Computing Wrench Cones for Planar Contact Tasks

Devin J. Balkcom *
Carnegie Mellon Robotics Institute
Pittsburgh, PA 15213.
devin@ri.cmu.edu

J.C. Trinkle and E.J. Gottlieb †
Sandia National Laboratories
Albuquerque, NM 87185-1004
jctrink, ejgottl@sandia.gov

Abstract

The successful execution of any contact task fundamentally requires the application of wrenches (forces and moments) consistent with the task. We develop an algorithm for computing the entire set of wrenches consistent with achieving a given *augmented contact mode* (*e.g.*, sliding at contact 1, rolling at contact 2, and approaching potential contact 3) for one fixed and one moving part in the plane.

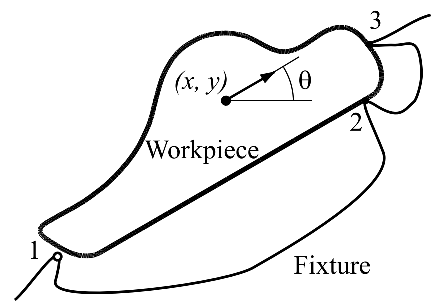

Figure 1: A workpiece nearly seated in a fixture.

1 Introduction

One of the most basic operations in robotic manufacturing is workpiece insertion. Consider the planar assembly task depicted in figure 1. The workpiece (a ratchet pawl) is to be fixtured for assembly. The goal is to achieve all three desired contacts (or fixels), so that the position and orientation of the workpiece will be uniquely determined. If the workpiece slides to the right across fixels 2 and 3 while maintaining contact with each, then the workpiece will eventually contact fixel 1.

This plan requires a device that applies specified wrenches. If the workpiece is positioned on a tray that is tilted to achieve the desired workpiece motion, then gravity provides the external wrench. If a spring is used to push the workpiece, then the deformation of the spring provides the wrench.

Regardless of the method used to apply the force, it is fundamental to the above plan that the set of external wrenches consistent with the desired contact mode (in this case, *sliding right over fixels 2 and 3*) be known. In this paper, we present an algorithm to determine the sets of external wrenches consistent with each possible contact mode.

Unfortunately, due to the nonuniqueness problem inherent in most mathematical models of dynamic rigid body systems (see [7], for example), it is possible that the wrenches in the calculated sets may be consistent not only with the desired mode, but also with another, undesireable mode. The companion paper [1] shows how unions and intersections of the sets of consistent external wrenches may be used to find wrenches consistent *only* with desired contact modes.

Relation to previous work

Our approach is based on previous theoretical results in rigid body mechanics [9, 11] and complementarity theory [2]. In Pang and Trinkle [9], examples are presented in which the polyhedral convex cones of external wrenches consistent with particular contact modes are calculated. We develop and extend their method into an algorithm that works with all contact modes. We represent polyhedral convex cones by matrices; our primary reference for operations on cones is Goldman and Tucker [5].

Mason [8] provides a good survey of previous work on manipulation planning from the perspective of rigid body dynamics with Coulomb friction. Erdmann's work on generalized friction cones in configuration space [3, 4] provided one of the first methods for computing the pos-

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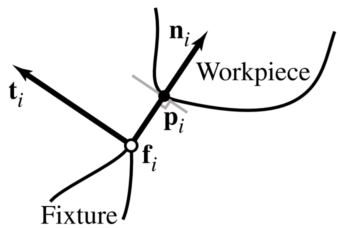

Figure 2: A local frame attached to fixel i.

sible motions of contacting rigid bodies under an applied wrench.

Apart from Pang and Trinkle [9], the work most similar to that presented here is probably Mason [8]. Mason describes a graphical method for finding the set of acceleration centers (and thus wrenches) consistent with a particular contact mode. The primary advantage of our algorithm is ease of implementation: the core of our sample implementation is less than one hundred lines of simple C code, not counting software by Hirai [6] used for conversion between representations of polyhedral convex cones. Our algorithm is also very flexible. For example, the set of external wrenches consistent with maintaining contact at some point can be found, without specifying whether the mode involves sliding right, sliding left, or rolling.

2 Mathematical model

The instantaneous dynamic model of a system of rigid bodies with unilateral, frictional contacts can be formulated and solved as a linear complementarity problem (LCP) [11]. Our problem may be seen as an inverse LCP. Given the rigid body configurations and a set of constraints on contact forces and accelerations imposed by the choice of contact mode, we want to find the set of consistent external wrenches. The derivation of our mathematical model therefore parallels that derived for LCP formulations. We assume that the workpiece is initially not moving, and either touching or infinitesimally distant from each fixel.

