SIAM J. SCI. COMPUT.
Vol. 33, No. 6, pp. 3201–3209

A NOTE ON THE CONVERGENCE OF SOR FOR THE PAGERANK PROBLEM*

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Abstract. A curious phenomenon when it comes to solving the linear system formulation of the PageRank problem is that while the convergence rate of Gauss–Seidel shows an improvement over Jacobi by a factor of approximately two, successive overrelaxation (SOR) does not seem to offer a meaningful improvement over Gauss–Seidel. This has been observed experimentally and noted in the literature, but to the best of our knowledge there has been no analytical explanation for this thus far. This convergence behavior is surprising because there are classes of matrices for which Gauss–Seidel is faster than Jacobi by a similar factor of two, and SOR accelerates convergence by an order of magnitude compared to Gauss–Seidel. In this short paper we prove analytically that the PageRank model has the unique feature that there exist PageRank linear systems for which SOR does not converge outside a very narrow interval depending on the damping factor, and that in such situations Gauss–Seidel may be the best choice among the relaxation parameters. Conversely, we show that within that narrow interval, there exists no PageRank problem for which SOR does not converge. Our result may give an analytical justification for the popularity of Gauss–Seidel as a solver for the linear system formulation of PageRank.

Key words. PageRank, linear systems, Gauss–Seidel, SOR

AMS subject classifications. 65F10, 65F15, 65C40

DOI. 10.1137/110823523

1. Introduction.

PageRank is a Web ranking method that barely needs an introduction. The "raw" PageRank x_i of page i is defined as $x_i = \sum_{j \to i} \frac{x_j}{n_j}$, where $j \to i$ indicates that page j links to page i, and n_j is the outdegree of page j. Thus the problem in its basic form can be mathematically formulated as follows: find a vector x that satisfies $x = \bar{P}x$, where

$$\bar{P}_{ji} = \begin{cases} \frac{1}{n_i} & \text{if } i \to j, \\ 0 & \text{if } i \to j. \end{cases}$$

Pages with no outlinks (or, in graph terminology, out-edges) produce columns of all zeros in $\bar{P}$; hence $\bar{P}$ is not necessarily a column-stochastic matrix. Eliminating the zero columns can be done by replacing $\bar{P}$ with $P = \bar{P} + ud^T$, where $d_i \neq 0$ if $n_i = 0$. Here the vector $u \geq 0$ is a probability vector, i.e., a vector with nonnegative elements summing to 1. This rank-one change of $\bar{P}$ is known as a *dangling node correction.* The PageRank vector, say, x, is defined as the vector satisfying

$$Ax = x, \tag{1.1}$$

where $A = \alpha P + (1 - \alpha)ve^T$, $\alpha \in (0, 1)$, is a *damping factor*, e is the vector of all ones, and v is another probability vector, often called a *teleportation vector*. The matrix

*Submitted to the journal's Methods and Algorithms for Scientific Computing section February 7, 2011; accepted for publication (in revised form) August 8, 2011; published electronically November 8, 2011. This work was supported in part by the Natural Sciences and Engineering Research Council of Canada.
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A is now column-stochastic; i.e., its entries are nonnegative real numbers, and all its columns sum to one. See, for example, [3] for a comprehensive description of the PageRank problem.

As pointed out in [1], since the PageRank vector x is nonnegative, the normalization $\|x\|_1 = e^T x = 1$ can be used to reformulate the eigenvalue problem (1.1) as a linear system:

$$(I - \alpha P)x = (1 - \alpha)v. \tag{1.2}$$

The linear system formulation (1.2) allows for applying a wealth of iterative methods. Because the problem is typically huge (the Web is estimated these days to have several tens of billions of pages), matrix decompositions are out of the question. Sophisticated solvers such as Krylov subspace methods are at a disadvantage, too, since they typically entail a nonnegligible overhead in terms of floating point operations (multiple inner products, and so on) and storage requirements (basis vectors for the Krylov subspace, and so on). Thus, for the linear system formulation of the PageRank problem, basic stationary schemes are very competitive.

