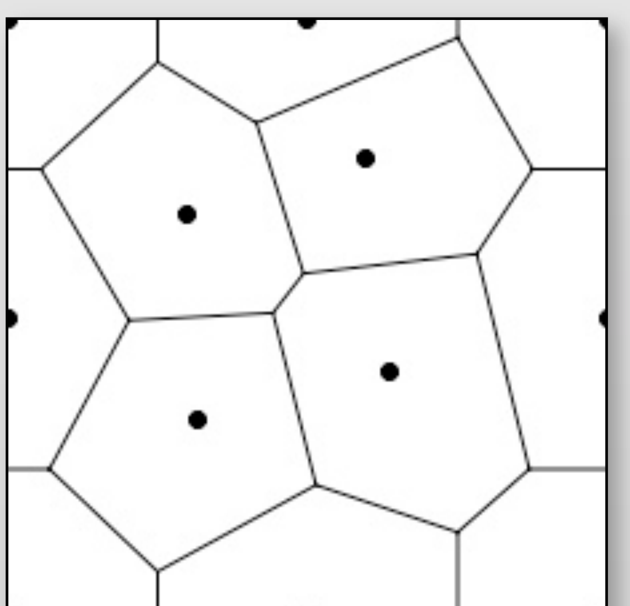

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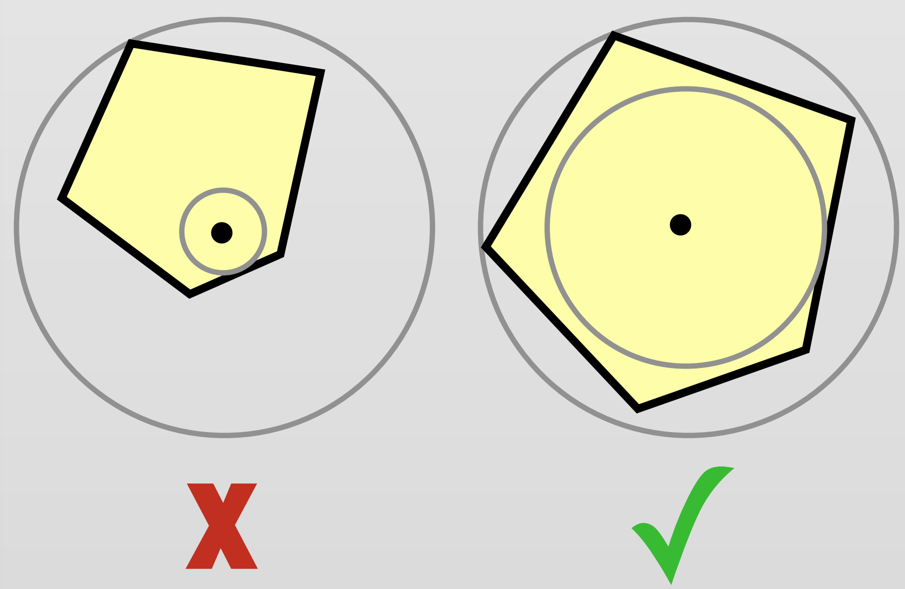

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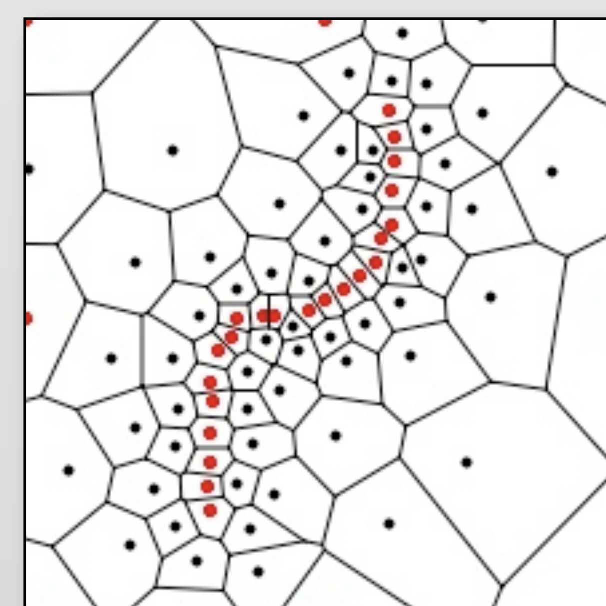

Meshing Points

Input: $P \subset \mathbb{R}^d$

Output: $M \supset P$ with a "nice" Voronoi diagram

$n = |P|, m = |M|$

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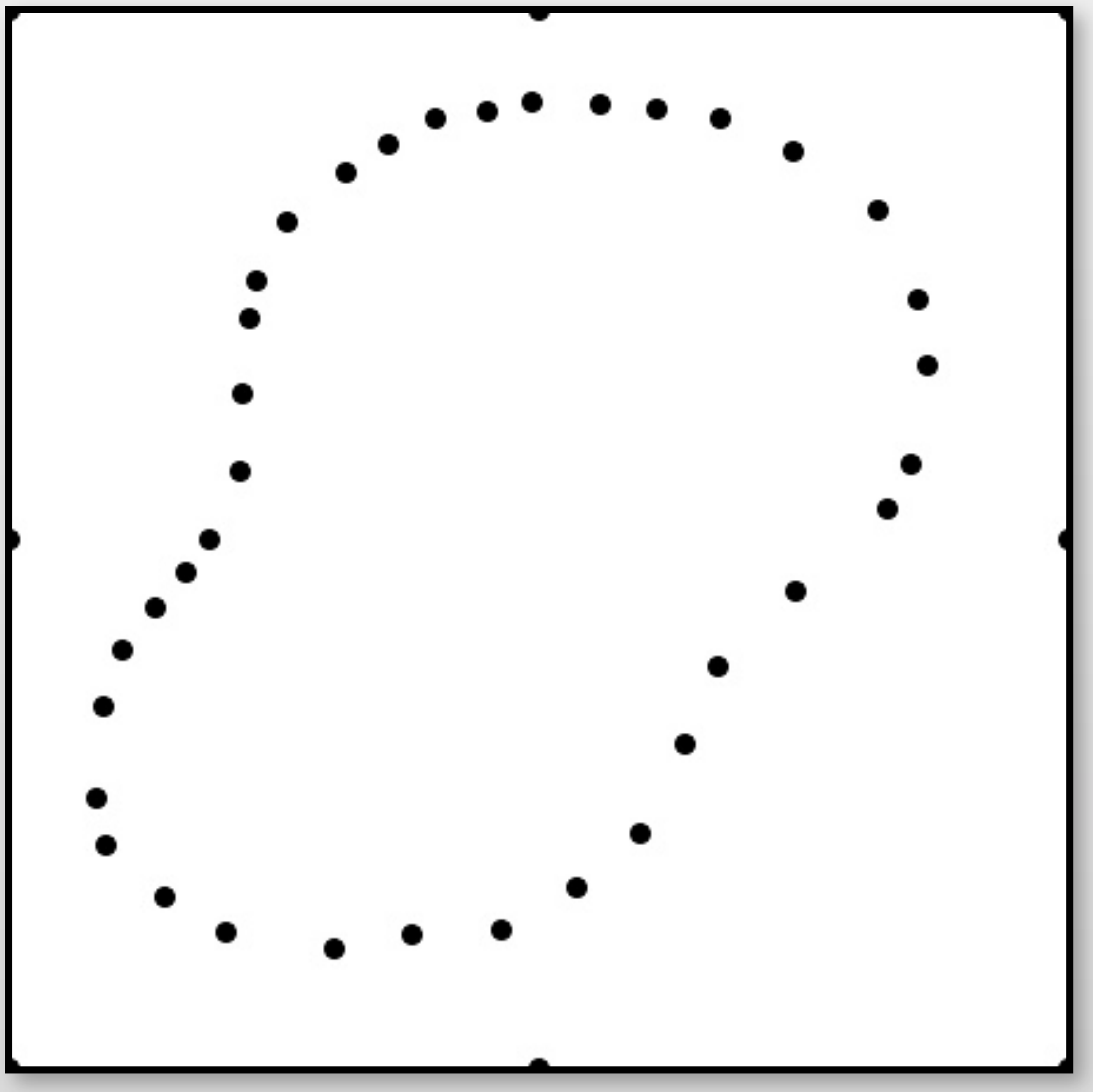

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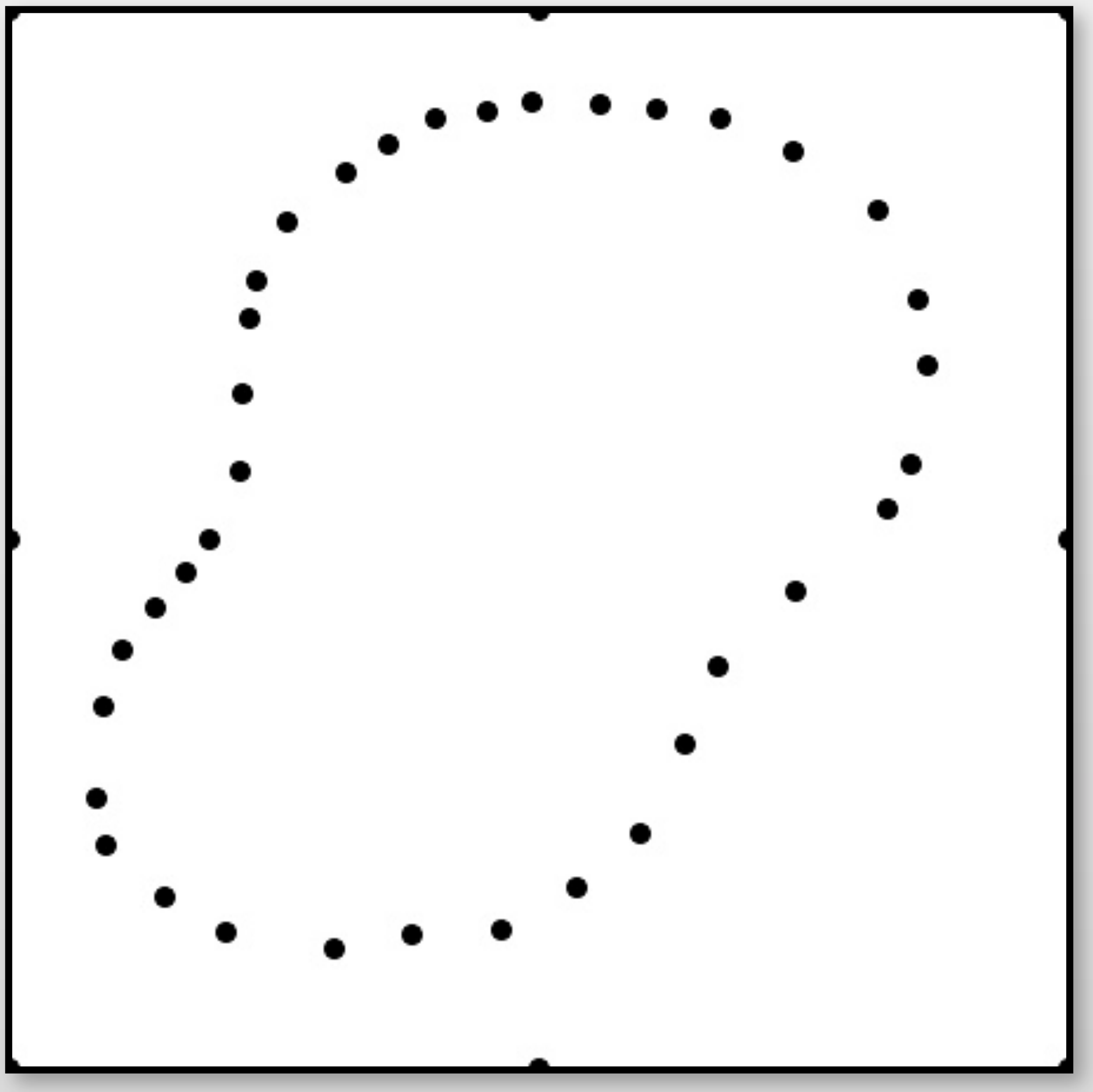