Variables and definitions

Let $\mathbf{f}_i$ be the location of fixel i. Let $\mathbf{p}_i$ be the unique closest point on the workpiece to $\mathbf{f}_i$, with signed distance function, or gap function, defined as $\langle \mathbf{n}_i(t), (\mathbf{p}_i - \mathbf{f}_i)\rangle$. We attach a right handed frame $(\mathbf{n}_i(t), \mathbf{t}_i(t))$ to each fixel such that the first axis points at $\mathbf{p}_i$ (see figure 2).

Let $v_{\mathrm{i}n}(t)$ and $v_{\mathrm{i}t}(t)$ denote the components of the velocity of $\mathbf{p}(i)$ in frame i:

$$v_{\mathrm{in}}(t) \quad = \quad \langle \dot{\mathbf{p}}_i(t), \mathbf{n}_i(t)\rangle \tag{1}$$

$$v_{\mathrm{it}}(t) \quad = \quad \langle \dot{\mathbf{p}}_i(t), \mathbf{t}_i(t)\rangle. \tag{2}$$

We now state four definitions. A *contact state* is the set of indices of fixels where contact has been achieved. A *contact interaction* is the relative motion at a point of contact: separating, rolling, sliding left, sliding right. A *contact mode* is the set of interactions at all the contacts. For the purpose of considering insertion tasks, we define an additional contact interaction: ‘approach’. We also extend the definition of a contact mode: an *augmented contact mode* is the extension of a contact mode that allows specification that the workpiece is approaching a nearby point of interest on the fixture. For example: the workpiece separates from fixel 1 and approaches fixel 2.

We may enumerate the possible contact interactions at fixel i based on the distance of the fixel from the workpiece, and the normal and tangential components of the velocity of the closest point on the workpiece. For example, if the interaction at fixel i is *left sliding*, then the gap is zero, $v_{in} = 0$ and $v_{\mathrm{it}} > 0$. Table 1 enumerates the cases.

Interaction	Abbrev.	gap	v_{in}	v_{it}
left sliding	l	0	0	> 0
right sliding	r	0	0	< 0
rolling	n	0	0	0
approaching	a	> 0	< 0	-
separating	s	$= 0$	> 0	-
		> 0	≥ 0	-

Table 1: Contact interactions.

Right sliding, *left sliding*, and *rolling* can occur only if contact has been achieved. *Approaching* can only occur if there is no contact, and would correspond to penetrating if contact had already been achieved. *Separating* may occur regardless of whether or not contact has been made.

We assume that the fixels have been ordered, and describe the augmented contact mode by a string, using the abbreviations from table 1. For example, the string ‘als’ should be read: the workpiece is approaching fixel 1, sliding left over fixel 2, and separating from fixel 3.

Newton-Euler equations

The Newton-Euler equations describe the dynamics of the system regardless of the contact mode. In this section we re-arrange the Newton-Euler equations into a form that will be useful in later sections.

The resultant wrench $\mathbf{w}$ applied to the workpiece and the generalized acceleration $\dot{\nu}$ of the workpiece are related through the three-by-three inertia matrix $\mathbf{M}$:

$$\mathbf{w} = \mathbf{M}\dot{\nu}. \tag{3}$$

Let $c_{\rm in}$ and $c_{\rm it}$ be the normal and tangential components of the force applied to the workpiece by contact i. Assume there are n contacts, and define $\mathbf{c} = [c_{1\rm n} ... c_{n\rm n} \; c_{1\rm t} ... c_{n\rm t}]^T$.

We partition the resultant wrench $\mathbf{w}$ into the contribution of the contact forces $\mathcal{J}\mathbf{c}$ and that of external loads $\mathbf{g}$, and solve for the generalized acceleration of the workpiece:

$$\dot{\nu} = \mathbf{M}^{-1}(\mathcal{J}\mathbf{c} + \mathbf{g}), \tag{4}$$

where $\mathcal{J}$ is the Jacobian matrix (also known as the wrench matrix) that transforms the contact forces into the inertial frame and sums their moments about the center of mass of the workpiece.