The Richardson iteration applied to (1.2) is equivalent to the power method for the analogous eigenvalue problem (1.1), and Jacobi is similar to Richardson in this setting. Ample practical evidence suggests that Gauss–Seidel converges almost twice as fast as Jacobi [2] for (1.2). Gauss–Seidel does not lend itself easily to parallel implementations, but it is very effective in sequential settings and is often used as the method of choice for solving moderate-size PageRank problems.

Given this, the natural next step would be to apply the successive overrelaxation (SOR) scheme, which is still very inexpensive in terms of the computational cost of a single iteration and is known in some cases to be far superior to Gauss–Seidel in terms of overall computational efficiency [5, 6]. However, in the case of the PageRank problem it seems that SOR is typically not superior to Gauss–Seidel. As pointed out in [2, 4], ordering may play a significant role in the speed of convergence, but in general little is known about the optimal relaxation parameter. Berkhin says in [2, p. 88] that "Existence of a good ω for large Web graphs with more than five to ten million nodes is questionable."

In this paper we prove analytically the observation made in practice that SOR does not seem to offer an improvement over Gauss–Seidel for the PageRank problem. To the best of our knowledge, this analytical result has not been made so far in the literature. Our result may further support the practical conclusion that for PageRank there is little reason to expect a significant improvement of other stationary solvers over Gauss–Seidel.

We stress that our results pertain to worst-case scenarios. That is, we do not claim that there are no PageRank problems for which SOR cannot perform superbly. Rather, we show that there *are* PageRank matrices for which SOR does not perform better than Gauss–Seidel and cannot be expected to converge for a relaxation parameter outside a narrow interval whose upper endpoint depends on the damping factor. Conversely, we show that inside that interval, SOR is guaranteed to converge for all PageRank matrices.

In section 2 we briefly state the iterative setup. In sections 3 and 4 we provide a sufficient and a necessary condition, respectively, for convergence of SOR for an arbitrary PageRank problem and show that the valid range of values of the damping factor is rather small. In section 5 we make an observation that may suggest that the optimal relaxation parameter for an arbitrary PageRank problem may be $\omega = 1$, which yields the Gauss–Seidel scheme. In section 6 we numerically illustrate and confirm our analytical results. In section 7 we briefly summarize our main findings.

2. SOR iteration. In general, Jacobi, Gauss–Seidel, and SOR solve a linear system $Cx = b$ by iterating, given an initial guess x_0 and a splitting $C = M - N$: $Mx_{k+1} = Nx_k + b$. Let L, D, and U be the strictly lower triangular, diagonal, and strictly upper triangular parts, respectively, of C. The splitting associated with SOR is $M = \frac{1}{\omega}D + L$ and $N = -(\frac{\omega-1}{\omega}D + U)$, where $\omega > 0$. The iteration matrix T is then

$$T = M^{-1}N = -\left(\frac{1}{\omega}D + L\right)^{-1}\left(\frac{\omega-1}{\omega}D + U\right).$$

A stationary method converges to a solution for any initial guess if and only if $\rho(T) < 1$, where $\rho(T)$ is the spectral radius of its iteration matrix T.

For the PageRank problem, the matrix of the linear system, $C = I - \alpha P$, has some special properties that are useful for our analysis. By virtue of P being column-stochastic, C is a diagonally dominant nonnegative M-matrix. Certain matrices known as consistently ordered [5, 6] possess the property that Gauss–Seidel converges asymptotically twice as fast as Jacobi, and SOR yields significantly faster convergence (for sufficiently large matrices) compared to Gauss–Seidel if one uses the optimal relaxation parameter, which is available by a closed-form formula for this family of matrices. However, PageRank matrices are not typically in this class, and while Gauss–Seidel seems to accomplish a similar improvement by a factor of two over Jacobi, using the SOR formula for the relaxation parameter (which approaches 2 as $\alpha \nearrow 1$) yields poor convergence. Some evidence reported in the literature for PageRank matrices and other finite Markov chains suggests that, in fact, a value of ω near 1 (say, $1 + \varepsilon$, ε small) may be the best practical choice; see [2, 4].

