

Approximate Centerpoints with Proofs*

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Abstract

We present the ITERATEDTVERBERG algorithm, the first deterministic algorithm for computing an approximate centerpoint of a set $S \in \mathbb{R}^d$ with running time sub-exponential in d . The algorithm is a derandomization of the ITERATEDRADON algorithm of Clarkson et al and is guaranteed to terminate with an $O(1/d^2)$ -center. Moreover, it returns a polynomial-time checkable proof of the approximation guarantee, despite the coNP-Completeness of testing centerpoints in general. We also explore the use of higher order Tverberg partitions to improve the runtime of the deterministic algorithm and improve the approximation guarantee for the randomized algorithm. In particular, we show how to improve the $O(1/d^2)$ -center of the ITERATEDRADON algorithm to $O(1/d^{\frac{r}{r-1}})$ for a cost of $O((rd)^d)$ in time for any integer $r > 1$.

1 Introduction

A centerpoint of a set $S \subset \mathbb{R}^d$ is a point c such that every closed half-space containing c also contains at least $\frac{n}{d+1}$ points of S . Intuitively, every hyperplane through a c divides S into roughly equal parts. A centerpoint is a natural generalization of the median to higher dimensions. In fact, a centerpoint is an approximate median of the set $f(S)$ for every linear projection $f : S \rightarrow \mathbb{R}$. They are used as robust estimators in statistics, because they are invariant under affine transformation and robust to outliers [5]. They are also used in mesh partitioning[7].

The *Tukey depth* of a point x is the minimum number of points of S contained in a half-space containing x . Thus, a centerpoint is a point with Tukey depth at least $\frac{n}{d+1}$. More generally, c is a β -center if it has depth at least βn . We consider the problem of finding an approximate centerpoint. In particular we will give a deterministic algorithm for computing a $\frac{n}{2(d+1)^2}$ -center.

The Centerpoint Theorem asserts that every set n points in $\mathbb{R}^d$ has a centerpoint. This was first established by a theorem of Rado [14], which deals with general measures of which point sets are a special case. The more direct proof for the case of centerpoints of point sets is due to Danzer et al[4]. Centerpoints Theorem is tight in that there exist point sets for which no point has depth greater than $\frac{n}{d+1}$.

The existence of centerpoints can be proven directly from either of two classic theorems of convexity theory, Helly's Theorem and Tverberg's Theorem. In Section 3, we discuss how these two proofs of the centerpoint theorem lead to different perspectives for designing algorithms for computing centerpoints.

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The exact complexity of computing centerpoints in higher dimensions is not known. The decision problem of testing if a given point is a centerpoint is coNP-Complete [16]. However, a simple corollary of Tverberg’s Theorem guarantees the existence of a subset of centerpoints, call them Tverberg points, that admit polynomial-time checkable proofs. Moreover, testing if a point is a Tverberg point is NP-Complete[16]. In this case, the decision problem is well understood but sheds little light on the hardness of the search problem of actually finding a centerpoint.

The fastest known algorithm for computing a centerpoint of $S \subset \mathbb{R}^d$ is due to Chan [1] and computes a β -center in time $O(n^{d-1})$ in expectation, where β is the maximum achievable for the set S . Such a β -center is also known as a Tukey median.

The ITERATEDRADON algorithm of Clarkson et al was the first algorithm that computes an approximate centerpoint in time sub-exponential in d [2]. The algorithm computes a $O(1/d^2)$ -center with high probability. Section 3 describes how this algorithm resembles the proof of the Centerpoint Theorem via Helly’s Theorem.

The main operation in the ITERATEDRADON algorithm is to replace sets of points by their Radon point, a point in the common intersection of the convex hull of two disjoint subsets. Radon’s Theorem guarantees the existence of such a point. Tverberg’s Theorem is a generalization of Radon’s Theorem that guarantees a common intersection for a larger collection of subsets.

In this paper, we use the intuition from Tverberg’s Theorem to iteratively construct a proof of depth for an approximate centerpoint. The result is a new approximation algorithm, ITERATEDTVERBERG, that derandomizes the ITERATEDRADON algorithm of Clarkson et al. In Section 4, we prove that the ITERATEDTVERBERG algorithm produces a $O(1/d^2)$ -center in time sub-exponential in d . Moreover, the center comes with a polynomial-time checkable proof of its depth.