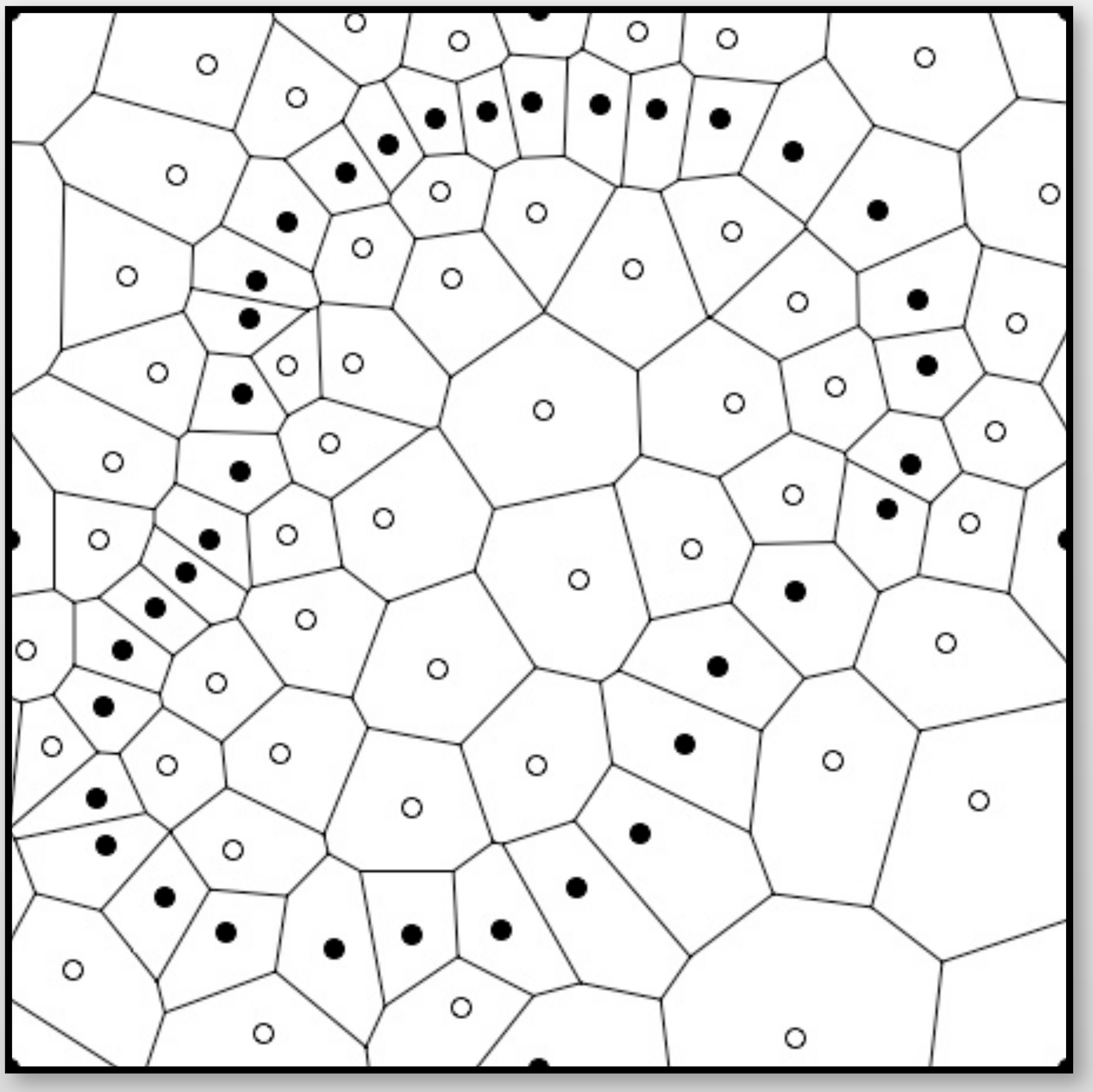

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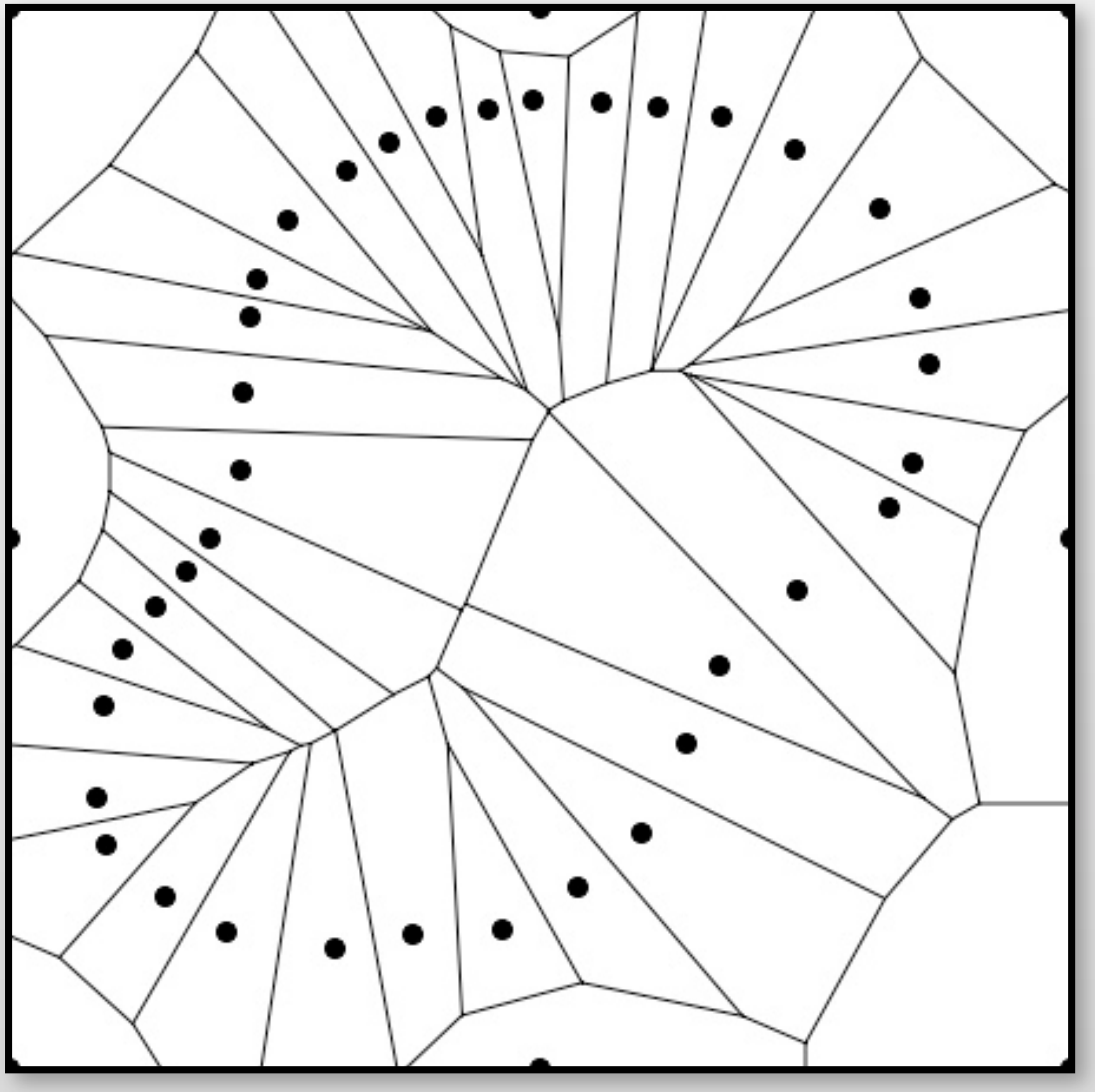
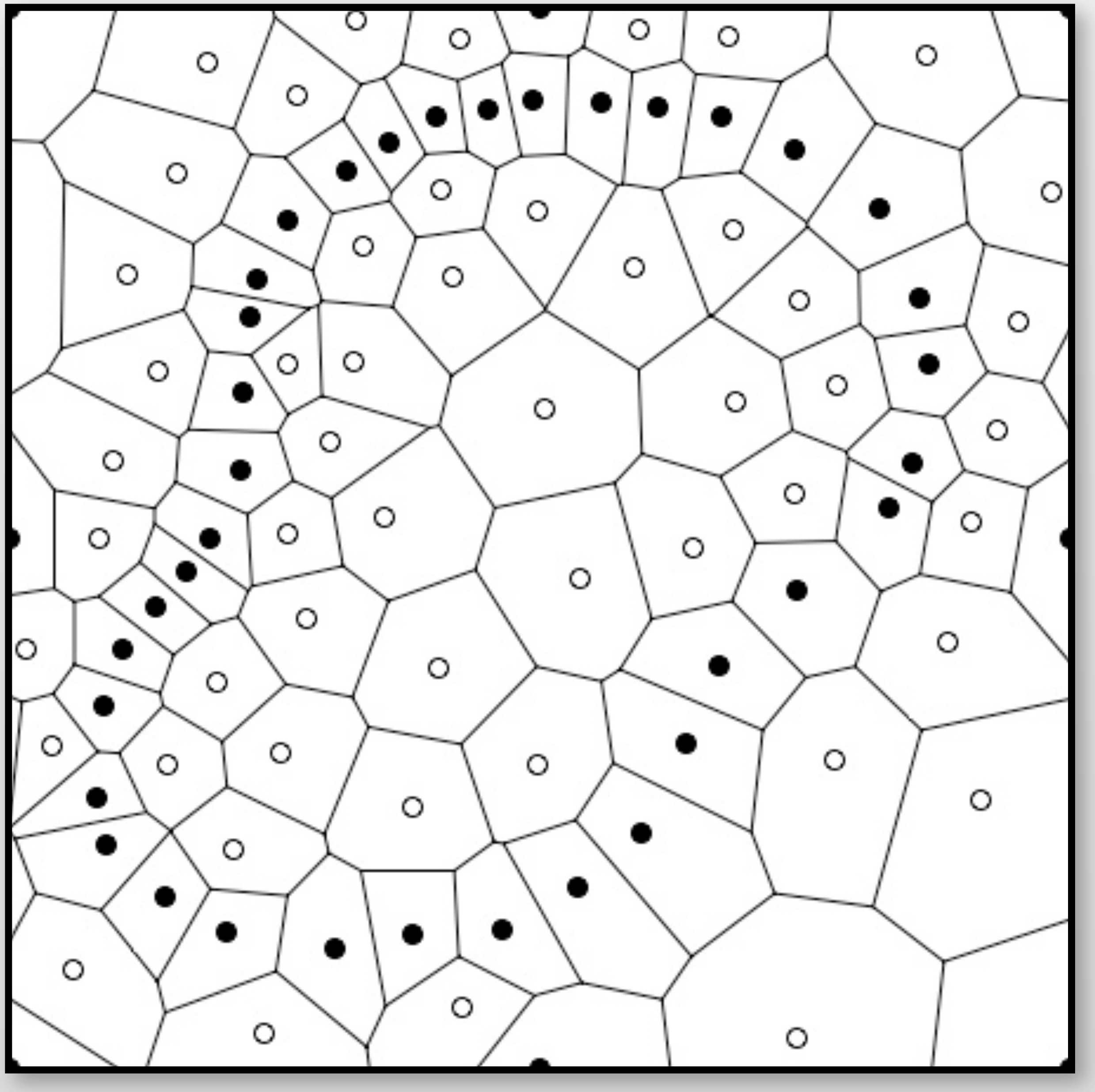
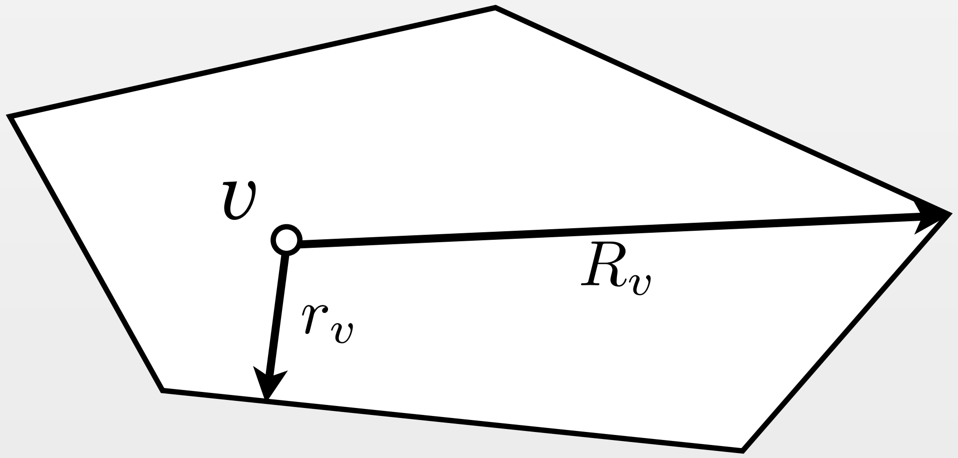




Meshing Guarantees



Aspect Ratio (quality):

$$\frac{R_v}{r_v} \leq \tau$$

Cell Sizing:

$$\operatorname{lfs}(x) := \mathbf{d}(x, P \setminus \{\operatorname{NN}(x)\})$$
$$R_v \leq \varepsilon \operatorname{lfs}(v)$$

Constant Local Complexity:

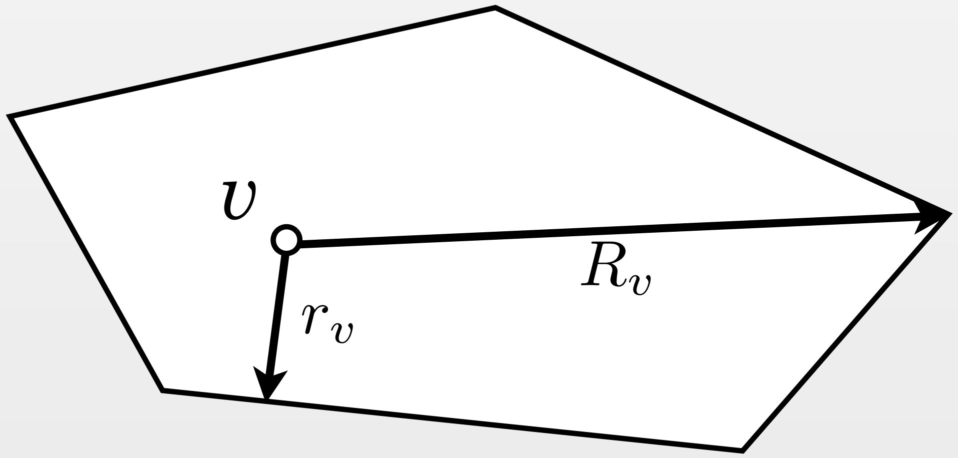
The degree of the 1-skeleton is $2^{O(d)}$.

Optimality and Running time:

$$|M| = \Theta(|\text{Optimal}|)$$

Running time: $O(n \log n + |M|)$

Meshing Guarantees



Aspect Ratio (quality):

$$\frac{R_v}{r_v} \leq \tau$$

Cell Sizing:

$$\text{lfs}(x) := \mathbf{d}(x, P \setminus \{\text{NN}(x)\})$$

ε -refined

$$R_v \leq \varepsilon \text{lfs}(v)$$

Constant Local Complexity:

The degree of the 1-skeleton is $2^{O(d)}$.

