# Dynamic Matching Markets and Voting Paths

David J. Abraham* Telikepalli Kavitha†

**Abstract**

We consider a matching market, in which the aim is to maintain a popular matching between a set of applicants and a set of posts, where each applicant has a preference list ranking a subset of posts in some order of preference. A matching $M$ is popular if there is no other matching $M'$ such that the number of applicants who prefer their partners in $M'$ to $M$ exceeds the number of applicants who prefer their partners in $M$ to $M'$. Popular matchings $M$ are stable in the sense that no coalition of applicants can force a switch to another matching $M'$ by requesting a pairwise election between $M$ and $M'$ with one vote per applicant (i.e. an up-or-down vote).

The setting here is dynamic: applicants and posts can enter and leave the market, and applicants can also change their preferences arbitrarily. After any change, the matching we have in place may no longer be popular, in which case we are required to update it.

However, we cannot simply recompute a popular matching from scratch after every such change. This is because there are instances in which no popular matching is directly more popular than the existing non-popular matching, and hence there would be no consensus for the applicants to agree to the switch. The aim then is to find a *voting path*, which is a sequence of matchings, each more popular than its predecessor, that ends in a popular matching. In this paper, we show that, as long as some popular matching exists, there exists a 2-step voting path from any given matching to some popular matching. Furthermore, given any popular matching, we show how to find a shortest-length such voting path in linear time.

## 1 Introduction

An instance of the *popular matching problem* consists of a bipartite graph $G = (\mathcal{A} \cup \mathcal{P}, E)$, together with a partition $E_1 \dot\cup E_2 \ldots \dot\cup E_r$ of the edge set $E$. For exposition purposes, we call $\mathcal{A}$ the set of *applicants*, $\mathcal{P}$ the set of *posts*, and $E_i$ the set of edges with rank $i$. If $(a, p) \in E_i$ and $(a, p') \in E_j$ with $i < j$, we say that $a$ prefers $p$ to $p'$. If $i = j$, then $a$ is indifferent between $p$ and $p'$. The ordering of posts adjacent to $a$ is called $a$'s preference list. We say that preference lists are strictly ordered if no applicant is indifferent between any two posts in its preference list.

A *matching* $M$ of $G$ is a subset of $E$, such that no two edges of $M$ share a common endpoint. A node $u \in \mathcal{A} \cup \mathcal{P}$ is either unmatched in $M$, or matched to some node denoted by $M(u)$. We say an applicant $a$ *prefers* matching $M'$ to $M$ if (i) $a$ is matched in $M'$ and unmatched in $M$, or (ii) $a$ is matched in both $M'$ and $M$, and $a$ prefers $M'(a)$ to $M(a)$.

**Definition 1** *$M'$ is* more popular than *$M$, denoted by $M' \succ M$, if the number of applicants preferring $M'$ to $M$ is greater than the number of applicants preferring $M$ to $M'$. A matching $M$ is* popular *if there is no matching $M'$ that is more popular than $M$.*

*Computer Science Department, Carnegie Mellon University,USA. Email: dabraham@cs.cmu.edu
†Indian Institute of Science, Bangalore, India. Email: kavitha@csa.iisc.ernet.in

Figure 1 contains an example instance in which $\mathcal{A} = \{a_1, a_2, a_3\}$, $\mathcal{P} = \{p_1, p_2, p_3\}$, and each applicant prefers $p_1$ to $p_2$, and $p_2$ to $p_3$. Consider the three symmetrical matchings $M_1 = \{(a_1, p_1), (a_2, p_2), (a_3, p_3)\}$, $M_2 = \{(a_1, p_3), (a_2, p_1), (a_3, p_2)\}$ and $M_3 = \{(a_1, p_2), (a_2, p_3), (a_3, p_1)\}$. None of these matchings is popular, since $M_1 \prec M_2$, $M_2 \prec M_3$, and $M_3 \prec M_1$. In fact, it turns out that this instance admits no popular matching, the problem being, of course, that the *more popular than* relation is not acyclic.

$$\begin{array}{llll} a_1: & p_1 & p_2 & p_3 \\ a_2: & p_1 & p_2 & p_3 \\ a_3: & p_1 & p_2 & p_3 \end{array}$$

Figure 1: An instance for which there is no popular matching.