Let $\mathbf{v} = [v_{1\rm n} ... v_{n\rm n} \; v_{1\rm t} ... v_{n\rm t}]^T = \mathcal{J}^T \nu$ be the vector of normal and tangential components of the contact velocities. Then the contact acceleration vector can be written as:

$$\mathbf{a} = \frac{d}{dt}\mathbf{v} = \frac{d}{dt}(\mathcal{J}^T \nu) = \dot{\mathcal{J}}^T \nu + \mathcal{J}^T \dot{\nu}. \tag{5}$$

Premultiplying equation 4 by $\mathcal{J}^T$, assuming velocity product terms are negligible, and combining with equation 5 yields the Newton-Euler equations mapped into the contact frames:

$$\mathbf{a} = \mathbf{A}\mathbf{c} + \mathbf{B}\mathbf{g}, \tag{6}$$

where $\mathbf{A} = \mathcal{J}^T \mathbf{M}^{-1} \mathcal{J}$ and $\mathbf{B} = \mathcal{J}^T \mathbf{M}^{-1}$.

We rearrange equation 6 to find a relation between external wrenches and the accelerations and forces at the contacts:

$$\mathbf{C}\mathbf{y} = \mathbf{B}\mathbf{g}, \tag{7}$$

where $\mathbf{C} = [\mathbf{I}_{2n \times 2n} \; -\mathbf{A}]$ is $(2n \times 4n)$ and $\mathbf{y} = [\mathbf{a}^T \; \mathbf{c}^T]^T$ has $4n$ elements. Equation 7 will be the starting point for our algorithm to find the set of external wrenches consistent with a given contact mode.

Constraints due to contact modes

In this section we will discuss how the current contact state and the contact mode to be achieved imply a set of constraints on $\mathbf{y}$, the vector of contact accelerations and forces.

Non-penetration If there is contact at fixel i, then $a_{\rm in} \geq 0$; otherwise the fixel and the workpiece would interpenetrate.

Unilateral force The force exerted by the fixel is unilateral: $c_{\rm in} \geq 0$.

Coulomb friction Let μ be the coefficient of friction. If the workpiece is sliding to the right over the fixel, then the frictional force will be on the left edge of the friction cone: $c_{\rm it} = \mu c_{\rm in}$. If the workpiece is sliding to the left, then the frictional force will be on the right edge of the friction cone: $c_{\rm it} = -\mu c_{\rm in}$. If the workpiece is rolling over the fixel, then the friction force may fall anywhere in the friction cone: $|c_{\rm it}| \leq \mu c_{\rm in}$.

Separation If there is no contact at fixel i, then $c_{\rm in} = c_{\rm it} = 0$. If there is contact but $a_{\rm in} > 0$, contact is breaking. The fixel cannot exert a force on the workpiece: $c_{\rm in} = c_{\rm it} = 0$.

Since we assume the workpiece is initially at rest, there are also constraints on the contact accelerations imposed by the choice of desired contact interaction at a fixel. If contact has been achieved, and we want the interaction to be *left sliding*, then we should choose $a_{\rm in} = 0$ and $a_{\rm it} < 0$.

We have collected constraints on $\mathbf{y}$ implied by a contact interaction in table 2. For simplicity, we do not distinguish between static and kinetic coefficients of friction.

Interaction	$a_{\rm in}$	$a_{\rm it}$	$c_{\rm in}$	$c_{\rm it}$
a	< 0	$-$	0	0
s	≥ 0	$-$	0	0
l	0	> 0	≥ 0	$-\mu c_{\rm in}$
r	0	< 0	≥ 0	$\mu c_{\rm in}$
n	0	0	≥ 0	$\lvert c_{\rm it}\rvert \leq \mu c_{\rm in}$

Table 2: Constraints on elements of $\mathbf{y}$ due to contact interaction.

3 Matrix representations of polyhedral convex cones

In this section, we review matrix representations of polyhedral convex cones. Our discussion is based on Goldman and Tucker [5].

Assume matrix $\mathbf{F}$ is given. The *polar* of $\mathbf{F}$ is defined as the set of solutions to the matrix inequality $\mathbf{F}\mathbf{g} \leq 0$.

$$\mathrm{polar}(\mathbf{F}) = \{\mathbf{g} : \mathbf{F}\mathbf{g} \leq 0\} \tag{8}$$

Any $\mathbf{g} \in \mathrm{polar}(F)$ makes a non-positive dot product with each row of $\mathbf{F}$. Each row of $\mathbf{F}$ may be interpreted as a normal describing a half-space; solutions lie in the intersection of the half-spaces. $\mathrm{polar}(\mathbf{F})$ is therefore a polyhedral convex cone, and we say that the inequality is a *face normal representation* of the cone.