Optimality and Running time:

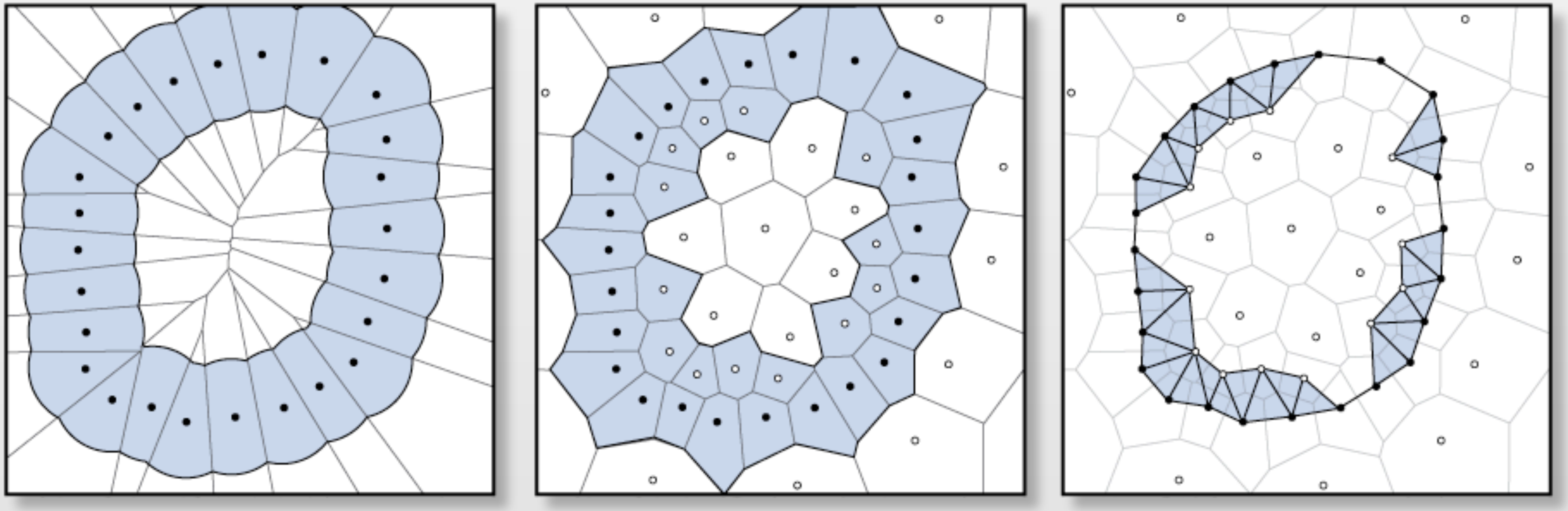
$$|M| = \Theta(|\text{Optimal}|)$$

Running time: $O(n \log n + |M|)$

Mesh Filtrations

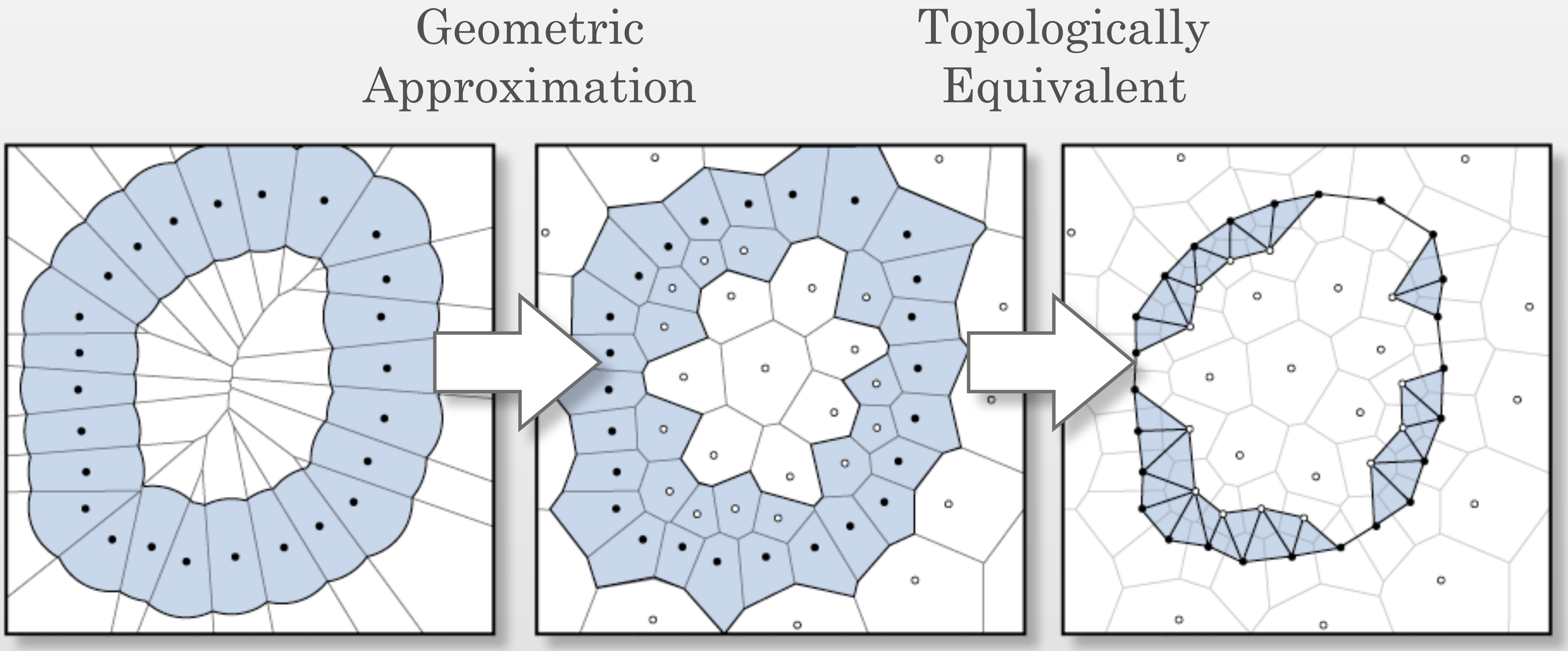
Geometric Approximation

Topologically Equivalent



1. Compute the function on the vertices.
2. Approximate the sublevel with a union of Voronoi cells.
3. Filter the Delaunay triangulation appropriately.

Mesh Filtrations



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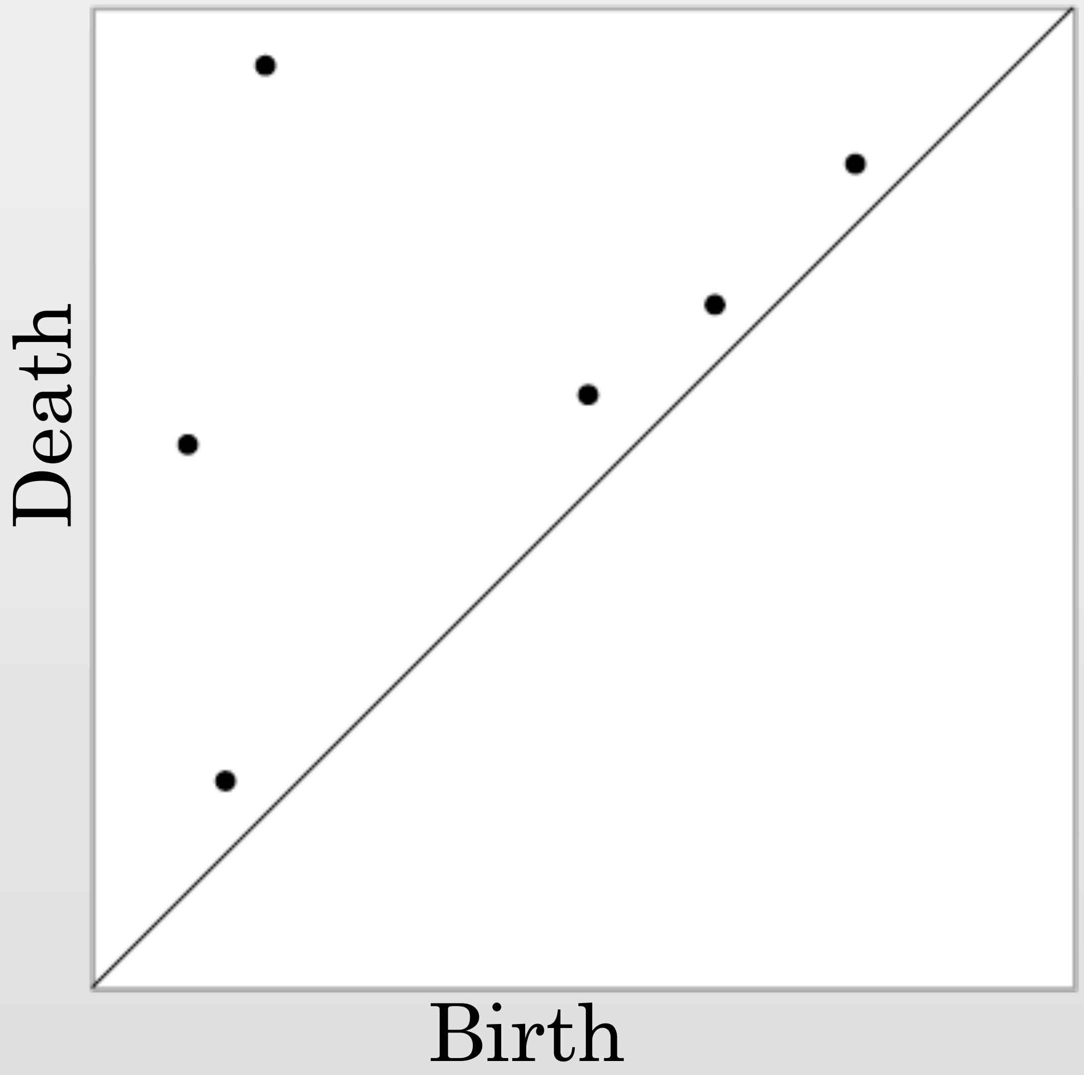
Persistence Diagrams



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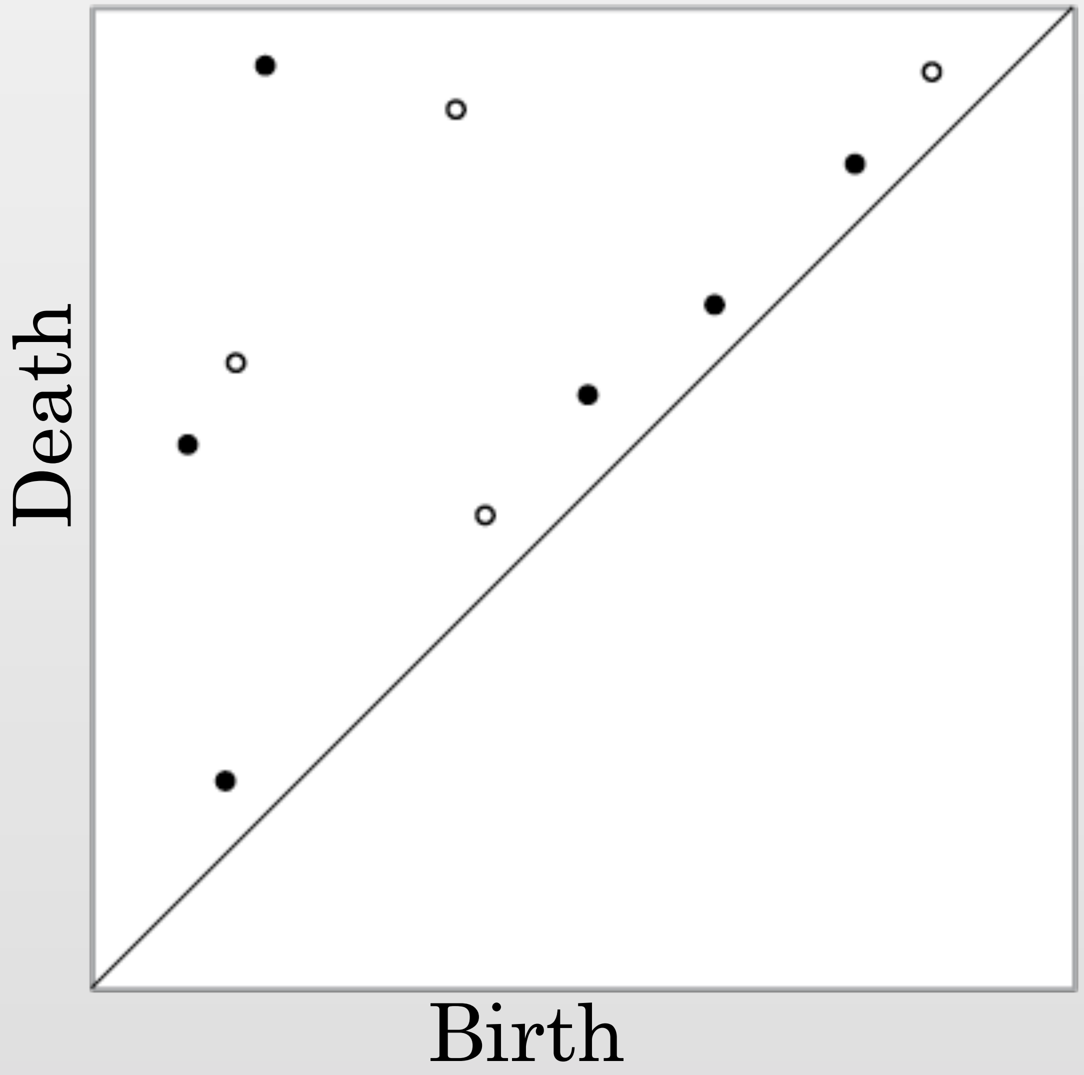