We elaborate on this intuition in Section 5, showing how solving larger sub-problems can be used to speed up the run time of the deterministic algorithm and to improve the approximation ratio of the randomized version.

2 Related Work

Centerpoints are the most well known definition of a geometric median [6]. Like many such medians, it can be computed via linear programming and the problem of finding a “best” centerpoint can be written as a maximum feasible subsystem problem (see [5] for a survey of computational aspects of data depth). The linear constraints are just the set of hyperplanes intersecting $d + 1$ points of S . There are $O(n^{d+1})$ of these. Consequently, any linear programming method will require time $n^{O(d)}$, limiting their usefulness to low-dimensional instances.

In the plane, centerpoints can be computed in linear time [9]. Several algorithms are known to compute centerpoints in $\mathbb{R}^3$ in $O(n^2 \text{ polylog } n)$ time [3, 13]. The best known algorithm for $d \geq 3$ is due to Chan and runs in $O(n^{d-1})$ randomized time[1]. Chan’s algorithm computes the deepest possible centerpoint, also known as a Tukey median. He conjectures that the $O(n^{d-1})$ runtime is optimal for this problem in the algebraic decision tree model. However, the exact complexity of computing centerpoints is not known. In particular, it is not known if it is possible to compute a centerpoint in time polynomial in n and d .

Several approximation algorithms for centerpoints exist in the literature. Many approaches using random sampling are known [11, 16, 2]. Verbarg showed that for dense points, the mean is a good approximate centerpoint[18]; it is a β -center where β depends on the density.

The only previously known algorithm to compute a centerpoint in time sub-exponential in d is the ITERATEDRADON algorithm of Clarkson et al [2]. The ITERATEDRADON al-

algorithm returns a $O(1/d^2)$ -center with high probability in time polynomial in n and d . ITERATEDRADON is a Monte Carlo algorithm and there is no way known to verify that the point returned by the algorithm is indeed a centerpoint. The inner loop of ITERATEDRADON depends on the following classic theorem [15].

Theorem 2.1 (Radon’s Theorem, 1921) *Given $n > d + 1$ points $S \subset \mathbb{R}^d$, there exists a partition $(U, \overline{U})$ of S such that $\text{conv}(U) \cap \text{conv}(\overline{U}) \neq \emptyset$.*

We call a partition of $d + 2$ points as described in the Theorem, a *Radon partition*, and we call a point in the intersection, a *Radon point*.

The simplest version of the ITERATEDRADON algorithm works as follows. Build a balanced $(d + 2)$ -ary tree of height h . Fill in the leaves with points from the input set S by sampling them uniformly at random. Each interior node of the tree is filled in with the Radon point of its children. A height of $h = \lg n$ is needed to compute a $O(1/d^2)$ -center with high probability, resulting in a runtime that is $O((d \lg n)^{\lg d})$. Thus it is sub-exponential in d but not polynomial.

Several different modifications to the basic ITERATEDRADON algorithm are presented in [2]. In one version, the running time is reduced to $O(\text{poly}(n, d))$ by reusing the Radon points the same way the input points are reused. Other variations of the algorithm incorporate linear programming to solve larger subproblems. These variants are able to produce $(1 + \varepsilon)\frac{n}{d+1}$ -centers. All of these results use random sampling and it is shown that the center of a sample is an approximate center for the whole set, allowing for sub-linear-time algorithms.

It is not known how to derandomize the other versions of ITERATEDRADON. We do, however, explore a variation on the linear programming based solution. We present a new way to leverage these larger subproblems and analyze the impact on both the randomized and the deterministic algorithms (see Section 5).

3 Two Proofs of the Centerpoint Theorem

Theorem 3.1 (The Centerpoint Theorem) *Given a set of n points $S \subset \mathbb{R}^d$, there exists a centerpoint $c \in \mathbb{R}^d$ such that every closed half-space containing c also contains at least $\frac{n}{d+1}$ points of S . (Rado [14], 1947, Danzer et al [4], 1963)*

The Centerpoint Theorem is most often presented as an easy consequence of Helly’s Theorem. It is also possible to prove the existence of centerpoints via Tverberg’s Theorem (see [12] for a comprehensive treatment). The relationship between these two proofs gives insight into the relationship between the ITERATEDRADON algorithm and its derandomization presented in this paper.