Similarly, assume matrix $\mathbf{G}$ is given and define the positive linear span of $\mathbf{G}$:

$$\mathrm{pos}(\mathbf{G}) = \{\mathbf{g} : \mathbf{g} = \mathbf{G}\mathbf{z} \text{ for some } \mathbf{z} \geq 0\} \tag{9}$$

Any vector $\mathbf{g} \in \mathrm{pos}(\mathbf{G})$ is in the *positive linear span* of the columns of $\mathbf{G}$, and we say that the inequality is a *span representation* of a polyhedral cone. The columns of $\mathbf{G}$ are referred to as *generators*.

Converting between representations

According to [5], Minkowski and Farkas first showed that for any face normal representation of a polyhedral convex cone, there is a corresponding span representation, and Weyl showed that the converse is true. If $\mathbf{F}$ or $\mathbf{G}$ is square and non-singular, then conversion between the two representations is easy and may be accomplished by matrix inversion. Goldman and Tucker [5] and Hirai [6] describe methods for performing the conversion in the general case. We introduce some new notation, and use the superscript F to denote conversion from a span representation of a cone to a face normal representation. If we are given a matrix $\mathbf{H}$, then $\mathbf{H}^F$ refers to a matrix such that $\mathrm{pos}(\mathbf{H}) = \mathrm{polar}(\mathbf{H}^F)$.

The following theorem makes use of the notation introduced above.

Theorem 1 *Assume we have a matrix equation and a set of constraints of the form*

$$\begin{array}{rcl} \mathbf{Kz} & = & \mathbf{Pg} \\ \mathbf{z} & \geq & \mathbf{k} \end{array}$$

where $\mathbf{K}$ and $\mathbf{P}$ are constant matrices, and $\mathbf{k}$ is a constant vector. For a given vector $\mathbf{g}$, there exists $\mathbf{z}$ satisfying the equation and the inequality if and only if

$$\mathbf{K}^F\mathbf{Pg} \leq \mathbf{K}^F\mathbf{Kk}$$

Proof: Define a change of variables $\mathbf{x} = \mathbf{z} - \mathbf{k}$. Then

$$\begin{array}{rcl} \mathbf{z} & = & \mathbf{x} + \mathbf{k} \\ \mathbf{K}(\mathbf{x} + \mathbf{k}) & = & \mathbf{Pg} \\ \mathbf{Pg} - \mathbf{Kk} & = & \mathbf{Kx} \\ \mathbf{x} & \geq & 0. \end{array}$$

The last two lines tell us that $\mathbf{Pg} - \mathbf{Kk}$ lies in a convex cone; this is the span representation of the cone. Therefore, we may convert to the face normal representation:

$$\begin{array}{rcl} \mathbf{K}^F(\mathbf{Pg} - \mathbf{Kk}) & \leq & 0 \\ \mathbf{K}^F\mathbf{Pg} & \leq & \mathbf{K}^F\mathbf{Kk} \end{array}$$

Verification of the 'only if' condition is similar. □

4 A simple example

Before presenting the complete algorithm, we present a simple example. Consider a disc-shaped workpiece (see figure 3) with unit radius, and inertia matrix $\mathbf{M}$ equal to the identity matrix. Fixel 1 is at the position $(0, -1)$, touching the workpiece, and fixel 2 is slightly to the right of $(1, 0)$, not quite touching the workpiece. We want the workpiece to roll on fixel 1, and approach fixel 2; that is, the contact mode is 'na'. What external wrenches are consistent with this mode?

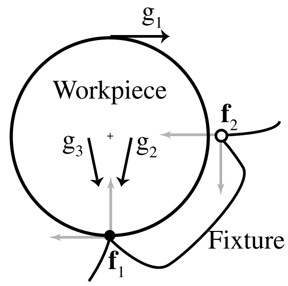

Figure 3: A simple example.