The popular matching problem is to determine if a given instance admits a popular matching, and to find such a matching, if one exists. The first polynomial-time algorithms for this problem were given in [3]: when preference lists are strictly ordered, the problem can be solved in $O(n+m)$ time, where $n = |\mathcal{A} \cup \mathcal{P}|$ and $m = |E|$, and more generally, the problem can be solved in $O(m\sqrt{n})$ time. Note that when $E = E_1$, a matching is popular if and only if it has maximum cardinality. Hence, the popular matching problem is at least as hard as the problem of finding a maximum matching in a bipartite graph.

### 1.1 Problem Definition

In this paper, we consider a matching market in which the aim is to *maintain* a popular matching. The setting is dynamic: applicants and posts can enter and leave the matching market, and applicants can change their preferences arbitrarily. More precisely, an instance of the *dynamic popular matching problem* consists of an instance $G$ of the popular matching problem, together with an existing (possibly empty) matching $M_0$.

It turns out that we cannot simply solve $G$ from scratch after each change, since any *particular* popular matching we find may not be more popular than $M_0$, and furthermore, it is possible that *no* popular matching is more popular than the existing matching $M_0$. Hence, in general, there may be no consensus amongst the applicants to move directly from $M_0$ to a popular matching. We show such an example below.

Consider the instance in Figure 2 with $M_0 = \{(a_1, p_5), (a_2, p_2), (a_3, p_3), (a_4, p_1)\}$. First note that $M_0$ is not popular, since it is less popular than $M = \{(a_1, p_2), (a_2, p_3), (a_4, p_1)\}$ (even with $a_3$ unmatched). We can show using Lemma 2 from Section 2 that the only popular matchings are $M^* = \{(a_1, p_1), (a_2, p_2), (a_3, p_3), (a_4, p_4)\}$ and $N^* = \{(a_1, p_1), (a_2, p_3), (a_3, p_2), (a_4, p_4)\}$. However, it is clear that neither $M^*$ nor $N^*$ is more popular than $M_0$.

$$\begin{array}{llll} a_1: & p_1 & p_2 & p_5 \\ a_2: & p_3 & p_2 & \\ a_3: & p_3 & p_2 & \\ a_4: & p_1 & p_4 & \end{array}$$

Figure 2: Instance motivating *voting-path* approach.

In order to arrive at a popular matching by consensus, [3] introduced the following generalization of the more popular than relation:

**Definition 2** *A matching $M_k$ is* reachable *from $M_0$ if there is a sequence of matchings $\langle M_0, M_1, \ldots, M_k\rangle$, such that each matching is more popular than its predecessor. Such a sequence is called a length-k* voting path *from $M_0$ to $M_k$.*

Note that the instance above has a length-2 voting path from $M_0$ to a popular matching, namely $\langle M_0, M, N^*\rangle$.

There is no a priori reason to expect that such a voting path must exist: the *more popular than* relation is not acyclic, and so perhaps there are some matchings $M_0$ from which we cannot avoid cycling. Even if such a path does exist, it may have length exponential in the size of $G$, since there can be an exponential number of matchings. In this paper, we show the following surprising result.