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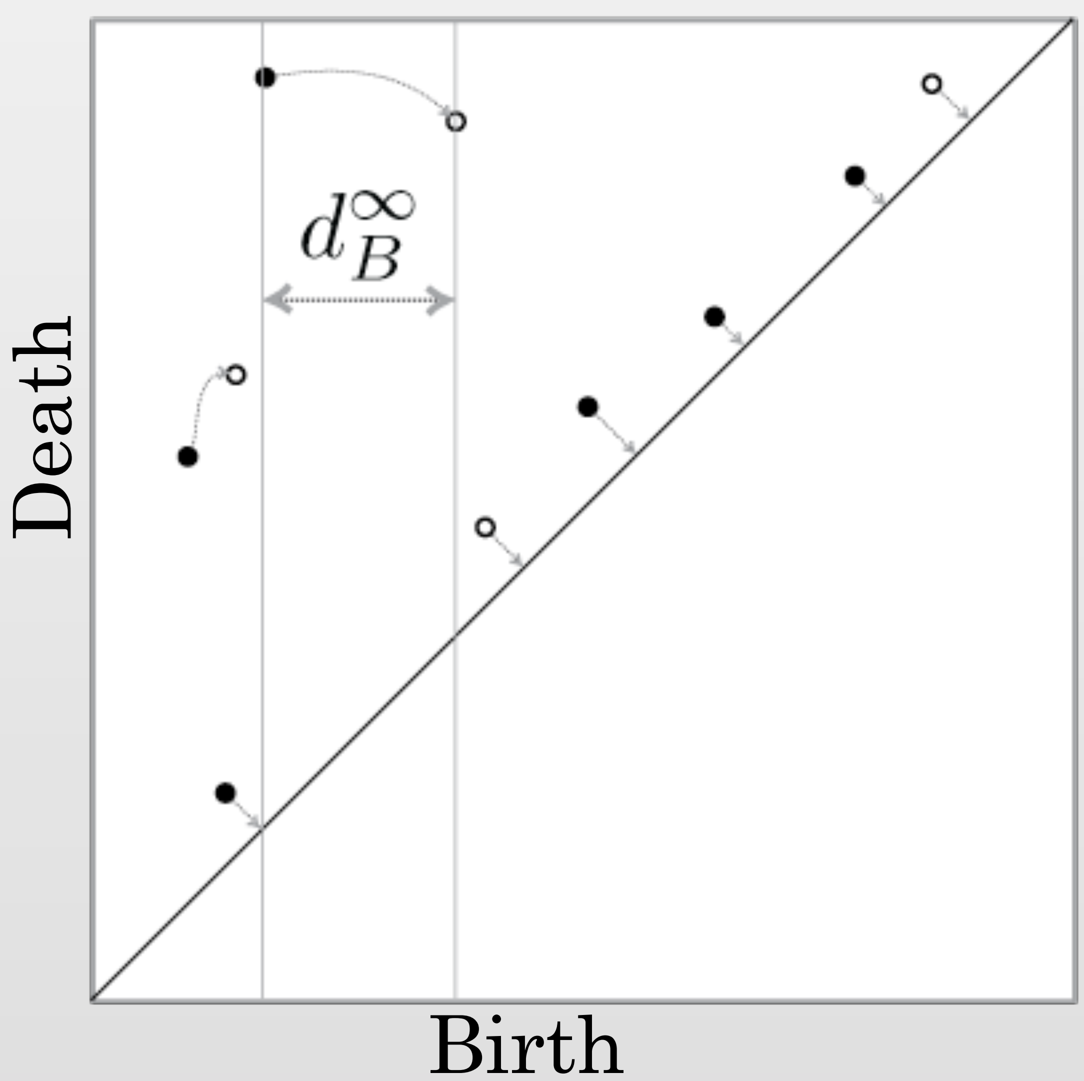
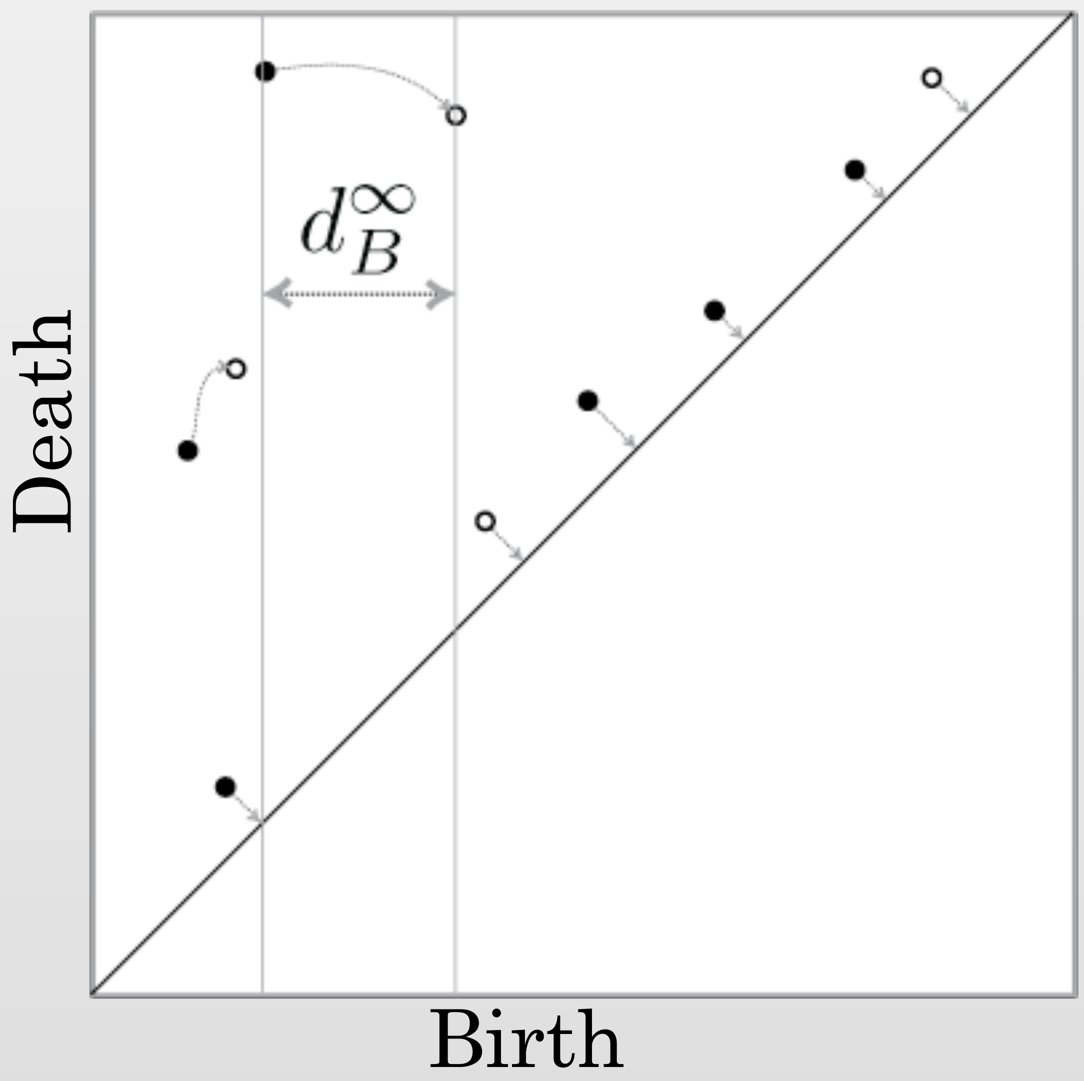


Bottleneck Distance

$$d_B^\infty = \max_i |p_i - q_i|_\infty$$

Approximate
Persistence Diagrams



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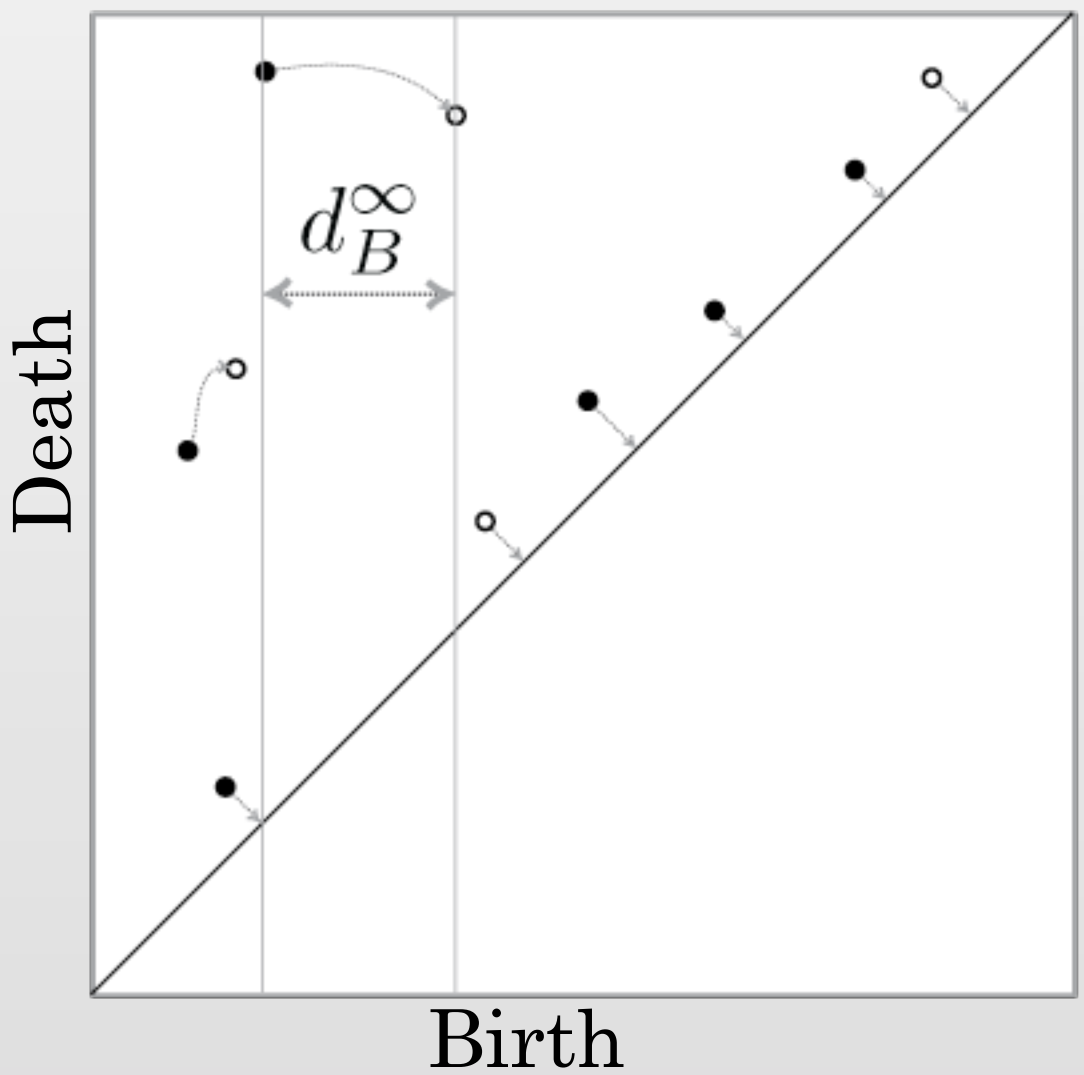

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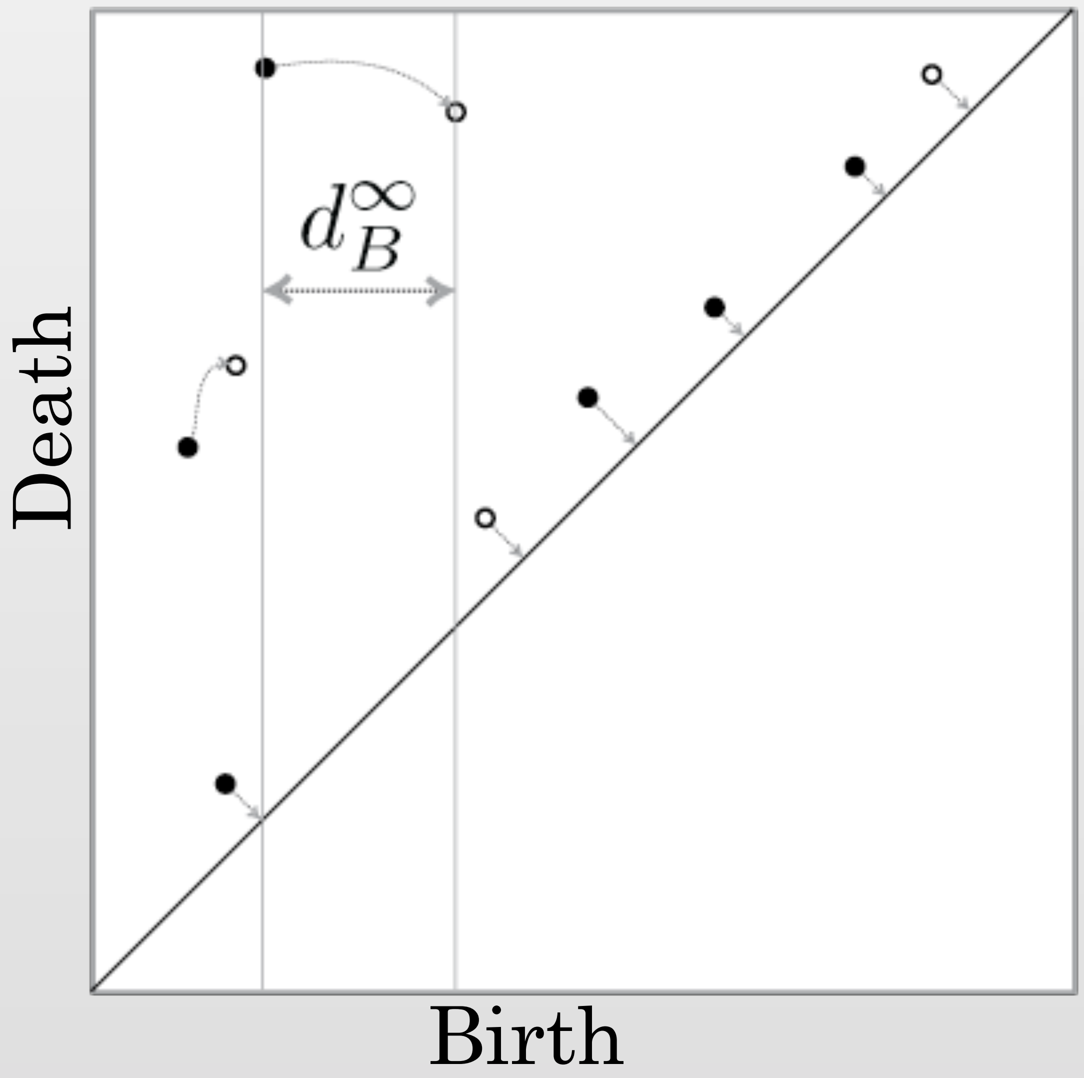
Birth and Death times differ by a constant factor.