We first construct the Jacobian $\mathcal{J}$. Each column of $\mathcal{J}$ may be thought of as the wrench corresponding to a unit force applied to the workpiece at a point near a fixel, in a local coordinate direction.

$$\mathcal{J} = \begin{bmatrix} \mathbf{n}_1 & \mathbf{n}_2 & \mathbf{t}_1 & \mathbf{t}_2 \\ \mathbf{p}_1 \times \mathbf{n}_1 & \mathbf{p}_2 \times \mathbf{n}_2 & \mathbf{p}_1 \times \mathbf{t}_1 & \mathbf{p}_2 \times \mathbf{t}_2 \end{bmatrix} = \begin{bmatrix} 0 & -1 & -1 & 0 \\ 1 & 0 & 0 & -1 \\ 0 & 0 & -1 & -1 \end{bmatrix}$$

We may calculate $\mathbf{C}$ and $\mathbf{B}$ and rewrite equation 7:

$$\begin{bmatrix} 1 & 0 & 0 & 0 & -1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 & -1 & -1 & 0 \\ 0 & 0 & 1 & 0 & 0 & -1 & -2 & -1 \\ 0 & 0 & 0 & 1 & 1 & 0 & -1 & -2 \end{bmatrix} \mathbf{y} = \begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ -1 & 0 & -1 \\ 0 & -1 & -1 \end{bmatrix} \mathbf{g} \tag{10}$$

We now turn to the constraints on $\mathbf{y}$. The contact interactions are *rolling* at fixel 1 and *approaching* at fixel 2. Table 2 gives constraints on four elements of $\mathbf{y}$ for each interaction; we collect the constraints in table 3.

$\mathbf{y}$	uncon	ineq	roll	slide	zero
$a_{1\mathrm{n}}$					$= 0$
$a_{2\mathrm{n}}$		< 0			
$a_{1\mathrm{t}}$					$= 0$
$a_{2\mathrm{t}}$	x				
$c_{1\mathrm{n}}$		≥ 0			
$c_{2\mathrm{n}}$					$= 0$
$\lvert c_{1\mathrm{t}} \rvert$			$\leq \mu c_{1\mathrm{n}}$		
$c_{2\mathrm{t}}$					$= 0$

Table 3: Constraints on $\mathbf{y}$ for contact mode 'na'.

We will take equation 10, together with the constraints listed in table 3, and find a new equation and simpler constraints of the form:

$$\mathbf{G}\mathbf{z} = \mathbf{g}, \qquad \mathbf{z} \geq 0$$

1. Since $a_{1\mathrm{n}} = a_{1\mathrm{t}} = c_{2\mathrm{n}} = c_{2\mathrm{t}} = 0$, we may remove these elements from $\mathbf{y}$, and we remove the first, third, sixth, and eighth columns of $\mathbf{C}$.
2. Since $a_{2\mathrm{n}}$ is constrained to be less than zero, we may change the signs of the elements of the second column of $\mathbf{C}$, replace $a_{2\mathrm{n}}$ by $-a_{2\mathrm{n}}$ in $\mathbf{y}$, and constrain $-a_{2\mathrm{n}} \geq 0$. (For now, we ignore the issue posed by the strict inequality.)
3. We may replace the constraints $|c_{1\mathrm{t}}| \leq \mu c_{1\mathrm{n}}$ by two equivalent constraints: $(\mu c_{1\mathrm{n}} + c_{1\mathrm{t}})/2\mu \geq 0$ and $(\mu c_{1\mathrm{n}} - c_{1\mathrm{t}})/2\mu \geq 0$. (We take $\mu > 0$, so this is well-defined.) We make a variable substitution in $\mathbf{y}$, and take the appropriate linear combinations of columns five and seven of $\mathbf{C}$.
4. Since $a_{2\mathrm{t}}$ is unconstrained, we may drop the equation involving it; we remove the fourth row of $\mathbf{C}$ and $\mathbf{B}$. Once this has been done, the fourth column of $\mathbf{C}$ is comprised only of zeros; we remove the column from $\mathbf{C}$ and $a_{2\mathrm{t}}$ from $\mathbf{y}$.

After applying the above steps to equation 10, we have a new equation and set of constraints in the form used in theorem 1:

$$\begin{bmatrix} 0 & -1 & -1 \\ -1 & \mu & -\mu \\ 0 & 2\mu & -2\mu \end{bmatrix} \begin{pmatrix} -a_{2\mathrm{n}} \\ (\mu c_{1\mathrm{n}} + c_{1\mathrm{t}})/2\mu \\ (\mu c_{1\mathrm{n}} - c_{1\mathrm{t}})/2\mu \end{pmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ -1 & 0 & -1 \end{bmatrix} \mathbf{g}$$

$$\begin{pmatrix} -a_{2\mathrm{n}} \\ (\mu c_{1\mathrm{n}} + c_{1\mathrm{t}})/2\mu \\ (\mu c_{1\mathrm{n}} - c_{1\mathrm{t}})/2\mu \end{pmatrix} \geq 0$$