3. A sufficient condition for the convergence of SOR for PageRank. Throughout we assume we are given a PageRank problem with arbitrary P and known α. Our goal for this section is to answer the following question: For what values of $\omega \in (0, 2)$ is SOR guaranteed to converge? We will show that for $\omega \in (0, 2/(1 + \alpha))$ we have the desired convergence. The following lemma characterizes the eigenvalues of the iteration matrix T. This result is known and can be found in old references such as [5, 6], but we present it because it is necessary for the proof of convergence that follows for our particular setting.

LEMMA 3.1. *λ is an eigenvalue of T if and only if* $\det\left(\frac{\lambda+\omega-1}{\omega}D + \lambda L + U\right) = 0$.

Proof. We have

$$\begin{aligned}
\det(\lambda I - T) = 0 \iff & \det\left(\lambda\left(\frac{1}{\omega}D + L\right)^{-1}\left(\frac{1}{\omega}D + L\right)\right. \\
& \left. + \left(\frac{1}{\omega}D + L\right)^{-1}\left(\frac{\omega-1}{\omega}D + U\right)\right) = 0 \\
\iff & \det\left(\left(\frac{1}{\omega}D + L\right)^{-1}\right)\det\left(\lambda\left(\frac{1}{\omega}D + L\right) + \frac{\omega-1}{\omega}D + U\right) = 0 \\
\iff & \det\left(\frac{\lambda+\omega-1}{\omega}D + \lambda L + U\right) = 0,
\end{aligned}$$

which completes the proof. ∎

The characterization in Lemma 3.1 of the eigenvalues of T allows us to use diagonal dominance arguments, which lead us to the following critical lemma.

LEMMA 3.2. *Given P and α, if λ is an eigenvalue of T, then* $|\frac{\lambda+\omega-1}{\omega}| \leq \max(\alpha, \alpha|\lambda|)$.

Proof. Let

$$X = \frac{\lambda + \omega - 1}{\omega} D + \lambda L + U.$$

By Lemma 3.1 we have $\det(X) = 0$, and hence X is necessarily not strictly column diagonally dominant. We will exploit this by showing that if the ith column is not strictly diagonally dominant, then the inequality stated in the lemma must hold. Denote by p and q the sum of the entries above and, respectively, below the diagonal in the ith column of P. Then we have that $p, q \geq 0$ and $p + q \leq 1$ and that this column is not strictly diagonally dominant if and only if

$$(1 - \alpha(1 - p - q)) \left| \frac{\lambda + \omega - 1}{\omega} \right| \leq \alpha \left(p + |\lambda| \, q \right)$$

or, equivalently,

$$\left| \frac{\lambda + \omega - 1}{\omega} \right| \leq \frac{\alpha \left(p + |\lambda| \, q \right)}{1 - \alpha(1 - p - q)}.$$

Denote the right-hand side of the above inequality by the function $f(p, q)$. Then we have

$$\begin{aligned} \frac{\partial f}{\partial p} &= \frac{\alpha \left(1 - \alpha(1 - p - q)\right) - \alpha^2 \left(p + |\lambda| \, q\right)}{\left(1 - \alpha(1 - p - q)\right)^2} \\ &= \frac{\alpha \left(1 - \alpha + \alpha q - \alpha \, |\lambda| \, q\right)}{\left(1 - \alpha(1 - p - q)\right)^2}. \end{aligned}$$

As the denominator is always positive for valid p, q and the numerator has no dependence on p, $\frac{\partial f}{\partial p}$ must have a fixed sign for fixed q. Thus, for a fixed q, f is at its maximum when $p = 0$ or $p = 1 - q$. A similar calculation for $\frac{\partial f}{\partial q}$ demonstrates that for a fixed p, f is at its maximum when $q = 0$ or $q = 1 - p$. Combining these two results, we arrive at the conclusion that f will attain its global maximum somewhere on the set $\{(p, q) \,|\, p = q = 0 \text{ or } p + q = 1\}$:

$$\begin{aligned} f(0, 0) &= 0; \\ f(p, 1 - p) &= \alpha p + \alpha \, |\lambda| \, (1 - p) \leq \max(\alpha, \alpha \, |\lambda|). \end{aligned}$$

Therefore, if the ith column is not strictly diagonally dominant, then $|\frac{\lambda+\omega-1}{\omega}| \leq \frac{\alpha(p+|\lambda|q)}{1-\alpha(1-p-q)} \leq \max(\alpha, \alpha|\lambda|)$, and so, more generally, if $\det(X) = 0$, then $|\frac{\lambda+\omega-1}{\omega}| \leq \max(\alpha, \alpha|\lambda|)$. ∎

With Lemma 3.2 in hand, we state our first main result.