**Theorem 1** *Let $\langle G, M_0\rangle$ be an instance of the dynamic popular matching problem, where G admits a popular matching. Then G admits a voting path of length at most 2 from $M_0$ to some popular matching. Additionally, given any popular matching, we can find a shortest-length such voting path in only linear time.*

Hence, by using the popular matching algorithms in [3], we can solve the dynamic popular matching problem in $O(m+n)$ time when preference lists are strictly ordered, and more generally in $O(m\sqrt{n})$ time [1]. This solves the problem of efficiently computing a shortest-length voting path to a popular matching, which was posed in [3]. We have also shown that such paths have length at most 2, which is better than the bound of 3 previously claimed. The proof of the length-3 bound quoted in [3] is unpublished, and only applies when preference lists are strictly ordered. This restriction greatly simplifies the more general problem, which is discussed in this paper.

Interestingly, the improvement from 3 to 2 implies a connection to the famous result in graph theory that every tournament has a king [14]. The *more popular than* relation is a directed graph on an exponential number of vertices. This graph is not a tournament though, since for any pair of matchings, there is no guarantee that one is more popular than the other. However, even without these edges, the set of popular matchings collectively act as a king, since every unpopular matching has a voting path of length at most 2 into this set.

### 1.2 Related Previous Work

The bipartite matching problem with a graded edge set is well-studied in both economics and computer science, see for example [1, 18, 22] and [6, 12, 2]. It models some important real-world problems, including the allocation of graduates to training positions [10], families to government-owned housing [21], and customers to rental DVDs [4, 17].

Gardenfors [8] first introduced the notion of a popular matching in the context of the stable marriage problem[2] [7, 9]. Of course, the *more popular than* concept can be traced back even further to the Condorcet voting protocol.

One drawback of the popularity criterion is that a popular matching may not exist. However, in recent work, Mahdian [15] showed that a popular matching exists with high probability, when (i) preference lists are randomly constructed, and (ii) the number of posts is a small multiplicative factor larger than of the number of applicants. Other recent work on popular matching includes Mestre's [16] generalization of the efficient popular matching characterization in [3] to the case where applicant votes carry different weights.

[1] We make the standard assumption in dynamic programming settings that the instance $G$ does not change while we are computing a voting path.

[2] A stable marriage instance is the same as a popular matching instance, except that *both* applicants (i.e. men) and posts (i.e. women) rank each other in order of preference.

We remark that the result in our paper is analogous to a series of papers [13, 19, 20, 5] on decentralized mechanisms in the stable matching literature. The well-known mechanisms for stable matching, due to Gale/Shapley [7] and Irving [11], require a central body to collect preferences and dictate the final matching. Alternatively, in a decentralized setting, a blocking pair (i.e. a man and woman who prefer each other to their current partners) will *act* locally by divorcing their current partners and marrying each other. Knuth [13] showed that if the divorced partners also marry each other, it is possible for this process to cycle. However, when divorced partners are not required to marry each other, and every blocking pair has some probability of acting next, Roth and Vande Vate [19] show by way of a potential argument that there is always a path to a stable matching.

In our setting, the analogue of a blocking pair is a coalition of applicants who (i) prefer some matching $M'$ to the current matching $M$, and (ii) have sufficient numbers to win a vote between $M'$ and $M$. It is not too difficult to give a potential argument to prove the existence of voting paths (at least for the restriction to strictly-ordered preference lists). However, in this paper, we use more powerful techniques from matching theory, which in addition to proving existence, also give the surprising length-2 bound. As with the result in [19], this means that as long as every matching more popular than the current one has some probability of an up-or-down vote, then in the limit, a decentralized mechanism will lead to a popular matching.

### 1.3 Organization of the Paper

In Section 2, we review the theory of popular matchings, and then use this to characterize the set of matchings that admit length-0 voting paths. In Section 3 and 4, we derive the central lemmas of the paper, using these to characterize the set of matchings that admit length-1 and length-2 voting paths respectively. Finally, in Section 5, we conclude with an open problem.

## 2 Preliminaries: Length-0 Voting Paths

In this section, we review the algorithmic characterization of popular matchings given in [3]. We then note that this characterization can be used to determine if an instance $\langle G, M_0 \rangle$ of the dynamic popular matching problem admits a length-0 voting path to a popular matching.