where the matrices on the left and right sides of the equation are $\mathbf{K}$ and $\mathbf{P}$, respectively, and the vector on the left hand side is $\mathbf{z}$. If we choose $\mu = .2$ and solve for $\mathbf{g}$ by inverting $\mathbf{P}$ and premultiplying both sides of the equation, we arrive at the desired form:

$$\begin{bmatrix} 1 & -.2 & .2 \\ 0 & -1 & -1 \\ -1 & -.2 & .2 \end{bmatrix} \mathbf{z} = \mathbf{g}$$

$$\mathbf{z} \geq 0.$$

This result is recognizable as a span representation of a polyhedral convex cone. A geometric interpretation is shown in figure 3. The generators of $\mathbf{G}$ (its columns) will be denoted by $\mathbf{g}_1$, $\mathbf{g}_2$, and $\mathbf{g}_3$. Each generator corresponds to a wrench, which we may view as a directed line of force. The lines of force corresponding to $\mathbf{g}_2$ and $\mathbf{g}_3$ lie on the edges of the friction cone of fixel 1; the positive linear combinations of these generators are the external wrenches that may be balanced by the contact force at fixel 1. The line of force corresponding to $\mathbf{g}_3$ points to the right and is at the top of the disc; forces along this line will cause the disc to approach fixel 2, without breaking contact or causing a load at fixel 1.

5 The algorithm

We may generalize the procedure used in the example into an algorithm that works for any contact mode. First, we calculate the matrices $\mathbf{C}$ and $\mathbf{B}$. Once a contact mode has been chosen, table 2 may be used to determine the set of constraints on $\mathbf{y}$. We then build a series of matrices that may be used to transform the equation and constraints into a face representation of a polyhedral convex cone.

Eliminate equations (Matrix $\mathbf{E}$)

Some elements of $\mathbf{y}$ may be unconstrained by the choice of augmented contact mode. We may eliminate the equations involving these variables by premultiplying $\mathbf{C}$ and $\mathbf{B}$ by a row selection matrix $\mathbf{E}$.

Let $\mathcal{E}$ be the set of indices of unconstrained elements of $\mathbf{y}$. We form $\mathbf{E}$ by removing the rows of $\mathbf{I}_{2n\times 2n}$ corresponding to elements of $\mathcal{E}$.

For the example problem discussed above, we examine the first column of table 3; the fourth variable $a_{2\mathrm{t}}$ is unconstrained. Therefore, $\mathcal{E} = \{4\}$, and we form $\mathbf{E}$ by removing the fourth row of $\mathbf{I}_{4\times 4}$.

Negative variables (Matrix $\mathbf{N}$)

The choice of augmented contact mode may constrain some elements of $\mathbf{y}$ to be negative. We change the sign of columns of $\mathbf{C}$ so that all inequalities may be expressed using $>$ or $\geq$.

Let $\mathcal{N}$ be the indices of the elements of $\mathbf{y}$ constrained to be negative, and let i_{ij} be the (i, j) element of an appropriately sized identity matrix. We postmultiply $\mathbf{C}$ by the diagonal $4n \times 4n$ matrix $\mathbf{N}$ defined as follows:

$$\begin{array}{ll} n_{ij} = -1 & \text{if } i = j \text{ and } i \in \mathcal{N} \\ n_{ij} = i_{ij} & \text{otherwise} \end{array}$$

For the example, we examine the second column of table 3, and see that $\mathcal{N} = \{2\}$, since only $a_{2\mathrm{n}} < 0$. We form $\mathbf{N}$ by changing the sign on the second diagonal element of $\mathbf{I}_{8\times 8}$.

Rolling friction (Matrix $\mathbf{R}$)

If the augmented contact mode involves 'rolling' interactions, some elements of $\mathbf{y}$ must satisfy constraints of the form $|c_{it}| \leq \mu c_{in}$. We replace the variables c_{in} and c_{it} by $(\mu c_{1\mathrm{n}} + c_{1\mathrm{t}})/2\mu$ and $(\mu c_{1\mathrm{n}} - c_{1\mathrm{t}})/2\mu$, both of which are constrained to be nonnegative. The variable substitution requires that we take appropriate linear combinations of columns of $\mathbf{C}$. We take the linear combination by postmultiplying $\mathbf{C}$ by a square matrix $\mathbf{R}$.