THEOREM 3.3. *If $\omega \in [1, \frac{2}{1+\alpha})$, then $|\lambda| < 1$ for all eigenvalues λ of T, and SOR is guaranteed to converge for such ω.*

Proof. The proof proceeds by contradiction. Assume there exists an eigenvalue λ of T such that $|\lambda| \geq 1$. Then by Lemma 3.2, we have that $|\frac{\lambda+\omega-1}{\omega}| \leq \max(\alpha, \alpha|\lambda|) =$

$\alpha|\lambda|$. We thus have

$$\left|\frac{\lambda+\omega-1}{\omega}\right| \leq \alpha\,|\lambda| \Rightarrow |\lambda+\omega+1| \leq \alpha\omega\,|\lambda|$$
$$\Rightarrow |\lambda| \leq \alpha\omega\,|\lambda| + |\omega-1|$$
$$\Rightarrow |\lambda| \leq \frac{\omega-1}{1-\alpha\omega},$$

where we have used the fact that $1-\alpha\omega > \frac{1-\alpha}{1+\alpha} > 0$ (because $\omega < \frac{2}{1+\alpha}$). Further extraction of the inequality $\omega < \frac{2}{1+\alpha}$ gives us $\omega - 1 < 1 - \alpha\omega$, and so

$$|\lambda| \leq \frac{\omega-1}{1-\alpha\omega} < 1,$$

which is a contradiction. This gives the desired result. $\square$

A similar proof by a contradiction argument yields the following result.

THEOREM 3.4. *If $\omega \in (0,1]$, then $|\lambda| < 1$ for all eigenvalues λ of T, and SOR is guaranteed to converge for such ω.*

With Theorems 3.3 and 3.4 we have established that for any PageRank problem, if $\omega \in (0, \frac{2}{1+\alpha})$, then we are guaranteed convergence of the SOR method.

4. A necessary condition for the convergence of SOR for PageRank. We have shown that for any PageRank problem with arbitrary P and known α, if ω is chosen to be in $(0, \frac{2}{1+\alpha})$, then SOR will converge. We now explore the behavior outside this interval. In this section we will demonstrate that this bound is, in some sense, optimal. That is, for any $\alpha \in (0,1)$ and $\omega \in [\frac{2}{1+\alpha}, +\infty)$ there exists a column-stochastic matrix P such that the corresponding SOR iteration matrix T will have a spectral radius greater than or equal to one.

DEFINITION 4.1. *Suppose R is the $n \times n$ circulant matrix*

$$R = \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \\ 1 & 0 & 0 & \cdots & 0 \end{pmatrix}.$$

Then we say that R is a reverse (or backward) cycle matrix *(of size n).*

LEMMA 4.2. *Suppose the column-stochastic matrix in a PageRank problem is a reverse cycle matrix of size n. Then the eigenvalues of the SOR iteration matrix T are the roots of $(\lambda+\omega-1)^n - \alpha^n\omega^n\lambda = 0$.*

Proof. Recall from Lemma 3.1 that λ is an eigenvalue of T if and only if $\det(\frac{\lambda+\omega-1}{\omega}D + \lambda L + U) = 0$. In this case we have

$$\det\begin{pmatrix} \frac{\lambda+\omega-1}{\omega} & -\alpha & 0 & \cdots & 0 \\ 0 & \frac{\lambda+\omega-1}{\omega} & -\alpha & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & -\alpha \\ -\alpha\lambda & 0 & 0 & \cdots & \frac{\lambda+\omega-1}{\omega} \end{pmatrix} = 0.$$