Theorem 3.2 (Helly’s Theorem [8], 1913) *Given a collection of compact, convex sets $X_1, \dots, X_n \subset \mathbb{R}^d$. If every $d + 1$ of these sets have a common intersection, then the whole collection has a common intersection.*

The Centerpoint Theorem follows from Helly’s Theorem as follows. Consider the set H of all open half-spaces that contain more than $\frac{dn}{d+1}$ points of S . For each such half-space $h \in H$, let P_h denote $\text{conv}(S \cap h)$, a compact, convex set containing the same points of S as h . Any $d + 1$ of the half-spaces have a common intersection at one of the points of S , and thus every $d + 1$ of the P_h ’s also have a common intersection. We apply Helly’s Theorem to the family of sets $P_H = \{P_h \mid h \in H\}$. The common intersection guaranteed by Helly’s Theorem is exactly the set of all centerpoints.

The most common elementary proof of Helly’s Theorem makes extensive use of Radon’s Theorem, despite that Helly’s Theorem technically came first (though published second). The proof first considers the case where there are only $d+2$ sets. The

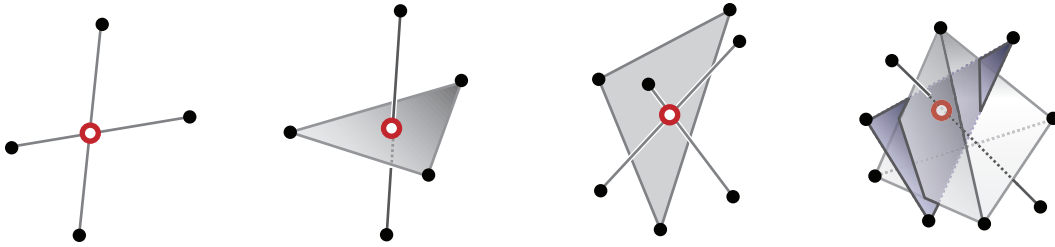


Figure 1: Radon points for $\mathbb{R}^2$ and $\mathbb{R}^3$. Tverberg points for $r = 3$ in $\mathbb{R}^2$ and $\mathbb{R}^3$.

hypothesis of the Theorem implies the existence of $d + 2$ points, each taken from the common intersection of $d + 1$ of the sets. The Radon point of these $d + 2$ points satisfies the conclusion of the Theorem. The proof for $n > d + 2$ sets uses induction. The inductive step again considers a set of points taken from each of the common intersections of $n - 1$ sets, and shows the Radon point of this set satisfies the Theorem. Unraveling this induction into an algorithm, we arrive at something very much like the ITERATEDRADON algorithm except with far too many Radon point computations to be computationally feasible. This is why ITERATEDRADON uses random sampling used to get avoid paying for the combinatorial blowup in the number of sets. The ITERATEDTVERBERG algorithm we present shows how to make this sampling deterministic (Section 4).

The Centerpoint Theorem can also be proven via Tverberg's generalization of Radon's Theorem.

Theorem 3.3 (Tverberg's Theorem [17], 1966) *Given $(d + 1)(r - 1) + 1$ points $S \subset \mathbb{R}^d$, there exists a partition of S into $S_1, \dots, S_r$, such that $\bigcap_{i=1}^r \text{conv}(S_i) \neq \emptyset$.*

Observe that Radon's Theorem is a special case of Tverberg's Theorem when $r = 2$.

Say that a point c is a Tverberg point if it is in the common intersection of the convex hulls in the Tverberg partition. Then, setting $r = \lceil n/d + 1 \rceil$ yields a Tverberg point contained in the convex hull of $\lceil n/d + 1 \rceil$ pairwise disjoint subsets of S . Any half-space containing c must also contain at least one point from each of the subsets and therefore, c is a centerpoint.