Let $\mathcal{R}$ be the indices of the elements c_{it} of $\mathbf{y}$ that must satisfy rolling friction constraints. Define $\mathbf{R}$ to be the $4n \times 4n$ matrix with:

$$
\begin{array}{ll}
r_{ij} = \mu & \text{if } i \in \mathcal{R} \text{ and } i = j \\
r_{ij} = 1 & \text{if } j \in \mathcal{R} \text{ and } i = j - n \\
r_{ij} = -\mu & \text{if } i \in \mathcal{R} \text{ and } i = j + n \\
r_{ij} = i_{ij} & \text{otherwise}
\end{array}
$$

For our example, we examine the third column of table 3, and find that the seventh element is subject to a rolling friction constraint. We build $\mathbf{R}$ from an 8×8 identity matrix, but set $r_{75} = -.2$, $r_{57} = 1$, and $r_{77} = .2$.

Sliding friction (Matrix S)

If the augmented contact mode involves 'sliding' interactions, some elements of $\mathbf{y}$ must satisfy constraints of the form $c_{it} = \pm\mu c_{in}$. We may eliminate c_{it} by replacing a column of $\mathbf{C}$ by an appropriate linear combination of columns, and removing a column of $\mathbf{C}$. These operations may be accomplished by postmultiplying by a square matrix $\mathbf{S}$ and a column selection matrix $\mathbf{V}$ defined below.

Let $\mathcal{S}_{\mathrm{r}}$ be the indices of the elements of $\mathbf{y}$ constrained to be a positive multiple of another element. Let $\mathcal{S}_{\mathrm{l}}$ be the indices of the elements of $\mathbf{y}$ constrained to be a negative multiple of another element. Define $\mathbf{S}$ to be the $4n \times 4n$ matrix with:

$$
\begin{array}{ll}
s_{ij} = \mu & \text{if } i \in \mathcal{S}_{\mathrm{l}} \text{ and } i = j + n \\
s_{ij} = -\mu & \text{if } i \in \mathcal{S}_{\mathrm{r}} \text{ and } i = j + n \\
s_{ij} = i_{ij} & \text{otherwise}
\end{array}
$$

For our example, we examine the fourth column of table 3, and find that there are no sliding friction constraints, so $\mathbf{S} = \mathbf{I}_{8\times 8}$.

Eliminate variables (Matrix V)

We need to remove the columns of $\mathbf{C}$ corresponding to the variables c_{it} subject to sliding friction constraints. Similarly, some elements of $\mathbf{y}$ will be constrained to be 0, and we may remove the corresponding columns of $\mathbf{C}$. Finally, since the unconstrained variables are accelerations appearing only in equations removed by the matrix $\mathbf{E}$, we may eliminate them. We may eliminate variables by postmultiplying by a column selection matrix $\mathbf{V}$.

Let $\mathcal{Z}$ be the indices of the elements of $\mathbf{y}$ constrained to be zero. Then define $\mathcal{V} = \mathcal{Z} \cup \mathcal{S}_{\mathrm{r}} \cup \mathcal{S}_{\mathrm{l}} \cup \mathcal{E}$. We form $\mathbf{V}$ by removing any columns of $\mathbf{I}_{4nx4n}$ that have an index contained in $\mathcal{V}$.

For our example, we examine the first column of table 3, and find that the fourth element of $\mathbf{y}$ is unconstrained and may be eliminated. The fourth column of the table tells us that there are no sliding constraints, and the fifth column tells us that elements one, three, six, and eight constrained to be zero and may be eliminated. We build $\mathbf{V}$ by removing the first, third, fourth, sixth, and eighth columns of $I_{8\times 8}$.

Cone form

We apply all of the above matrices in sequence to equation 7 and find that

$$
\begin{aligned}
\mathbf{Kz} &= \mathbf{Pg} && (11)\\
\mathbf{z} &\geq 0 && (12)
\end{aligned}
$$

where

$$
\begin{aligned}
\mathbf{K} &= \mathbf{ECNRSV} && (13)\\
\mathbf{z} &= \mathbf{V}^T\mathbf{S}^{-1}\mathbf{R}^{-1}\mathbf{N}^{-1}\mathbf{y} && (14)\\
\mathbf{P} &= \mathbf{EB}. && (15)
\end{aligned}
$$

We may calculate $\mathbf{K}$ and $\mathbf{P}$; then equation 11 and inequality 12 'almost' give the cone containing the external wrench. In special cases, we may either take an inverse or pseudoinverse to find a cone form, as we did in the simple example. In general, we apply theorem 1, choose $\mathbf{k} = 0$ and define $\mathbf{F} = \mathbf{K}^F\mathbf{P}$. Then

$$
\mathbf{Fg} \leq 0. \tag{16}
$$

This is the face normal representation of the polyhedral convex cone containing the external wrench $\mathbf{g}$.