Approximate Persistence Diagrams



Bottleneck Distance

$$d_B^\infty = \max_i |p_i - q_i|_\infty$$

Birth and Death times differ by a constant factor.

This is just the bottleneck distance of the log-scale diagrams.

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Dgm $\mathcal{F}$ is a γ**-approximation** to Dgm $\mathcal{G}$ iff

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Lemma. *Let $\mathcal{F} = \{F_\alpha\}$ and $\mathcal{G} = \{G_\alpha\}$ be filtrations. If $F_{\alpha/\gamma} \subseteq G_\alpha \subseteq F_{\alpha\gamma}$ for all $\alpha \geq 0$, then $\mathrm{Dgm}\,\mathcal{F}$ is a γ-approximation to $\mathrm{Dgm}\,\mathcal{G}$.*

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We will show $\mathrm{Dgm}\, \mathcal{V}$ is a good approximation to $\mathrm{Dgm}\, \mathcal{F}$.

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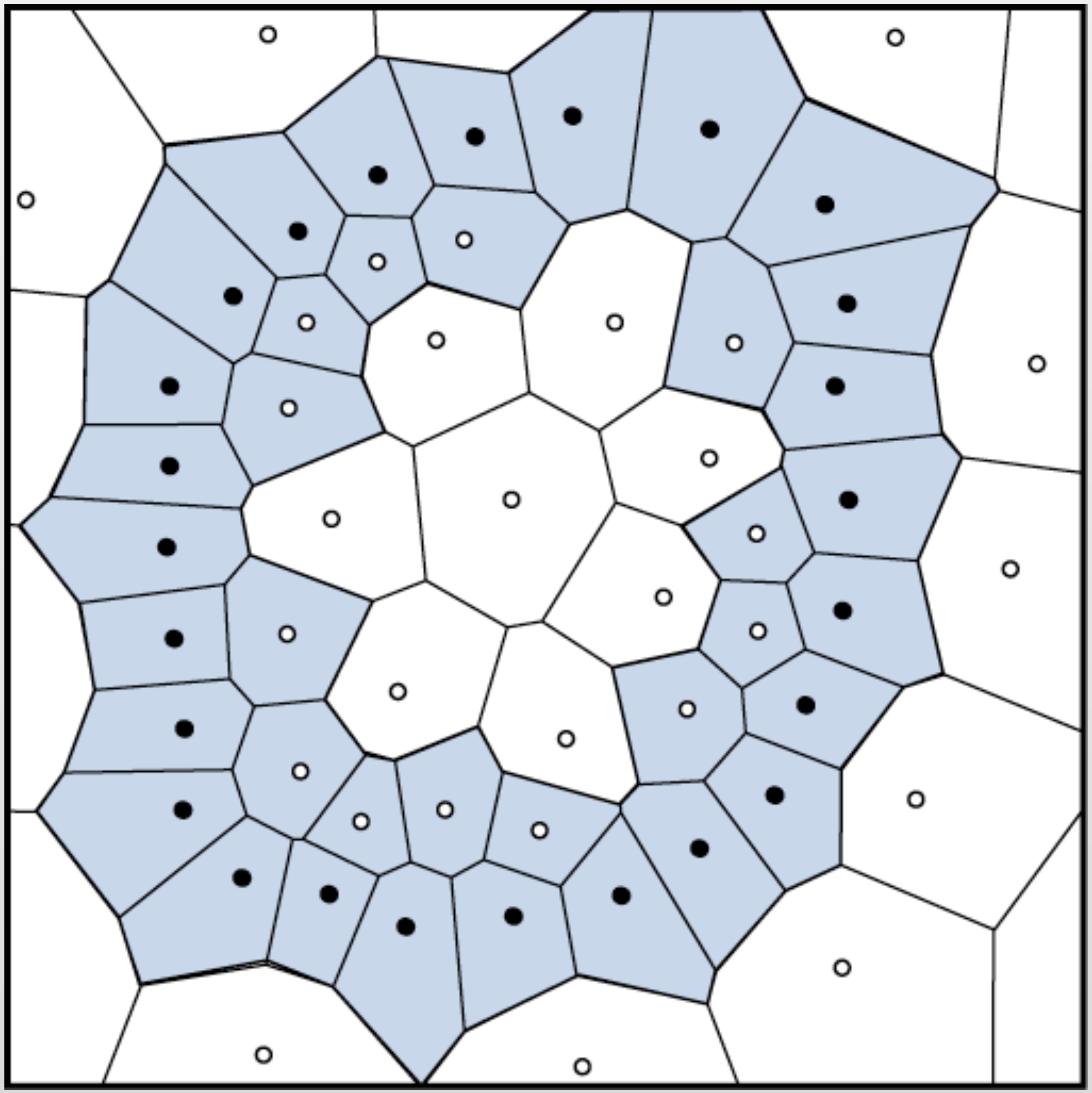

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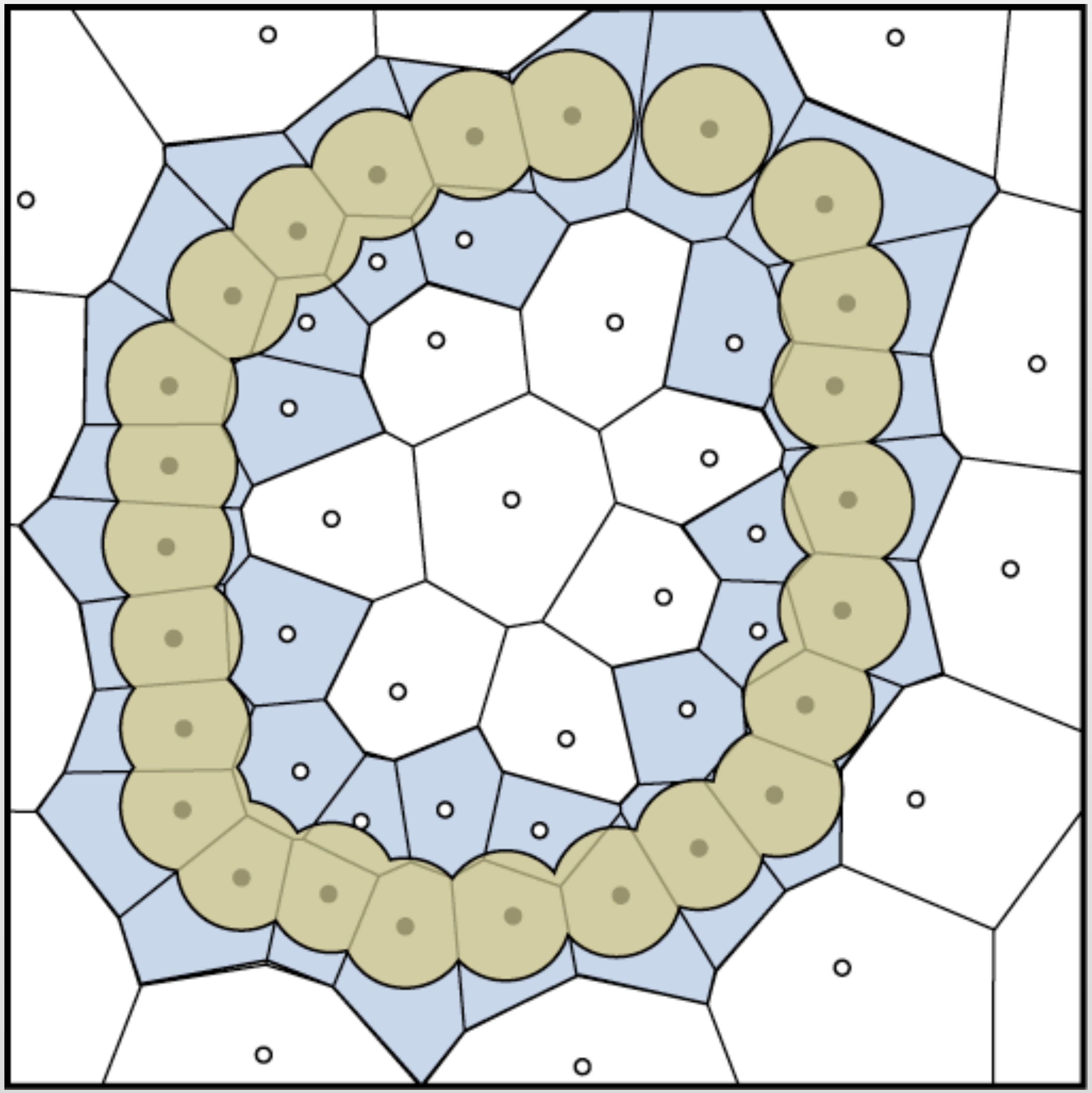

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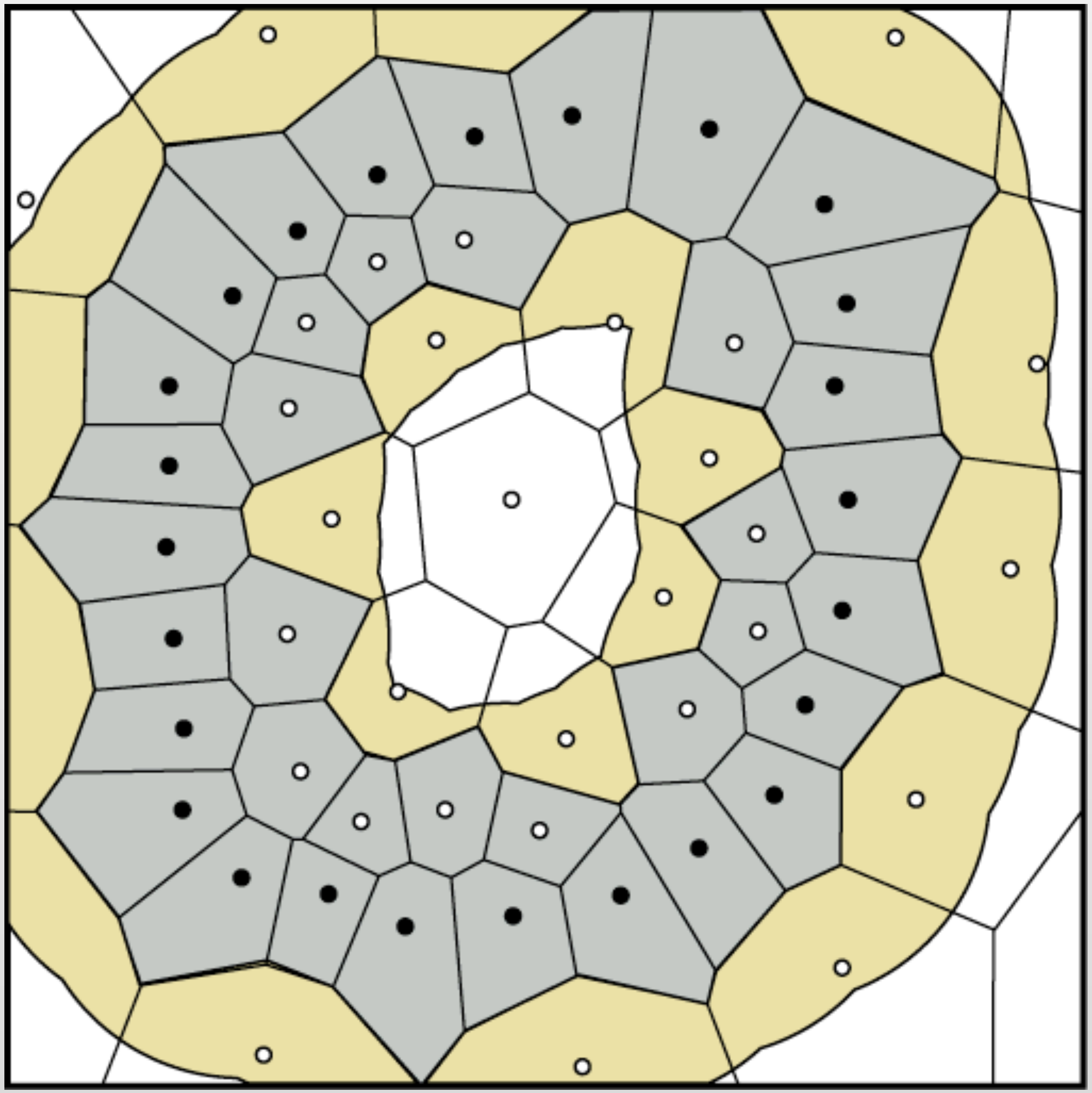