Observe that the centerpoints guaranteed by Tverberg's Theorem come equipped with a polynomial-time checkable proof. Given the partition, we need only verify that the point is in the convex hull of each part. If any part in the partition has more than $d + 1$ points then by Carathéodory's Theorem, there is a subset of size $d + 1$ that contains the Tverberg point in its convex hull. We may therefore assume the convex hulls are simplices of dimension at most d , so checking can be done quickly. The key insight in derandomizing the ITERATEDRADON algorithm is to actively construct these Tverberg partitions for the intermediate points used in the algorithm.

4 Derandomizing the IteratedRadon algorithm

The ITERATEDTVERBERG algorithm looks very similar to the ITERATEDRADON algorithm. The key difference is that each successive approximation computed along the way carries with it a proof of its quality as a centerpoint. The proof is in the form of a Tverberg partition of a subset of the inputs. Define the *depth* of a Tverberg point to be the number of parts in the corresponding Tverberg partition. The depth is at least the Tukey depth.

When we combine $d + 2$ points of depth r into a Radon point c , we can rearrange the proofs to get a new proof that c has depth $2r$ as shown in the following Lemma.

Lemma 4.1 *Given a set P of $d + 2$ Tverberg points of depth at least r with disjoint partitions, the Radon point of P has depth at least $2r$.*

Proof: Let (P_1, P_2) be the Radon partition for P , and let c be the Radon point. For each $p_i \in P$, order the parts in the proof partition of p_i and call the j th part $U_{i,j}$. We build a proof that c has depth at least $2r$. The parts in the new proof are of the form $\bigcup_{p_i \in P_k} U_{i,j}$ for $k \in \{1, 2\}$ and $j \in \{1, \dots, r\}$.

To show that the new proof is correct, it suffices to show that for any choice of j and k , the new approximation c is contained in $\text{conv}(\bigcup_{p_i \in P_k} U_{i,j})$. What follows is the long proof of the intuitive statement that a convex combination of convex combinations is itself a convex combination of the base set.

Because c is a Radon point, we know that $c \in \text{conv}(P_k)$. Also, the Tverberg points $p_i \in P_k$ are each contained in $\text{conv}(U_{i,j})$. So, we can write $c = \sum_{p_i \in P_k} \lambda_i p_i$ and $p_i = \sum_{u_m \in U_{i,j}} \alpha_m u_m$, where $\sum \lambda_i = \sum \alpha_m = 1$ and $\lambda_i, \alpha_m \geq 0$. Combining these two convex combinations, we see that

$$c = \sum_{p_i \in P_k} \lambda_i \sum_{u_m \in U_{i,j}} \alpha_m u_m \quad (1)$$

$$= \sum_{u_m \in \bigcup_{p_i \in P_k} U_{i,j}} \lambda_i \alpha_m u_m. \quad (2)$$

To show that this is indeed a convex combination, we note that

$$\sum_{i,m} \lambda_i \alpha_m = \sum_i \lambda_i \left(\sum_m \alpha_m \right) = \sum_i \lambda_i (1) = 1.$$

■

The preceding Lemma implies a simple deterministic algorithm for computing an approximate centerpoint. Construct a $(d + 2)$ -ary tree with n leaves. Fill the leaves with the points of S . Fill in each interior node of the tree by the Radon point of its children. The height of the tree is $\log_{d+2} n$, so Lemma 4.1 implies that the depth of the root is $2^{\log_{d+2} n} = O(n^{1/\lg(d+2)})$. Not too shabby for such a simple algorithm, but the depth of the output is only sub-linear in n . To get an $O(\frac{1}{d^2})$ -center, we need to find a way to build this tree higher, and in order to do that, we need more leaves. The following Lemma gives a hint as to where we can look to find some more points to stick in the leaves.

Lemma 4.2 *If there is a proof that a point p has depth r , there exists such a proof that contains at most $r(d + 1)$ points of S .*

Proof: Let $P_1, \dots, P_r$ be the sets in the proof for p . This means that $p \in \text{conv}(P_i)$ for each $i = 1 \dots r$. By Carathéodory's Theorem, there exists a subset $P'_i \subset P_i$ of at most $d + 1$ points such that $p \in \text{conv}(P'_i)$. So, the sets $P'_1, \dots, P'_r$ is the desired proof of the correct size. ■

We refer to this economizing of proofs as *pruning*. In the algorithm, pruning is applied to the proofs generated by combining smaller proofs as in Lemma 4.1. In such instances, the convex combination is known. Moreover, if the combined proofs were each pruned, then the total number of points in the combined sets is at most $(d + 1)^2$, and the pruned set can be found by subtracting $O(d^2)$ affine dependences, each requiring a linear system solve at a cost of $O(d^3)$ time. So, each pruning operation takes $O(d^5)$ time.