For exposition purposes, we create a unique strictly-least-preferred post $l(a)$ for each applicant $a$. In this way, we can assume that every applicant is matched, since any unmatched applicant $a$ can be paired with $l(a)$. From now on then, matchings are $\mathcal{A}$-perfect. Also, without loss of generality, we assume that preference lists contain no gaps, i.e. if $a$ is incident to an edge of rank $i$, then $a$ is incident to an edge of rank $i-1$, for all $i > 1$.

Let $G_1 = (\mathcal{A} \cup \mathcal{P}, E_1)$ be the graph containing only rank-one edges. [3, Lemma 3.1] shows that a matching $M$ is popular in $G$ only if $M \cap E_1$ is a maximum matching of $G_1$. Maximum matchings have the following important properties, which we use throughout the rest of the paper.

$M \cap E_1$ defines a partition of $\mathcal{A} \cup \mathcal{P}$ into three disjoint sets: a node $u \in \mathcal{A} \cup \mathcal{P}$ is *even* (respectively *odd*) if there is an even (respectively odd) length alternating path in $G_1$ (with respect to $M \cap E_1$) from an unmatched node to $u$. Similarly, a node $u$ is *unreachable* if there is no alternating path from an unmatched node to $u$. Denote by $\mathcal{E}$, $\mathcal{O}$ and $\mathcal{U}$ the sets of even, odd, and unreachable nodes, respectively.

**Lemma 1 (Gallai-Edmonds Decomposition)** *Let $\mathcal{E}$, $\mathcal{O}$ and $\mathcal{U}$ be the sets of nodes defined by $G_1$ and $M \cap E_1$ above. Then*

*(a) $\mathcal{E}$, $\mathcal{O}$ and $\mathcal{U}$ are pairwise disjoint, and independent of the maximum matching $M \cap E_1$.*

*(b) In any maximum matching of $G_1$, every node in $\mathcal{O}$ is matched with a node in $\mathcal{E}$, and every node in $\mathcal{U}$ is matched with another node in $\mathcal{U}$. The size of a maximum matching is $|\mathcal{O}|+|\mathcal{U}|/2$.*

*(c) No maximum matching of $G_1$ contains an edge between a node in $\mathcal{O}$ and a node in $\mathcal{O}\cup\mathcal{U}$. Also, $G_1$ contains no edge between a node in $\mathcal{E}$ and a node in $\mathcal{E}\cup\mathcal{U}$.*

Using this node partition, we make the following definitions: for each applicant $a$, define $f(a)$ to be the set of most-preferred odd/unreachable posts in $a$'s preference list[3]. Also, define $s(a)$ to be the set of most-preferred even posts in $a$'s preference list.

We refer to posts in $\cup_{a\in\mathcal{A}} f(a)$ as *$f$-posts* and posts in $\cup_{a\in\mathcal{A}} s(a)$ as *$s$-posts*. Note that $f$-posts and $s$-posts are disjoint, and that $s(a)\neq\emptyset$ for any $a$, since $l(a)$ is always even. Also note that there may be posts in $\mathcal{P}$ that are neither $f$-posts nor $s$-posts. The next lemma characterizes the set of all popular matchings.

**Lemma 2 ([3])** *A matching $M$ is popular in $G$ if and only if (i) $M\cap E_1$ is a maximum matching of $G_1=(\mathcal{A}\cup\mathcal{P},E_1)$, and (ii) for each applicant $a$, $M(a)\in f(a)\cup s(a)$.*

Using this lemma, we can check if $\langle G, M_0\rangle$ admits a length-0 voting path to a popular matching: $M_0\cap E_1$ is a maximum matching of $G_1$ if $G_1$ admits no augmenting path. Also, given that $M_0\cap E_1$ is a maximum matching of $G_1$, it is trivial to compute the Gallai-Edmonds decomposition and then to check that each applicant $a$ is matched to $M(a)\in f(a)\cup s(a)$. These checks can clearly be performed in linear time. Henceforth, we assume then that $M_0$ is not popular, for otherwise $\langle G, M_0\rangle$ admits a length-0 voting path, and we are done. We also assume that $\langle G, M_0\rangle$ admits a popular matching, for otherwise no voting path can end in a popular matching.

