15-863 Hint for Computer Assignment #2: HAIL STORM!

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Due: Tuesday, March 11, 2003 (EXTENDED!)

Since some of you have asked, I will give you all a hint by deriving the special case of a single particle ($s = 1$) applying a constraint on a triangle of the bicycle seat. The general case you have to handle involves s point-like contact constraints, and is only a little more complicated.

Let the triangle in question have vertex indices 1, 2 and 3 for simplicity (i_1, i_2 and i_3 in general), and let $\beta = (\beta_1, \beta_2, \beta_3)$ denote the barycentric coordinate of the *surface contact point (SCP)*. The deformation quantities to be determined are the triangle's vertex displacements u^Δ and forces f^Δ, i.e.,

$$\mathsf{u}^\Delta = \begin{bmatrix} \mathsf{u}_1 \\ \mathsf{u}_2 \\ \mathsf{u}_3 \end{bmatrix}, \qquad \mathsf{f}^\Delta = \begin{bmatrix} \mathsf{f}_1 \\ \mathsf{f}_2 \\ \mathsf{f}_3 \end{bmatrix},$$

as well as the SCP's displacement and contact force,

$$\mathsf{u}^\beta, \qquad \mathsf{f}^\beta.$$

Note that the vertex forces are related to the vertex tractions by multiplying by the vertex area,

$$\mathsf{f}_i = a_i \mathsf{p}_i \quad \Leftrightarrow \quad \mathsf{f}^\Delta = \mathsf{A}^\Delta \mathsf{p}^\Delta,$$

where A^Δ is the equivalent diagonal matrix. In general, there are vertex variables for contacted triangles, and SCP variables for the s contacts.

For $s = 1$ we have 24 unknowns to determine, since

$$\begin{aligned} dim(\mathsf{u}^\Delta) &= 9 && (1) \\ dim(\mathsf{f}^\Delta) &= 9 && (2) \\ dim(\mathsf{u}^\beta) &= 3 && (3) \\ dim(\mathsf{f}^\beta) &= 3, && (4) \end{aligned}$$

and we therefore need 24 constraints to determine all variables.

The constraints you have are as follows:

1. *(9 Constraints) The Green's function relationship*

$$\mathsf{u}^\Delta = \Xi^\Delta \mathsf{p}^\Delta = \Xi^\Delta (\mathsf{A}^\Delta)^{-1} \mathsf{f}^\Delta = \mathsf{D}^\Delta \mathsf{f}^\Delta$$

   where D^Δ is the *compliance matrix* of the contacted triangle vertices.

2. *(9 Constraints) Barycentric force distribution*

$$\mathsf{f}^\Delta = \mathsf{B}\mathsf{f}^\beta$$

where B is a matrix expressing $\mathsf{f}_i = \beta_i \mathsf{f}^\beta$, i.e.,

$$\mathsf{B} = \begin{bmatrix} \beta_1 \mathsf{I}_3 \\ \beta_2 \mathsf{I}_3 \\ \beta_3 \mathsf{I}_3 \end{bmatrix}$$

where I_3 is the 3-by-3 identity matrix.

3. *(3 Constraints) SCP displacement constraint*

$$\mathsf{u}^\beta = \bar{\mathsf{u}}^\beta$$

assuming you specify displacement constraints arising from your particle time-stepping results.

4. *(3 Constraints) Definition of SCP displacement*

$$\mathsf{u}^\beta = \sum_{i=1}^{3} \beta_i \mathsf{u}_i = \mathsf{B}^T \mathsf{u}^\Delta$$

These equations can be combined to determine the relationship between the SCP contact displacement and force as

$$\begin{aligned} \mathsf{f}^\beta &= \left(\mathsf{B}^T \mathsf{D}^\Delta \mathsf{B}\right)^{-1} \bar{\mathsf{u}}^\beta && (5) \\ &= \mathsf{K}^\beta \bar{\mathsf{u}}^\beta && (6) \end{aligned}$$

where K^β is the effective SCP stiffness matrix. Once this equation is solved (trivially), we can use the contact force to determine the traction constraints p^Δ and thus compute the deformation using the Green's function matrix.

As you know, in your assignment, you need to address the case with s barycentric contact constraints, derive the equations and simulate the results. We just did the $s = 1$ case, and the generalization is not much harder–in fact it is strikingly similar–and you only have to find the constraints that mirror #1–#4 above.