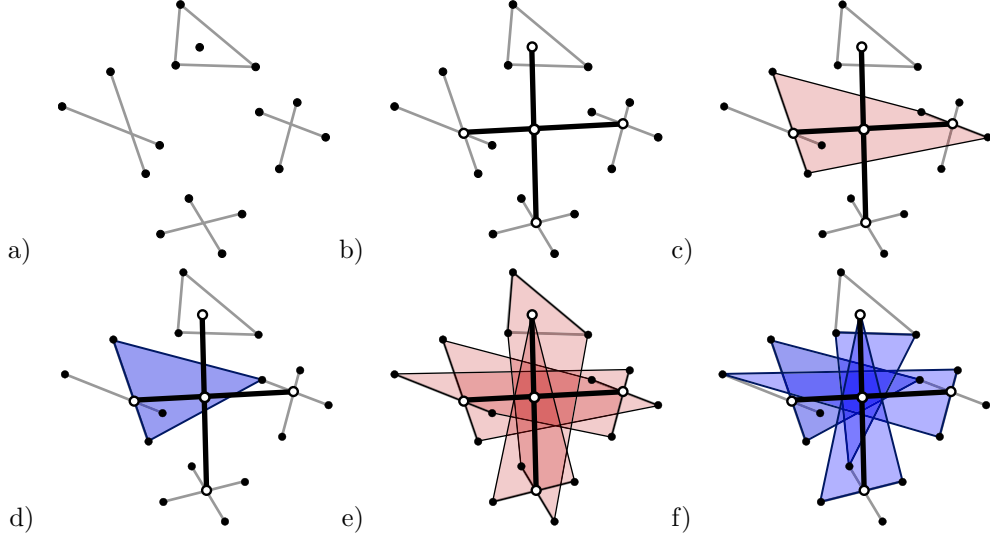


Figure 2: The ITERATEDTVERBERG algorithm: (a) Sets of $d + 2$ points are divided into Radon partitions. (b) $d + 2$ Radon points are combined into a second-order Radon partition. (c) A proof polygon is formed by taking the convex hull of two subpartitions, one from each of the Radon points in the second order partition. (d) The proof polygon is pruned to a simplex. (e) All of the proof polygons before the pruning phase. (f) The proof simplices after the reduction.

Algorithm 1 ITERATEDTVERBERG

Input: $S \in \mathbb{R}^d : |S| = n$
while $B_{\lceil \frac{n}{2(d+1)^2} \rceil}$ is empty **do**
 Let i be the max such that bucket B_i has at least $d + 2$ points
 Pop $d + 2$ points $q_1, \dots, q_{d+2}$ from B_i
 Let $\langle c, \{U_1, U_2\} \rangle = \text{RADON}(q_1, \dots, q_{d+2})$
 Let S_{jk} be the k th part of the proof for q_j
for $\ell = 1, 2$ **do**
for $k = 1, \dots, 2^i$ **do**
 Add $\text{PRUNE}(c, \bigcup_{q_j \in U_\ell} S_{jk})$ to *proof*
 Push the excess points to B_0
end for
end for
 Push $\langle c, \text{proof} \rangle$ to B_{i+1}
end while
return any $\langle c, \text{proof} \rangle$ from $B_{\lceil \frac{n}{2(d+1)^2} \rceil}$

Theorem 4.3 *The ITERATEDTVERBERG algorithm always returns a β -center of depth at least $\left\lceil \frac{n}{2(d+1)^2} \right\rceil$.*