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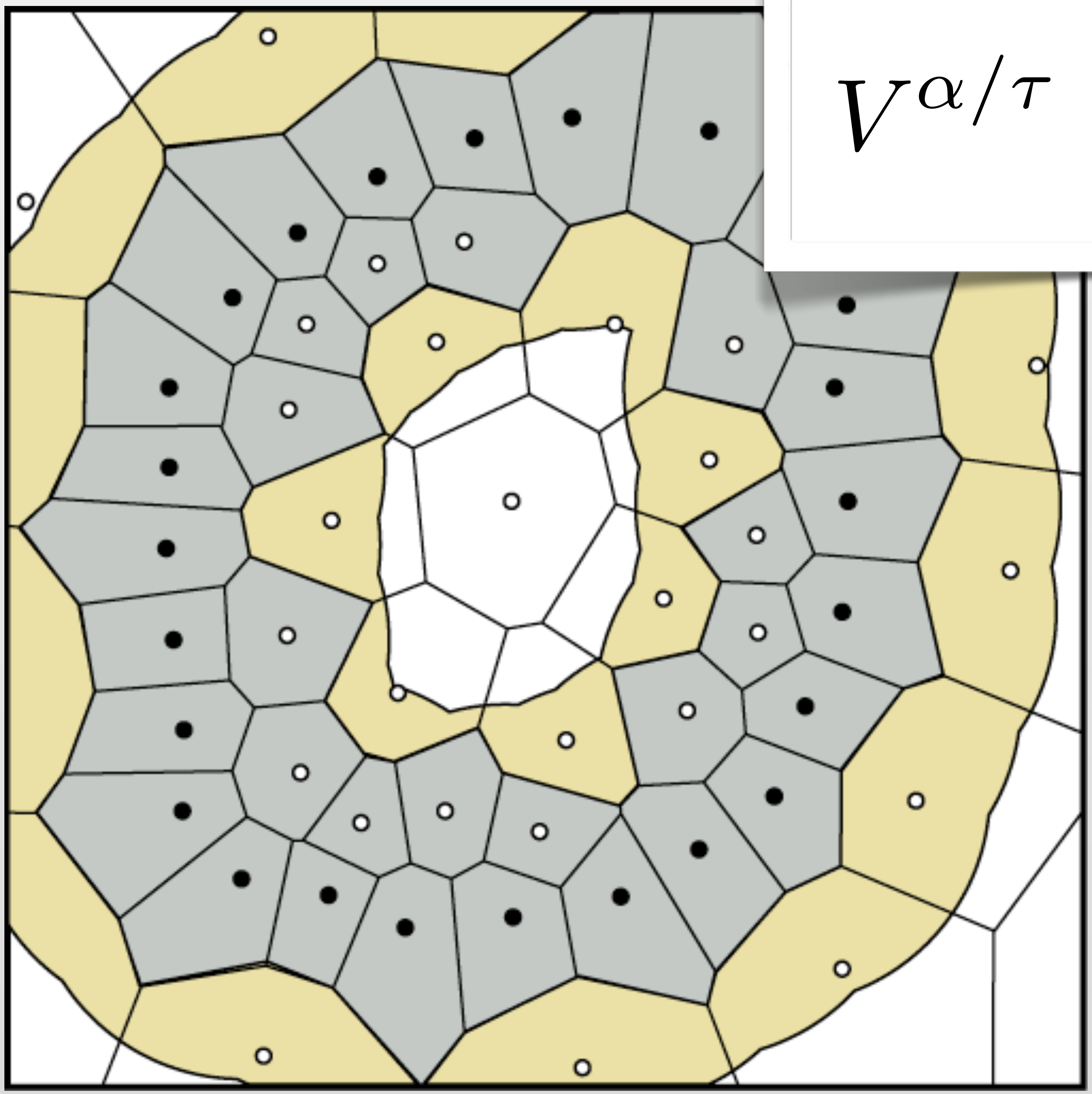

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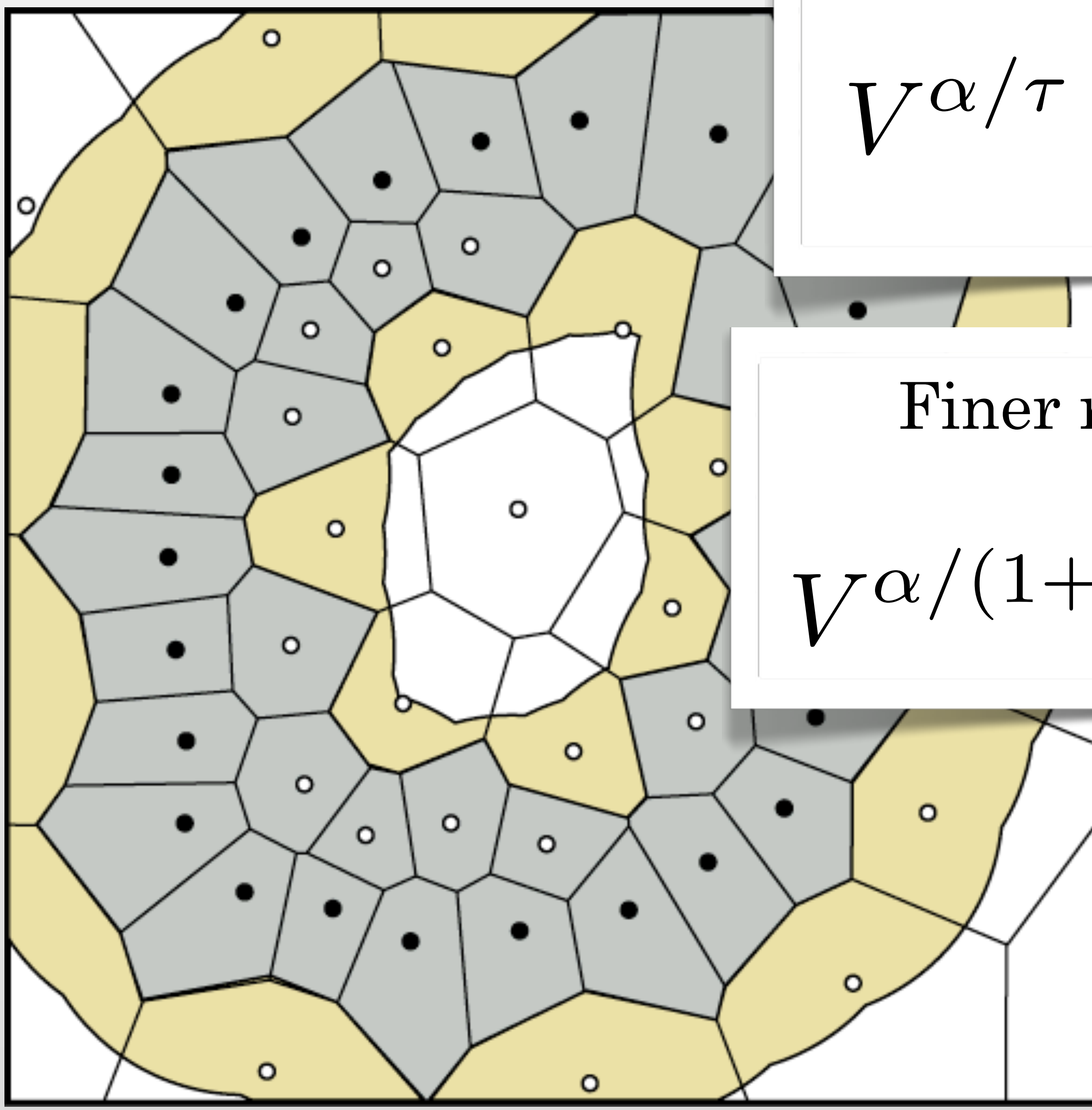

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Finer refinement yields a tighter interleaving.

$$V^{\alpha/(1+\varepsilon)} \subseteq P^{\alpha} \subseteq V^{\alpha(1+\varepsilon)}$$

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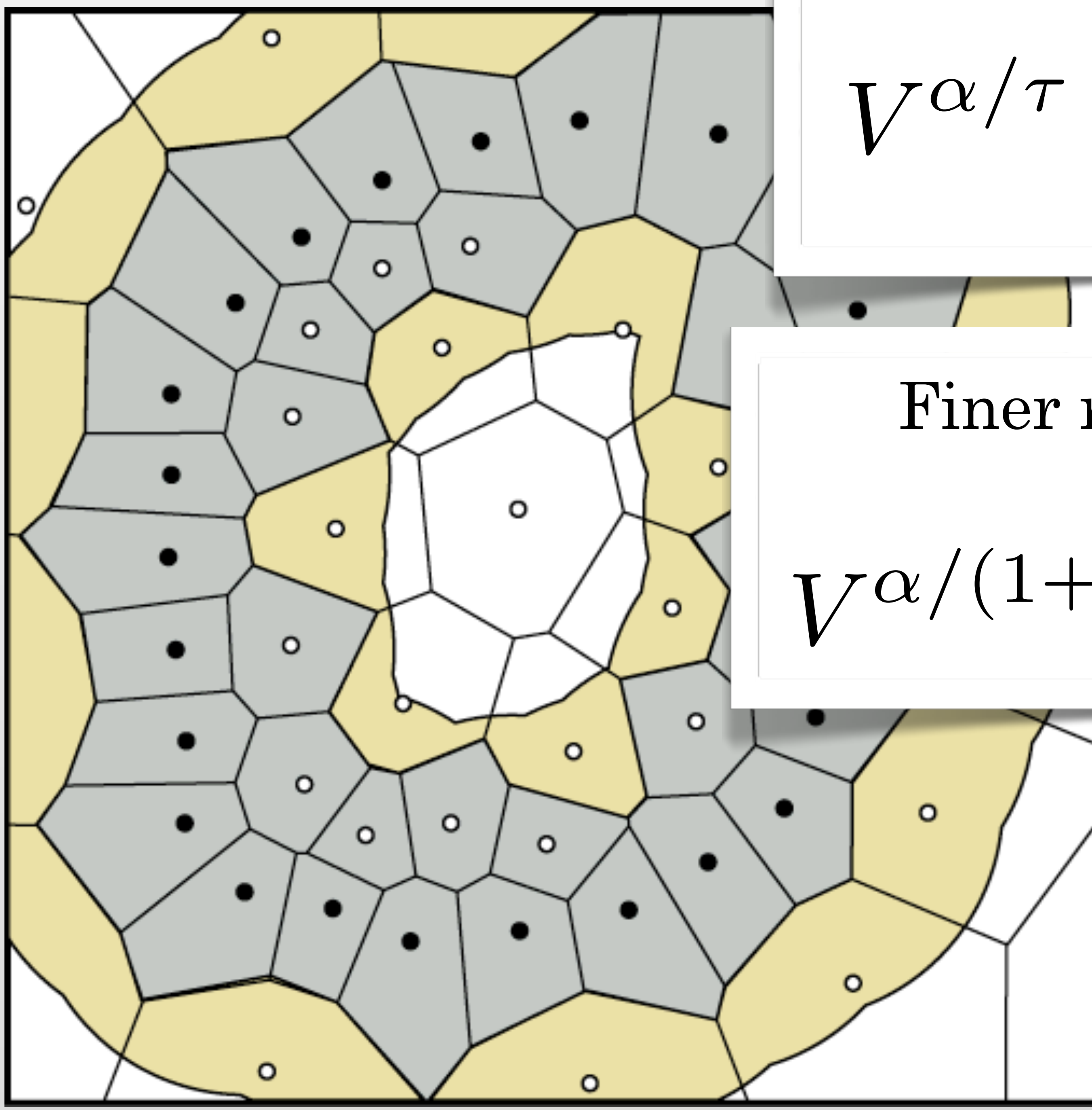

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caveat: Special case for small scales.

The Not Obvious.

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1 Why meshing is not obviously a good idea for TDA.

The Not Obvious.

1. Why meshing is not obviously a good idea for TDA.

2. Why meshing is a nonobvious good idea for TDA.

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image credit: Pointwise

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On the other hand, meshing for FEA often has stricter requirements than those for TDA.

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- Approximate gradients

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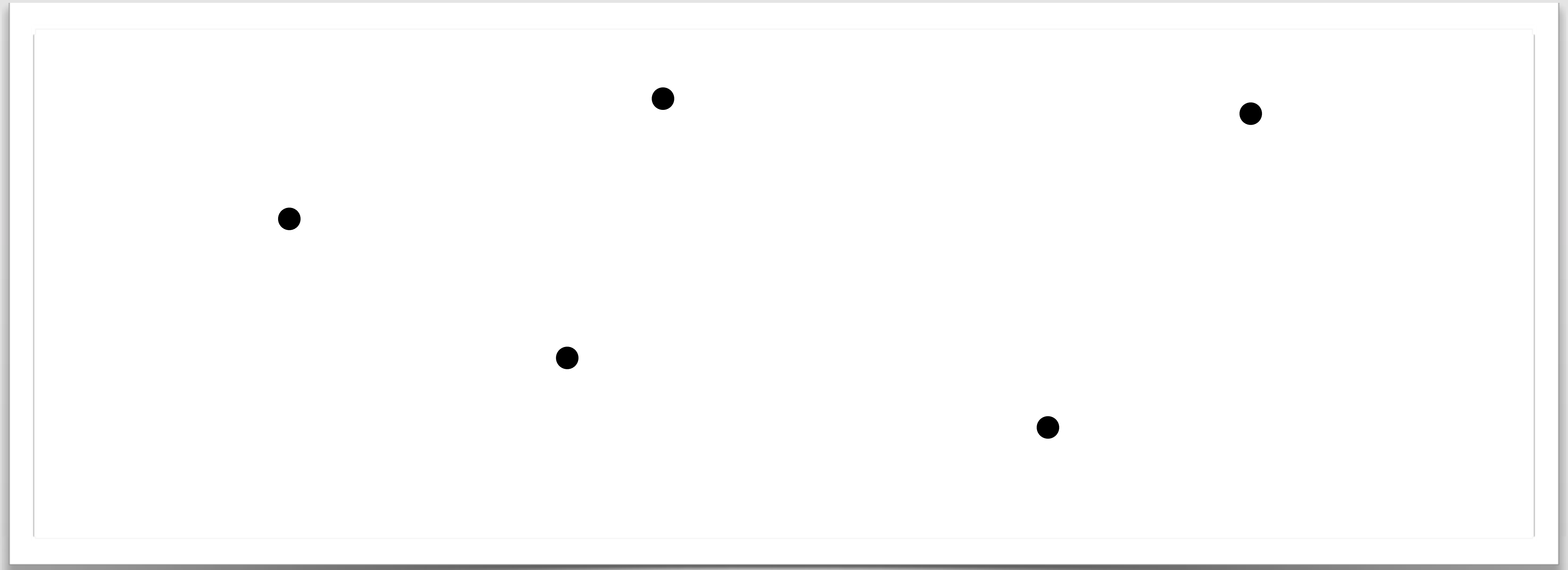