We conclude this section by giving the algorithm based on Lemma 2 for solving the popular matching problem.

**Popular-Matching**($G=(\mathcal{A}\cup\mathcal{P},E)$)

1. Construct the graph $G'=(\mathcal{A}\cup\mathcal{P},E')$, where $E'=\{(a,p): a\in\mathcal{A} \text{ and } p\in f(a)\cup s(a)\}$.
2. Construct a maximum matching $M$ of $G_1=(\mathcal{A}\cup\mathcal{P},E_1)$.
   (Note that $M$ is also a matching in $G'$).
3. Remove any edge in $G'$ between a node in $\mathcal{O}$ and a node in $\mathcal{O}\cup\mathcal{U}$.
   (No maximum matching of $G_1$ contains such an edge).
4. Augment $M$ in $G'$ until it is a maximum matching of $G'$.
5. Return $M$ if it is $\mathcal{A}$-perfect, otherwise return *"no popular matching"*.

# 3 Length-1 voting paths

In this section, we show that, given any popular matching of $G$, the problem of finding a length-1 voting path from $M_0$ to a popular matching, or proving that no such path exists, can be solved in linear time. First though, we work towards characterizing the set of all popular matchings that are more popular than $M_0$.

[3] In [3], $f(a)$ is defined as the set of rank-1 posts in $a$'s preference list. We find the definition above more suitable.

Let $r_a(p)$ be the rank of edge $(a,p) \in E$. Also, let $r_a(s(a))$ be the rank of any edge $(a,p)$, where $p \in s(a)$. We define the *signature* of a matching $M$ as the 4-tuple $(|A_f^M|, |A_m^M|, |A_s^M|, |A_l^M|)$ [4], where:

(i) $A_f^M = \{a \in \mathcal{A} | r_a(M(a)) = 1$, and $a$ is even/unreachable, i.e. $a \in \mathcal{E} \cup \mathcal{U}\}$.

(ii) $A_m^M = \{a \in \mathcal{A} | 1 < r_a(M(a)) < r_a(s(a))\}$.

(iii) $A_s^M = \{a \in \mathcal{A} | r_a(M(a)) = r_a(s(a))\}$.

(iv) $A_l^M = \{a \in \mathcal{A} | r_a(M(a)) > r_a(s(a))\}$.

Note that an odd applicant $a$ (i.e. one in $\mathcal{O}$) can only belong to $A_s^M \cup A_l^M$, even if $r_a(M(a)) = 1$, for $a \notin A_f^M$ by definition, and $a \notin A_m^M$, since $r_a(s(a)) = 1$. Also note that for any even/unreachable applicant $a$ (i.e. one in $\mathcal{A} \setminus \mathcal{O}$), $r_a(s(a)) \neq 1$. Hence $A_f^M, A_m^M, A_s^M$, and $A_l^M$ are pairwise disjoint and partition $\mathcal{A}$. This gives us $|A_f^M| + |A_m^M| + |A_s^M| + |A_l^M| = |\mathcal{A}|$. Finally, note that $A_f^M = \{a \in \mathcal{A} : M(a) \in f(a)\}$.

Now, let $\mathcal{F}$ be the set of $f$-posts - i.e. $\mathcal{F} = \cup_{a \in \mathcal{A}} f(a)$. The following lemma characterizes the set of all popular matchings in terms of their signatures.

**Lemma 3** *A matching M is popular if and only if its signature is* $(|\mathcal{F}|, 0, |\mathcal{A}| - |\mathcal{F}|, 0)$.

