Written Assignment #1: Transformation and Viewing

15-462 Graphics I, Fall 2003
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Due: Thursday, September 18, 2003 (**before lecture**)
60 POINTS

October 7, 2003

- The work must be all your own.
- The assignment is due before lecture on Thursday, September 18.
- Be explicit, define your symbols, and explain your steps. This will make it a lot easier for us to assign partial credit.
- Use geometric intuition *together* with trigonometry and linear algebra.
- Verify whether your answer is meaningful with a simple example.

1 Angel, Chapter 4, Exercise 4.2

Two transformations, A and B, are said to commute if $AB = BA$. Show that the following transformation sequences commute:

1. A rotation and a uniform scaling;

   If the scaling matrix is uniform then $S = \alpha I$, so that

   $$RS = RS(\alpha, \alpha, \alpha) = R\alpha = \alpha R = SR$$

2. Two rotations about the same axis;

   It is sufficient to consider $R_x(\theta)$. If we multiply and use the standard trigonometric identities for the sine and cosine of the sum of two angles, we will find

   $$R_x(\theta)R_x(\phi) = R_x(\theta + \phi).$$

3. Two translations.

   This can be shown by simply multiplying the translation matrices and observing that

$$T(x_1, y_1, z_1)T(x_2, y_2, z2) = T(x_1 + x_2, y_1 + y_2, z_1 + z_2)$$

2 Angel, Chapter 4, Exercise 4.7

Show that any sequence of rotations and translations can be replaced by a single rotation about the origin, followed by a translation.

We can show by simply multiplying 4-by-4 matrices that the concatenation of two rotations yields a rotation and that the concatenation of two translations yields a translation. By looking at the product of a rotation and a translation, we find left three columns of RT are the left three columns of R and the right column of RT is the right right column of the translation matrix. If we now consider RTR' where R' is a 4-by-4 rotation matrix, the left three columns are exactly the same as the left three columns of RR' and the and right column still has 1 as its bottom element. Thus, the form is the same as RT with an altered rotation (which is the concatenation of the two rotations) and an altered translation. Inductively, we can see that any further concatenations with rotations and translations do no alter this form.

3 Angel, Chapter 4, Exercise 4.8

Derive the shear transformation from the rotation, translation, and scaling transformations.

A shear transformation A can be interpreted as a rotation, followed by a nonuniform scaling, followed by another rotation. In fact, you can see this immediately if you consider the Singular Value Decomposition (see "Matrix Computations" by Golub and Van Loan), $A = USV^T$ where both U and V are orthogonal (rotation) matrices, and S is a diagonal scaling matrix.

It is sufficient to consider 2D shear (3D is analogous), and for the case of a shear along a particular axis. A particularly nice identity for shear in the y-direction (x-direction is just the transpose) factorized into rotation*scaling*rotation is

$$\begin{bmatrix} 1 & 0 \\ a - \frac{1}{a} & 1 \end{bmatrix} = \left(\frac{1}{\sqrt{1+a^2}} \begin{bmatrix} 1 & -a \\ a & 1 \end{bmatrix} \right) \begin{bmatrix} a & 0 \\ 0 & \frac{1}{a} \end{bmatrix} \left(\begin{bmatrix} a & 1 \\ -1 & a \end{bmatrix} \frac{1}{\sqrt{1+a^2}} \right).$$

This can be interpreted as follows. If we consider an angle α such that $\tan\alpha = \frac{1}{a}$, then (using $\sin\alpha = \frac{1}{\sqrt{1+s^2}}$ and $\cos\alpha = \frac{s}{\sqrt{1+s^2}}$) the rotation matrix

$$R(\alpha) = \begin{bmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{bmatrix} = \frac{1}{\sqrt{1+a^2}} \begin{bmatrix} a & -1 \\ 1 & a \end{bmatrix}$$

can be used to observe that the last rotation matrix is just $R(-\alpha)$. We can also observe that

$$R(\frac{\pi}{2} - \alpha) = \begin{bmatrix} \sin\alpha & -\cos\alpha \\ \cos\alpha & \sin\alpha \end{bmatrix} = \frac{1}{\sqrt{1+a^2}} \begin{bmatrix} 1 & -a \\ a & 1 \end{bmatrix}.$$

Therefore the shear factorization is

$$R(-\alpha)\begin{bmatrix} a & 0 \\ 0 & \frac{1}{a} \end{bmatrix} R(\frac{\pi}{2}-\alpha).$$

Note that unless $\alpha = \frac{\pi}{4} \Leftrightarrow a = 1$, and hence the shear is simply the identity, the second rotation is never simply the inverse of the first.

For more details see the article: Ned Greene, "Transformation Identities," in Graphics Gems I, Edited by Andrew Glassner, 1990, p.485.

4 Angel, Chapter 4, Exercise 4.9

In two dimensions, we can specify a line by the equation $y = mx + b$.