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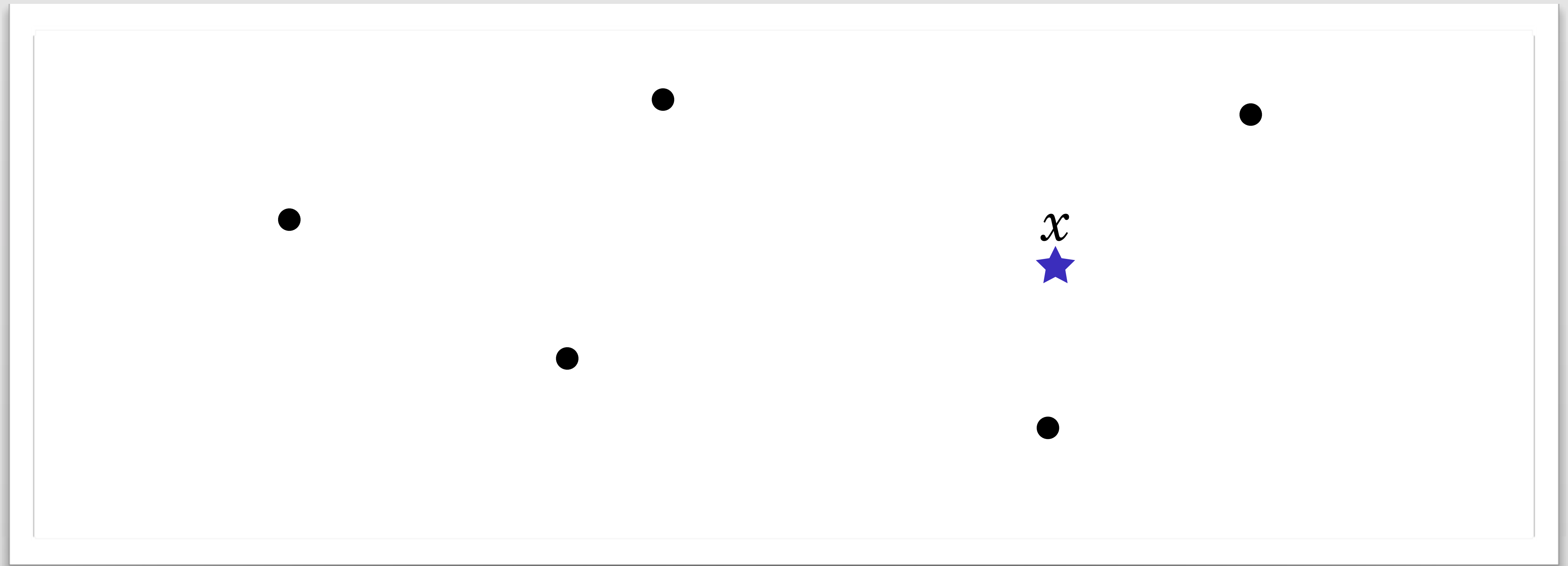

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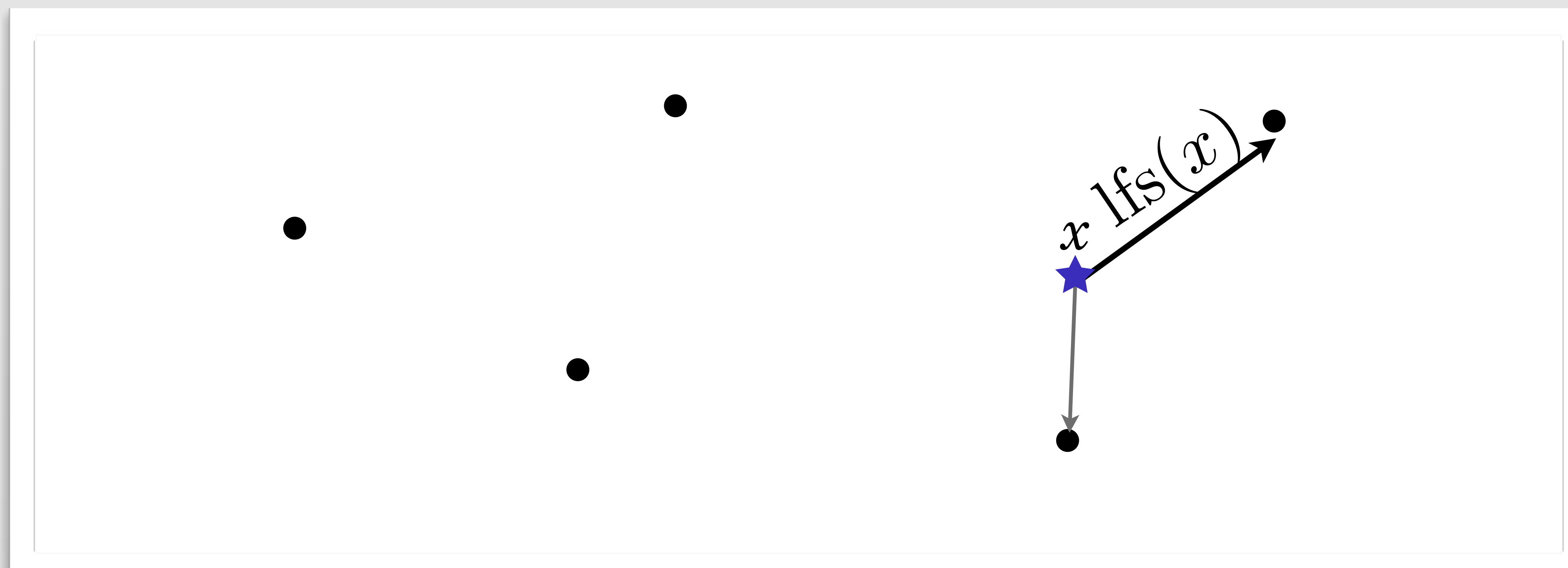

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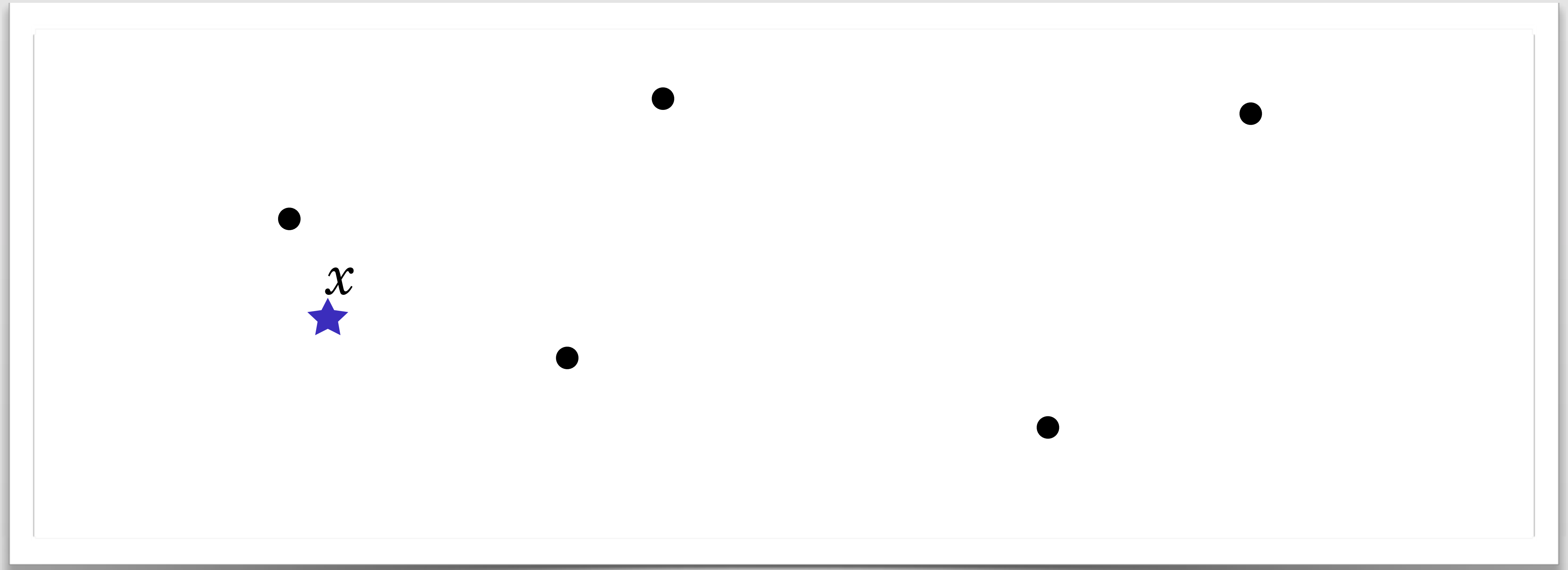

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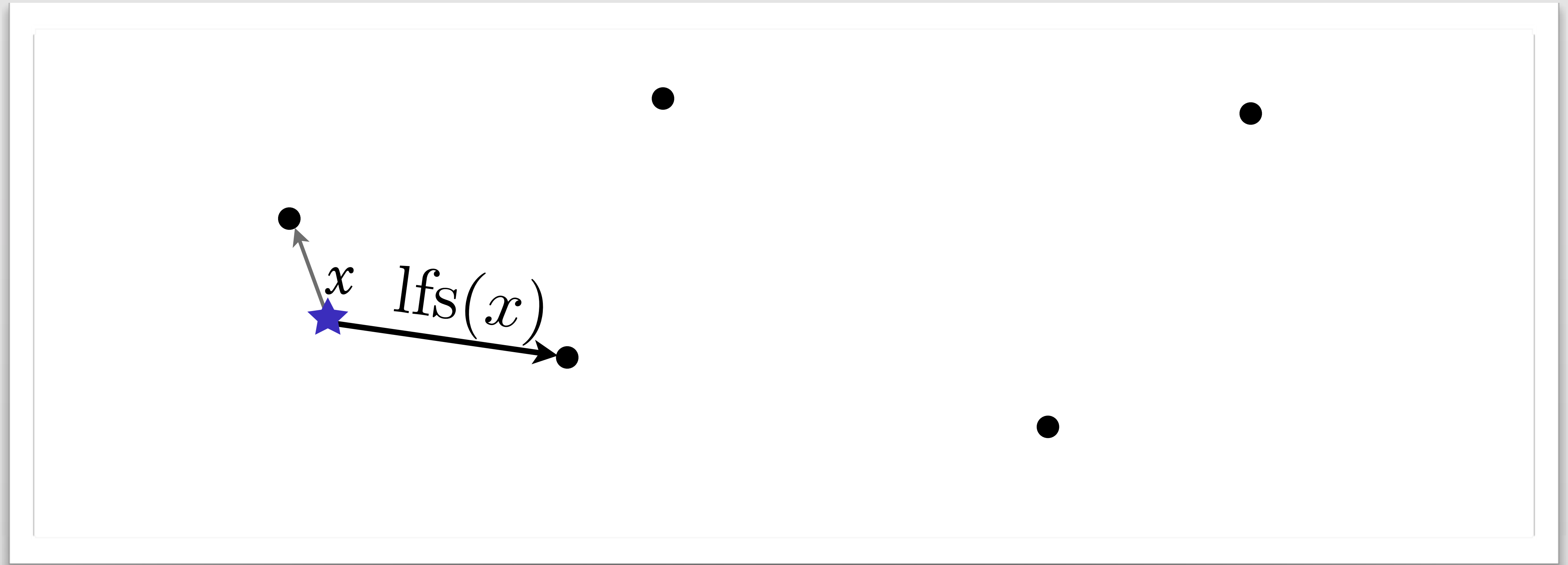

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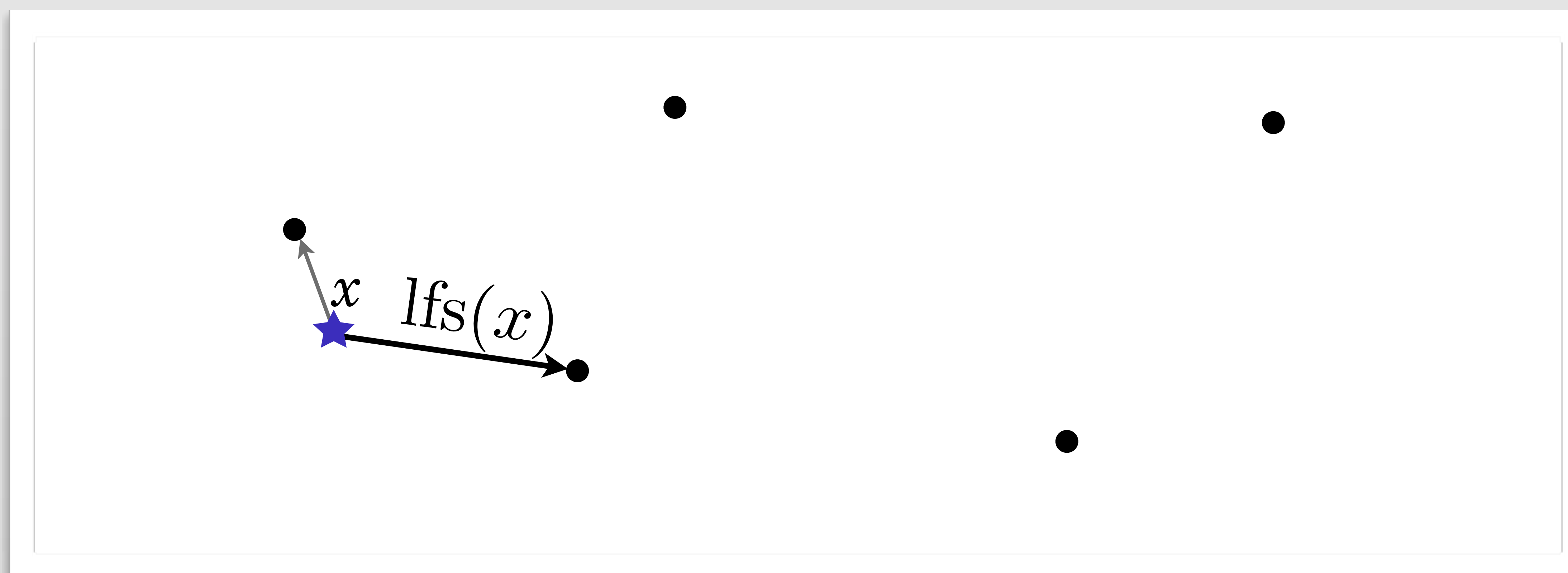