Proof: Clearly, if the algorithm terminates then it returns a point of the desired depth and its corresponding proof. Suppose for contradiction that the algorithm does not terminate. Then, at some time t , it is impossible to pick i because every bucket has fewer than $d+2$ points. Let h be the maximum such that B_h is nonempty at time t . For any point p in a bucket B_i , the proof for p contains at most $2^i(d+1)$ input points. Since all of the proofs in all of the buckets are disjoint, we have

$$\sum_{i=0}^h 2^i(d+1)^2 \geq n. \quad (3)$$

Therefore,

$$2^{h+1} - 1 \geq \frac{n}{(d+1)^2}, \quad (4)$$

and thus,

$$2^h > \frac{n}{2(d+1)^2}. \quad (5)$$

However, this is impossible because it implies that $h > \lg \frac{n}{2(d+1)^2}$, and therefore $B_{\lg \left\lceil \frac{n}{2(d+1)^2} \right\rceil}$ is nonempty. This would have caused the algorithm to break out of the while loop and terminate, contrary to our assumption. ■

Theorem 4.4 *The ITERATEDTVERBERG algorithm requires $O\left(\frac{n^{1+\lg(d+2)}}{(d+1)^{2\lg(d+2)-1}}\right)$ time.*

Proof: Let t_k be the time required to compute a point of depth 2^k . In the base case $t_0 = 1$. Let R be the time to compute the Radon point and partition for $d+2$ points in $\mathbb{R}^d$. Let P be the time to Prune a set of proofs where the base set has at most $(d+1)^2$ points.

We bound t_n as follows.

$$t_k \leq (d+2)t_{k-1} + R + 2^k P \quad (6)$$

$$\leq (d+2)^k + \sum_{i=0}^{k-1} (d+1)^i (R + 2^k P) \quad (7)$$

$$\leq (d+2)^k \left(1 + \frac{R + 2^k P}{d+1}\right) \quad (8)$$

Observe that $R = O(d^3)$ and $P = O(d^5)$ by naïve algorithms for computing Radon partitions and Carathéodory subsets respectively. By Theorem 4.3, the final depth is $\frac{n}{2(d+1)^2}$, so we are interested in $k = \lg \frac{n}{2(d+1)^2}$. Plugging in these values yields the following.

$$t_k \leq (d+2)^{\lg \frac{n}{2(d+1)^2}} (d^3(n+1)/(d+1) + 1) \quad (9)$$

$$= O\left(\frac{n^{1+\lg(d+2)}}{(d+1)^{2\lg(d+2)-1}}\right) \quad (10)$$

■

5 Leveraging larger subproblems

The Radon point of a set of $d+2$ points has Tukey depth at least 2 and, thus, is a centerpoint for that set. Both the ITERATEDRADON algorithm and the ITERATEDTVERBERG algorithm replace sets of $d+2$ points with their centerpoint using Radon's Theorem. In this section we address the result on these algorithms if we instead solve larger subproblems. Rather than combining points in sets of $d+2$, we look at sets of size $(d+1)(r-1)+1$ for some fixed r . The centerpoint of such a set has Tukey depth at least r . It is not known how to solve these larger problems in time sub-exponential in d . However, if n is large and d is not too large, it may be feasible to solve subproblems in $O(n^d)$ time even though $O(n^d)$ is prohibitive. In fact, we will see that this can even lead to an asymptotic speedup for the ITERATEDTVERBERG algorithm.

5.1 Improving the approximation for the IteratedRadon Algorithm

The Radon point of $d+2$ points is a centerpoint of the subset. Consider the following modified version of the ITERATEDRADON Algorithm.

We can run the same iterative algorithm as before except using r -partitions instead of 2-partitions. In fact, it is not necessary to keep around the partition, it actually suffices just to find any centerpoint of $(d+1)(r-1)+1$ points at each round. We can go through the analysis from [2] and see the impact of r in the quality of the centerpoint achieved.

The analysis works by looking at any projection of the point set to the line. We compute the probability $f_h(x)$ that the tree of iterations with height h returns a center of depth at most x . Without loss of generality, the projections of the points of S land on $\frac{1}{n}, \frac{2}{n}, \dots, 1$. It follows that $f_0(x) \leq x$. The quality of the center will be nx where x is such that $f_h(x)$ is very small.