1. Find an affine transformation to reflect two-dimensional points about this line.

   This transformation can be constructed in stages. If we do a translation, T, by $-b\hat{y}$ we convert the problem to reflection about a line passing through the origin; the translation matrix is

$$T = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & -b \\ 0 & 0 & 1 \end{bmatrix}.$$

   From the slope, $m = \tan\theta$, we can find a rotation so the line is aligned with the x (or y) axis:

$$R = R(-\theta) = \begin{bmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

   Now apply a reflection, S, about the x (or y) axis,

$$S = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

   Finally we undo the rotation and translation so the sequence is of the form

$$T^{-1}R^{-1}SRT = \begin{bmatrix} \cos 2\theta & \sin 2\theta & -b\sin 2\theta \\ \sin 2\theta & -\cos 2\theta & b\cos 2\theta \\ 0 & 0 & 1 \end{bmatrix}. \quad (1)$$

   For verification purposes, notice that when $b = m = 0$, the matrix correctly simplifies to S.

2. Extend your result to reflection about a plane in three dimensions.

   Let the plane be given by

$$0 = ax + by + cz + d = \mathbf{n^T p + d}$$

   where $\mathbf{n} = \mathbf{(a, b, c)^T}$ and $\mathbf{p} = \mathbf{(x, y, z)^T}$. Note that some (but not all) of a,b, or c may be zero. Let's do something slightly different (and more robust) than with the 2D example. First, given $\mathbf{p}$, we can find its projection $\bar{\mathbf{p}}$ on the plane,

$$\bar{\mathbf{p}} = \mathbf{p} - \mathbf{tn}$$

so that

$$\mathbf{p} = \bar{\mathbf{p}} + \mathbf{tn}$$

where we find t by substituting $b\bar{f}p$ into the plane equation,

$$t = \frac{d + \mathbf{n^T p}}{\mathbf{n^T n}}.$$

Next we simply flip the sign on the distance offset, t, to determine the reflection:

$$\begin{aligned}
\mathbf{p}^{\text{reflected}} &= \bar{\mathbf{p}} - t\mathbf{n} \\
&= \mathbf{p} - \mathbf{2tn} \\
&= \mathbf{p} - \mathbf{2}\frac{\mathbf{d} + \mathbf{n^T p}}{\mathbf{n^T n}}\mathbf{n} \\
&= \left(\mathbf{I} - \mathbf{2}\frac{\mathbf{nn^T}}{\mathbf{n^T n}}\right)\mathbf{p} - \frac{\mathbf{d}}{\mathbf{n^T n}}\mathbf{n}
\end{aligned}$$

which can be written as an affine transformation.

5 Angel, Chapter 4, Exercise 4.20

Given two nonparallel three-dimensional vectors, u and v, how can we form an orthogonal coordinate system in which u is one of the basis vectors?

The vector $a = u \times v$ is orthogonal to u and v. The vector $b = u \times a$ is orthogonal to u and a. Hence, u, a and b form an orthogonal coordinate system.

6 Angel, Chapter 4, Exercise 4.22

Find the quaternions for 90-degree rotations about the x- and y-axes. Determine their product. What rotation is this?

Using $r = (\cos\frac{\theta}{2}, \sin\frac{\theta}{2}\mathbf{v})$ with $\theta = \frac{\pi}{2}$ and $\mathbf{v} = (\mathbf{1}, \mathbf{0}, \mathbf{0})$, we find for rotation about the x-axis

$$r_x = \frac{1}{\sqrt{2}}(1, 1, 0, 0).$$

Likewise, for rotation about the y-axis,

$$r_y = \frac{1}{\sqrt{2}}(1, 0, 1, 0).$$

From the definition of quaternion multiplication (see §4.11.1), we have that

$$r_x r_y = (\frac{1}{2}, \frac{1}{2}(\hat{x} + \hat{y} + \hat{z}))$$

or, depending on how the question was interpreted, you may have

$$r_y r_x = (\frac{1}{2}, \frac{1}{2}(\hat{x} + \hat{y} - \hat{z})).$$

Note that these are both unit quaternions, and they correspond to 30 degree rotations (since $\cos\frac{\theta}{2} = \frac{1}{2} \rightarrow \theta = \frac{2\pi}{3}$) about the axes $(\hat{x} + \hat{y} + \hat{z})$ and $(\hat{x} + \hat{y} - \hat{z})$, respectively.

7 Angel, Chapter 5, Perspective Projection

In §5.9, it is shown that the OpenGL perspective transformation can be factored as

$$\mathbf{P} = \mathbf{NSH} = \begin{bmatrix} \left(\frac{2z_{min}}{x_{max}-x_{min}}\right) & 0 & \left(\frac{x_{max}+x_{min}}{x_{max}-x_{min}}\right) & 0 \\ 0 & \left(\frac{2z_{min}}{y_{max}-y_{min}}\right) & \left(\frac{y_{max}+y_{min}}{y_{max}-y_{min}}\right) & 0 \\ 0 & 0 & \left(-\frac{z_{max}+z_{min}}{z_{max}-z_{min}}\right) & \left(\frac{2z_{max}z_{min}}{z_{max}-z_{min}}\right) \\ 0 & 0 & -1 & 0 \end{bmatrix}.$$

1. State each of the homogeneous matrix factors, $\mathbf{N}$, $\mathbf{S}$ and $\mathbf{H}$.
2. In your own words and pictures, explain the role of each of the factors.

Taken directly from text. See explanation and pictures in §5.9.