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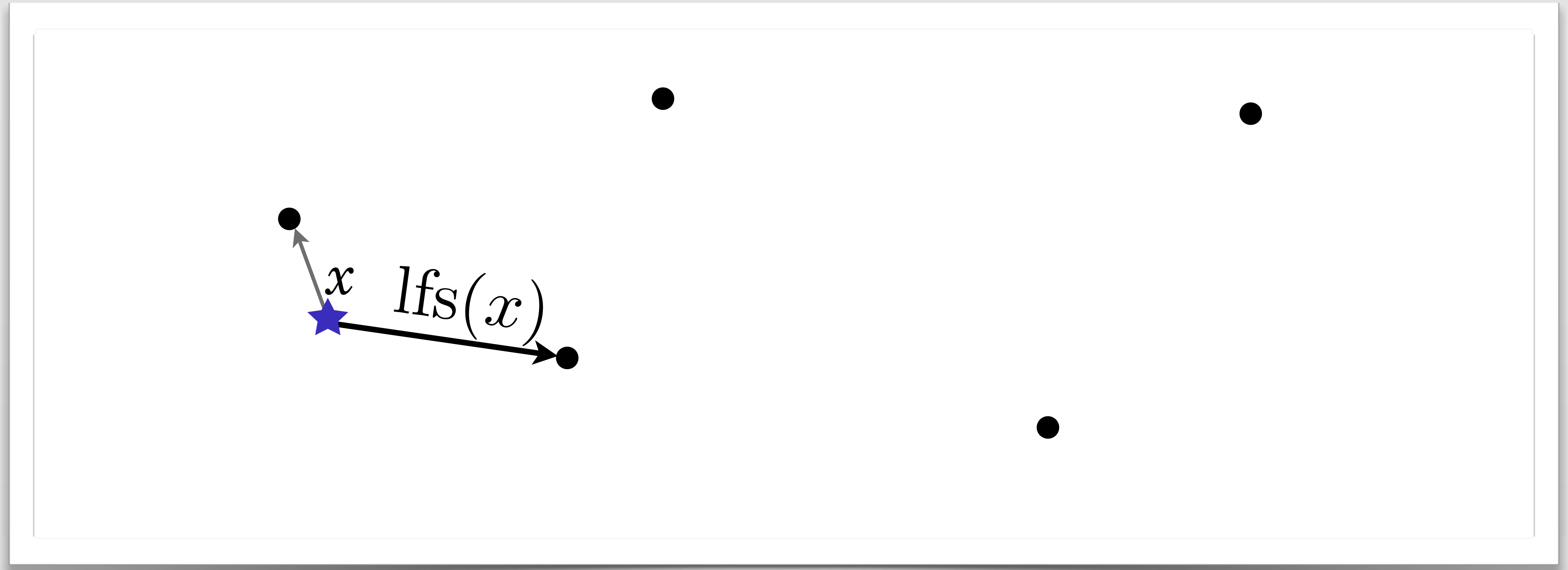

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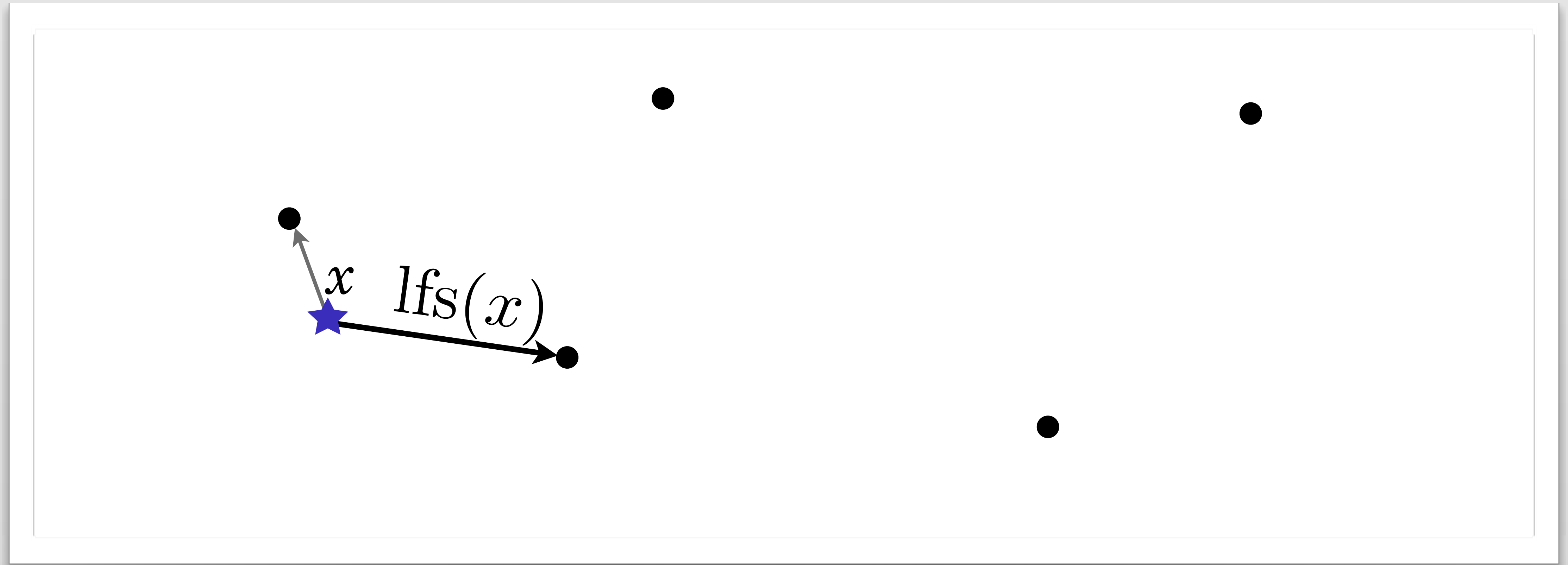

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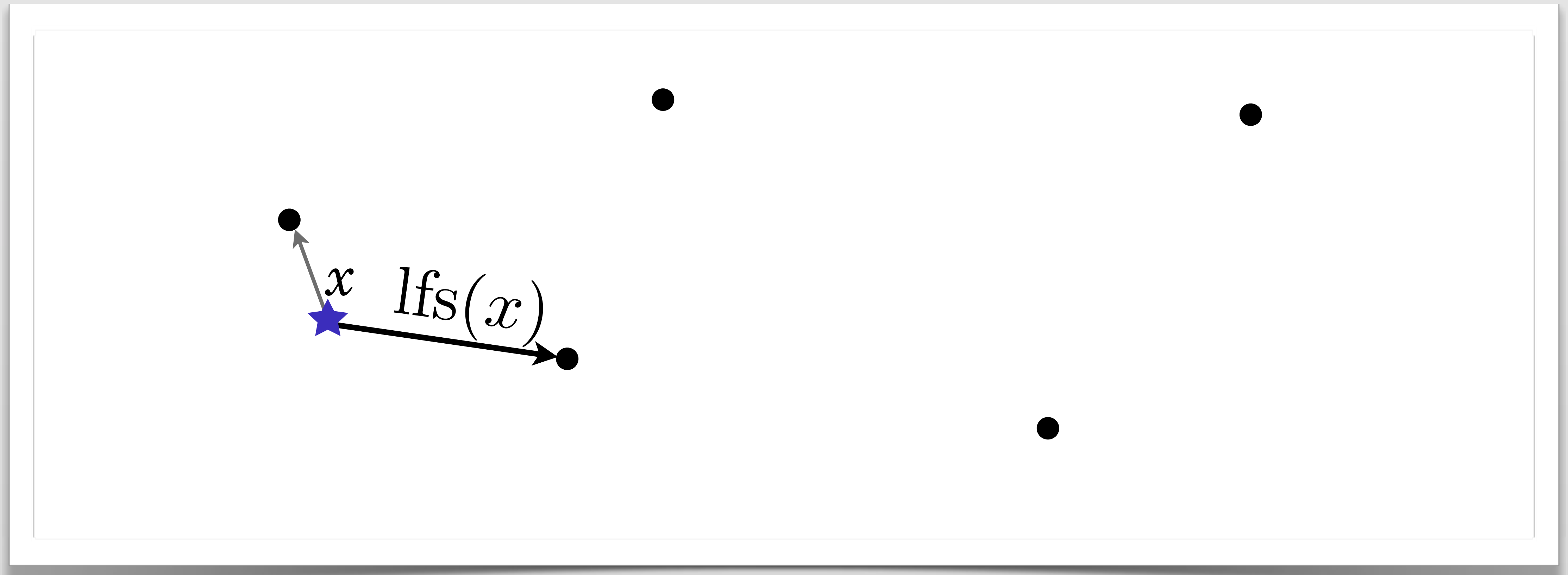

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When is $\mu_P(\Omega) = O(n)$?

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Pacing and the Empty Annulus Condition [S. 2012]

The Main Approximation Theorem

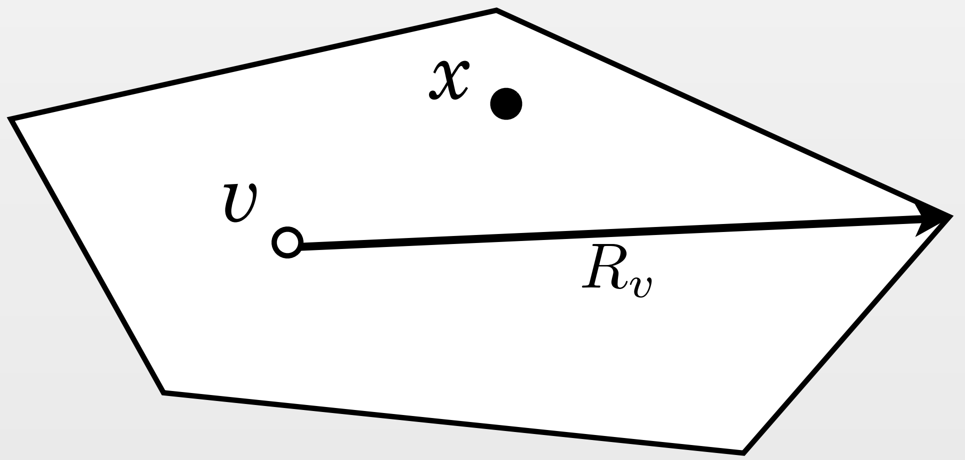
Theorem. *Let P be a point cloud and let M be an ε-refined mesh of P. Let $f \geq \frac{1}{c}\mathrm{lfs}_P$ be a t-Lipschitz function $\mathbb{R}^d \to \mathbb{R}$ for some constant $c > 0$. Let $\mathcal{F}$ be the sublevel filtration of f and let $\mathcal{V}$ be the Voronoi filtration of f on M. Then* $\mathrm{Dgm}\,\mathcal{V}$ *is a* $\left(1 + \frac{ct\varepsilon}{1-\varepsilon}\right)$*-approximation to* $\mathrm{Dgm}\,\mathcal{F}$.

Proof of the main theorem

Fix any $v \in M$ and any $x \in \mathrm{Vor}(v)$

$$f \geq \frac{1}{c}\mathrm{lfs}$$

$$R_v \leq \varepsilon \mathrm{lfs}(v)$$



$$\|v - x\| \leq R_v \leq \varepsilon \mathrm{lfs}(v) \leq c\varepsilon f(v)$$

$$f(x) \leq f(v) + t\|v - x\| \leq (1 + ct\varepsilon) f(v)$$

$$\forall v \in M, \mathrm{Vor}(v) \subseteq F_{(1+ct\varepsilon)f(v)}$$

$$\forall \alpha \geq 0, V_\alpha \subseteq F_{(1+ct\varepsilon)\alpha}$$

$$\|v - x\| \leq \frac{c\varepsilon}{1-\varepsilon} f(x)$$

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$$\forall \alpha \geq 0, F_\alpha \subseteq F_{\left(1+\frac{ct\varepsilon}{1-\varepsilon}\right)\alpha}$$

$$\forall \alpha \geq 0, \quad V_{\alpha/\gamma} \subseteq F_\alpha \subseteq V_{\alpha\gamma} \quad \text{where } \gamma = 1 + \frac{ct\varepsilon}{1-\varepsilon}$$

Dgm $\mathcal{V}$ is a γ-approximation to Dgm $\mathcal{F}$.

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Observations:
M can be made to have size *O(n)*.
The construction of *M* does not depend on *f*.
One mesh works for many functions.

We know how to do many things with meshes.

(Some of these should be useful for TDA).

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Dealing with anisotropy.

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Geometric Separators

Mesh generation is a natural preprocess for TDA in low-dimensional Euclidean space.

Thank You!