This equation, when expanding the determinant along the first column, is

$$\left(\frac{\lambda+\omega-1}{\omega}\right)^n - \alpha^n\lambda = 0.$$

Multiplying by ω^n gives the result. $\square$

THEOREM 4.3. *Suppose the column-stochastic matrix in a PageRank problem is a reverse cycle matrix of size n, where n is odd. Then if $\omega \geq \frac{2}{1+\alpha}$, the spectral radius of its SOR iteration matrix T is greater than or equal to one, with equality only if $\omega = \frac{2}{1+\alpha}$.*

Proof. From Lemma 4.2, we know any eigenvalue λ of T must satisfy $p_n(\lambda) = 0$ where $p_n(\lambda) = (\lambda + \omega - 1)^n - \alpha^n \omega^n \lambda$. It is trivial to check that if $\omega = \frac{2}{1+\alpha}$, then $\lambda = -1$ is a solution. Now suppose $\omega > \frac{2}{1+\alpha}$. We have $p_n(-1) = (\omega-2)^n + (\omega\alpha)^n > 0$, since $\omega - 2 > -\frac{2\alpha}{1+\alpha}$ and $\omega\alpha > \frac{2\alpha}{1+\alpha}$. Moreover, $\lim_{x\to-\infty} p_n(x) = -\infty$. Thus, by the intermediate value theorem there is a root λ of p_n such that $\lambda < -1$. ∎

The significance of Theorem 4.3 is that although solving a PageRank problem for a reverse cycle matrix of odd size is pathological, it is easy to see that so long as there exists such a submatrix (in an otherwise ordinary PageRank problem), the same logic applies. Thus, for example, if a given PageRank problem has a reverse triangle—a largely local phenomenon—then SOR will not converge for $\omega \geq \frac{2}{1+\alpha}$. In our numerical experiments (section 6) we will show that, in fact, the presence of other cycles as well may have a negative effect on the convergence of SOR.

5. Optimal ω for convergence.

In this section we give a theoretical reason why $\omega = 1$ (i.e., Gauss–Seidel) may be ideal in some situations for the PageRank problem. The two critical inequalities of section 3 were

$$\left|\frac{\lambda+\omega-1}{\omega}\right| < \alpha\,|\lambda| \ \text{ or } \ \left|\frac{\lambda+\omega-1}{\omega}\right| < \alpha.$$

Using these, we saw that by setting ω favorably, we could ensure that $|\lambda| < 1$. Now we will consider a somewhat opposite problem: For what ω will $|\lambda|$ be most strictly bounded? Using the approach from the proof of Theorem 3.3, we get that

$$\left|\frac{\lambda+\omega-1}{\omega}\right| < \alpha\,|\lambda| \Rightarrow |\lambda| < \frac{|\omega-1|}{1-\alpha\omega};$$
$$\left|\frac{\lambda+\omega-1}{\omega}\right| < \alpha \Rightarrow |\lambda| < \alpha\omega + |\omega-1|\,.$$

Thus, we wish to find the ω associated with

$$\min_{\omega}\left(\max\left(\frac{|\omega-1|}{1-\alpha\omega}, \alpha\omega + |\omega-1|\right)\right).$$

Clearly, the second term is a piecewise linear function with a unique global minimum at $\omega = 1$. Moreover, at $\omega = 1$ the first term is smaller than the second term, and so $\omega = 1$ is indeed the value we seek.

6. Numerical experiments.

In this section we present a few experimental results that confirm our analytical observations. We start by confirming the main claim in Theorem 4.3, namely that for a "pure" reverse cycle matrix of odd size (see Definition 4.1), the SOR iteration matrix has a spectral radius equal to one when $\omega = \frac{2}{1+\alpha}$. The left-hand plot of Figure 6.1 illustrates this. In that plot it is also evident that for even-sized reverse cycles, as the matrix size grows larger the spectral radius goes to one. The right-hand plot in the figure shows that for a *forward* cycle (which we have not formally defined but whose meaning is obvious), the spectral radius tends to one as the matrix size grows, albeit more slowly compared to the backward cycle case. A conclusion that can be drawn from this is that the presence