The careful reader will have noticed that we ignored the distinction between inequality constraints and strict inequality constraints which should be made in inequality 12. Although this approximation seems reasonable, we point out that it may be removed by choosing $\mathbf{k}$ to have small positive elements, and considering limits as $\mathbf{k} \to 0$.

We summarize the algorithm as follows:

1. Calculate the Jacobian $\mathcal{J}$ and the matrices $\mathbf{C}$ and $\mathbf{B}$.
2. Determine the constraints on $\mathbf{y}$ implied by the mode.
3. Calculate the matrices $\mathbf{E}$, $\mathbf{N}$, $\mathbf{S}$, $\mathbf{R}$, and $\mathbf{V}$.
4. Calculate the matrices $\mathbf{K}$ and $\mathbf{P}$.
5. Find $\mathbf{K}^F$, as discussed in Section 3.
6. Calculate $\mathbf{F} = \mathbf{K}^F\mathbf{P}$.

A given wrench $\mathbf{g}$ is consistent with the contact mode if it satisfies $\mathbf{Fg} \leq 0$.

6 Implementation and examples

We implemented the algorithm in C, and used software described in Hirai [6] for the conversion between face normal and span reprentations of convex cones. Four examples problems are shown in figure 4. For each example,

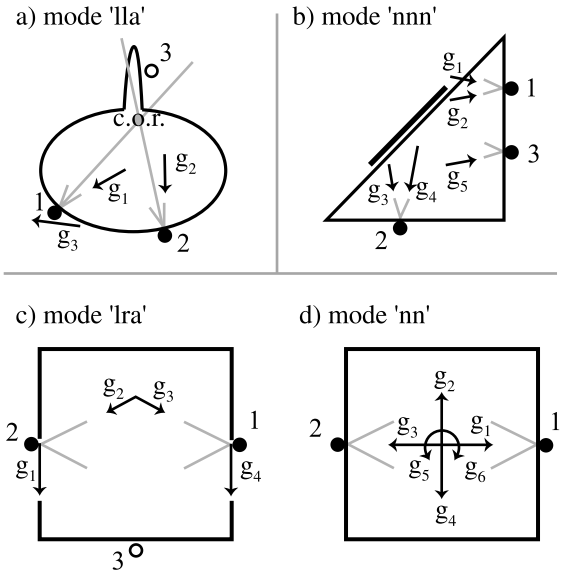

Figure 4: Four examples.

the output of the algorithm was a matrix **F** of the form described above. We used Hirai's software to convert to span form so that we could display the wrench cone generators.

Figure 4a shows an example for the contact mode 'lla'. The goal is to achieve contact at fixel 3, while maintaining contact at fixels 1 and 2. $\mathbf{g}_1$ and $\mathbf{g}_2$ saturate the right edges of the friction cones, and $\mathbf{g}_3$ provides the negative torque about the center of rotation to cause contact to be achieved at fixel 3. This example suggests how our algorithm might be used as part of a manipulation planner; the companion paper [1] describes one approach.

Figure 4b illustrates the problem of determining where a frictionless finger should be placed to achieve force closure. The problem can be solved using Reuleaux's graphical method [10]. Our algorithm finds the solution if we choose the contact mode 'nnn'. The thick black line on the surface of the workpiece shows places where the finger could be placed.

Figures 4c and 4d show an example of the frictional indeterminacy problem that arises in a peg-in-hole insertion. There is a cone of wrenches consistent with seating the workpiece through mode 'lra', but all of these wrenches are also consistent with the mode 'nn', in which jamming occurs. (Note that the wrench cone for figure 4d has six generators. The two that are pure moments are drawn as arcs about the center of gravity.)

7 Conclusion

We developed an aglorithm to find the polyhedral convex cone of external wrenches consistent with a contact mode between two rigid bodies, one moving and one motionless. The formulation of the model closely follows the formulation of the rigid body dynamics problem as a linear complementarity problem, and we also used results from the theory of polyhedral convex cones. We implemented the algorithm, and presented some example results.

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