At each iteration, the centerpoint is at least r deep in the projection. There are $\binom{(d+1)(r-1)+1}{r}$ choices for the r points less than the centerpoint in the projection. By the union bound,

$$f_h(x) \geq \binom{(d+1)(r-1)+1}{r} f_{h-1}(x)^r. \quad (11)$$

$$\text{Say, } \beta = \binom{(d+1)(r-1)+1}{r}^{-1}.$$

$$f_h(x) \geq \beta^{-1} f_{h-1}(x)^r \quad (12)$$

$$\geq \beta^{-1} (\beta^{-r} f_{h-2}(x)^{r^2}) \quad (13)$$

$$\geq \beta^{-1} \cdot \beta^{-r} \dots \beta^{-r^h} f_0(x)^{r^h} \quad (14)$$

$$\geq \beta^{\frac{1-r^h}{r-1}} x^{r^h} \quad (15)$$

$$\geq \beta^{\frac{1}{r-1}} \left(\frac{x}{\beta^{\frac{1}{r-1}}} \right)^{r^h} \quad (16)$$

Now, since $\beta = O((rd)^{-r})$, we can choose x smaller than $O((rd)^{\frac{-r}{r-1}})$ and the probability $f_h(x)$ vanishes as desired. So, even for a choice of $r = 3$, we can improve the quality of the resulting centerpoint by a $O(\sqrt{d})$ factor.

5.2 Speeding up the IteratedTverberg Algorithm

In this section, we show how the same trick of solving larger subproblems can speed up the run time of the deterministic algorithm. Tverberg’s Theorem guarantees the existence of a partition of S into r sets whose convex hulls have a common intersection as long as $|S| > (d+1)(r-1)+1$. Say $T(r)$ is the time required to compute a Tverberg partition into r parts. To the best of our knowledge, nothing better than brute force is known for computing Tverberg partitions for $r > 2$.

We will show that a slight modification to the ITERATEDTVERBERG algorithm to use Tverberg r -partitions instead of Radon partitions results in a $n^{\lg r}/T(r)$ speedup. Thus, for n large enough, we get an asymptotic speedup.

The modified algorithm simply makes recursive calls on sets of $\lceil n/r \rceil$ points and combines them in sets of $(r-1)(d+1)+1$. The analysis is virtually identical to the original version except we give up a factor of $r/2$ in the depth of the output. As for the running time, the new algorithm now has a recursion tree with higher fan out and the resulting run time is $O((d+2)^{\lg_r n} T(r)) = O(n^{\lg(d+2)/\lg r} T(r))$.

6 Conclusions and Open Problems

We have presented the ITERATEDTVERBERG algorithm, the first algorithm that deterministically computes an approximate centerpoint in time sub-exponential in d . By combining intuition from both Helly’s Theorem and Tverberg’s Theorem, our method sheds an interesting new light on the problem of computing centerpoints. It still remains open whether it is possible to compute approximate centerpoints deterministically in time polynomial in n and d . We conjecture that it is.

We also extended both our algorithm and the ITERATEDRADON algorithm by looking at the impact of solving larger subproblems. One consequence of this work is that any new results on quickly computing centerpoints for small point sets can be used to improve these algorithms. Currently, it is not known how to compute centerpoints of more than $2d+2$ points in time polynomial in d . However, we conjecture that computing the centerpoint of $2d+3$ points in $\mathbb{R}^d$ is NP-hard.

In the ITERATEDRADON algorithm, it was not clear if the factor of d appearing in the output depth approximation ratio was intrinsic to the problem or merely an artifact of the analysis. This same factor of d shows up in our analysis of the ITERATEDTVERBERG algorithm in a completely different way, implying that perhaps it is intrinsic to the problem. Finding the optimal approximation ratio that can be achieved by a deterministic algorithm in sub-exponential time remains an open problem.

It is also open as to whether the ITERATEDTVERBERG algorithm can be significantly sped up by doing the pruning step lazily. In such a variant, the “extra” points that get pruned out would be points with proofs of their own, possibly allowing the reuse of more computation. Thus far we have been unable to analyze the lazy pruning version of the algorithm.

The reader is directed to the survey of Kalai [10] for many other interesting problems related to Tverberg points.

The computation of centerpoints draws a compelling correspondence between fundamental theorems in convexity theory, Helly’s Theorem and Tverberg’s Theorem, and fundamental complexity classes of NP and coNP. It is our hope that future work will further elucidate this correspondence.

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