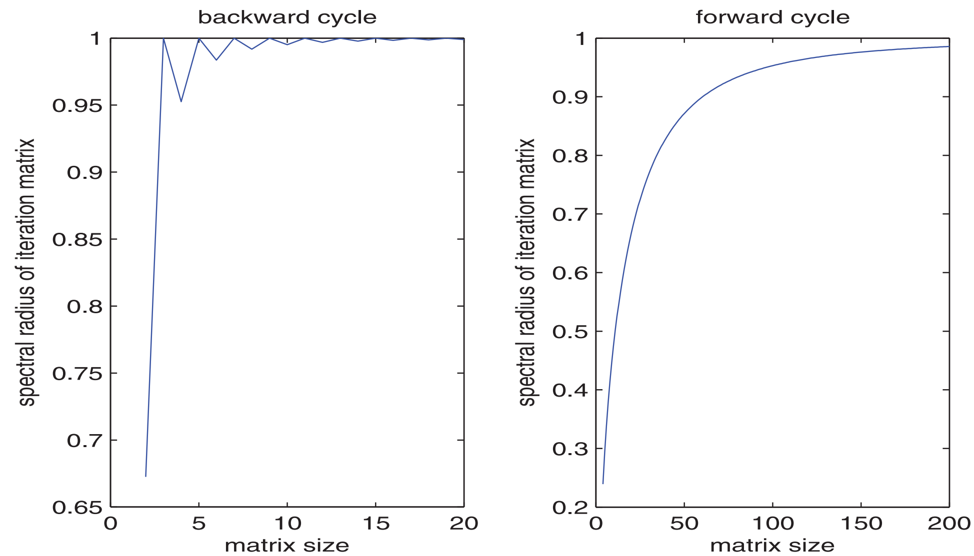

FIG. 6.1. *Spectral radius of the SOR iteration matrix for backward (reverse) and forward cycles, where* $\omega = \frac{2}{1+\alpha}$.

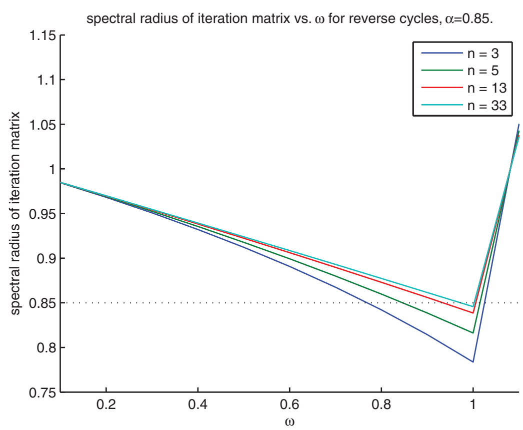

FIG. 6.2. *Spectral radius of the SOR iteration matrix versus the relaxation parameter* ω *for reverse odd cycles of various sizes.*

of *any* cycle in a graph may have a detrimental effect on the convergence of SOR, beyond the case of a reverse odd cycle, which we have analytically proved.

Figure 6.2 supports our observations in sections 3–5 about the range of ω values for convergence and the optimality of Gauss–Seidel in this case of reverse cycles. Figures 6.3–6.4 provide further evidence for a real-world example. In Figure 6.2 it is shown that for a reverse cycle with the typical value of the damping factor $\alpha = 0.85$, the optimal SOR parameter is $\omega = 1$, namely, the Gauss–Seidel scheme. Note that underrelaxation ($0 < \omega < 1$) yields a spectral radius smaller than one, which implies convergence. Note also that as expected from our analysis, the spectral radius becomes larger than one (i.e., no convergence) at the same value of ω for all matrix sizes, and that critical value is the one predicted by Theorem 4.3, namely, $\frac{2}{1+\alpha} \approx 1.0811$. The figure also validates the sufficient conditions in Theorems 3.3 and 3.4.

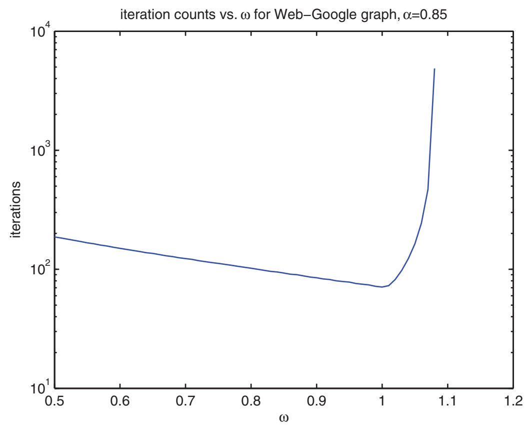

FIG. 6.3. *SOR convergence for the* `Web-Google` *matrix: The number of iterations it took to reach a tolerance of* 10^{-10} *in terms of the* ℓ_1*-norm of iterate difference for* $\omega = [0.50 : 0.01 : 1.08]$.

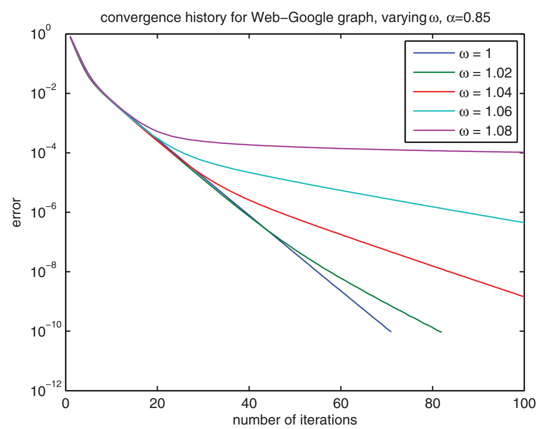

FIG. 6.4. *SOR convergence for the* `Web-Google` *matrix. The term "error" on the y-axis refers to* ℓ_1*-norm differences between adjacent iterates.*

Finally, we consider the performance of SOR for a mid-sized Web link graph. The matrix `Web-Google` has 875,713 vertices (matrix dimensions) and 5,105,039 edges (number of nonzeros). The matrix contains cycles, including a reverse odd cycle. We therefore expect Theorems 3.3, 3.4, and 4.3 to hold. That is, we expect to have convergence for $0 < \omega < 1$ and also in the narrow overrelaxation range $\omega \in [1, \frac{2}{1+\alpha})$. Also, based on section 5, we expect Gauss–Seidel to perform optimally among the relaxation parameters in this range. Indeed, Figures 6.3 and 6.4 confirm our expectations. The iteration is stopped when two adjacent iterates are within an ℓ_1-norm difference of 10^{-10}. We have performed additional numerical experiments to confirm that the convergence behavior is similar when this stopping criterion is replaced by a termination condition on the ℓ_1-norm of the residual associated with the eigenvalue problem (1.1).

In Figure 6.3 we test convergence for various values of ω. It takes 71 iterations for Gauss–Seidel to converge; this is faster than any other choice of the relaxation parameter. For the value $\omega = 1.08$, which is very close to the limit value $\frac{2}{1+\alpha} \approx 1.0811$, it takes 4,869 iterations to converge. In Figure 6.4 we plot the convergence history for ω in the domain of interest and observe the fastest convergence in the case of Gauss–Seidel versus a much slower convergence rate in the case of values of ω close to (but not exceeding) $\frac{2}{1+\alpha}$. We see that Theorems 3.3, 3.4, and 4.3 hold for this example as well as the claim in section 5.

7. Conclusions. We have derived necessary and sufficient conditions for convergence of SOR for the linear system formulation of the PageRank problem. The conditions feature an interval for the relaxation parameter that depends on the damping factor: $\omega \in (0, \frac{2}{1+\alpha})$. For α close to 1, for which the problem is harder to solve, the interval for overrelaxation (i.e., if $\omega \geq 1$ is required) is very narrow, and the smallest bound on the spectral radius of the iteration matrix is obtained for $\omega = 1$, which may suggest that the best strategy for choosing the relaxation parameter is to actually apply Gauss–Seidel. It is important to stress again that this result does not apply to *any* PageRank matrix and that there could be situations in which a different choice of ω may yield faster convergence. What we have shown is that there *exist* PageRank problems for which SOR does not converge outside the above-mentioned interval—for example, if the underlying graph contains a reverse cycle of odd length—and that in such situations Gauss–Seidel may be the best practical choice. Conversely, we have shown that within the bound $\omega \in (0, \frac{2}{1+\alpha})$, there exists no PageRank problem for which SOR does not converge.

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