**Proof:** Suppose $M$ is popular. Then by Lemma 2, $|A_m^M| = |A_l^M| = 0$, and so $|A_f^M| + |A_s^M| = |\mathcal{A}|$. Now, $|A_f^M| \leq |\mathcal{F}|$, since every applicant in $A_f^M$ is matched with some post in $\mathcal{F}$. But since $M \cap E_1$ is a maximum matching of $G_1$, Lemma 1(b) requires that every post in $\mathcal{F}$ is matched with some applicant in $A_f^M$. Hence, $|A_f^M| = |\mathcal{F}|$, $|A_s^M| = |\mathcal{A}| - |\mathcal{F}|$, and $M$ has signature $(|\mathcal{F}|, 0, |\mathcal{A}| - |\mathcal{F}|, 0)$.

Conversely, suppose $M$ is a matching with signature $(|\mathcal{F}|, 0, |\mathcal{A}| - |\mathcal{F}|, 0)$. Then $|A_m^M| = |A_l^M| = 0$, and so for every $a \in \mathcal{A}$, $M(a) \in f(a) \cup s(a)$. It remains to show that $M \cap E_1$ is a maximum matching of $G_1$. We have that $|M \cap E_1| = |A_f^M| + |\{a \in A_s^M : a \text{ is odd}\}|$. Since $|A_f^M| = |\mathcal{F}|$, $|M \cap E_1| = |\mathcal{F}| + |\{a \in A : a$ is odd$\}|$. So, $|M \cap E_1| = |\{v \in \mathcal{A} \cup \mathcal{P} : v \text{ is odd}\}| + |\{p \in \mathcal{P} : p \text{ is unreachable}\}|$, and the result follows from Lemma 1(b). ∎

**Lemma 4** *For any matching M,* $|A_f^M| + |A_m^M| \leq |\mathcal{F}|$.

**Proof:** For each $a \in A_f^M$, $M(a)$ is odd/unreachable (i.e. belongs to $\mathcal{O} \cup \mathcal{U}$), for otherwise, $G_1$ contains an edge contradicting Lemma 1(c). Also, for each $a \in A_m^M$, $M(a)$ is odd/unreachable, since $s(a)$ contains $a$'s most preferred *even* posts, and by definition of $A_m^M$, $a$ prefers $M(a)$ to posts in $s(a)$ (i.e. $r_a(M(a)) > r_a(s(a))$). Hence, $|A_f^M| + |A_m^M| \leq |\mathcal{F}|$. ∎

Finally, we come to the main technical lemma in this section, which characterizes the set of all popular matchings that are more popular than a given matching $M$.

**Lemma 5** *Let $M^*$ be a popular matching. Then $M^*$ is more popular than M if and only if (i)* $|A_f^M| + |A_m^M| < |\mathcal{F}|$, *or (ii)* $|A_m^M \cap A_f^{M^*}| > 0$, *or (iii)* $|A_l^M \cap A_s^{M^*}| > 0$.

**Proof:** Let $\Delta(M^*, M)$ be the difference between the number of applicants who prefer $M^*$ to $M$, and the number of applicants who prefer $M$ to $M^*$. That is,

$$\Delta(M^*, M) = \left| [A_m^M \cup A_s^M \cup A_l^M] \cap A_f^{M^*} \right| + |A_l^M \cap A_s^{M^*}| - \left| [A_f^M \cup A_m^M] \cap A_s^{M^*} \right|.$$

[4] The subscripts $f, m, s$, and $l$ stand for *first, middle, second*, and *last* respectively.

Now, since $M^*$ is popular, by Lemma 3 we have:

$$|A_f^{M^*}| = \left|[A_f^M \cup A_m^M \cup A_s^M \cup A_l^M] \cap A_f^{M^*}\right| = |\mathcal{F}|$$
$$= \left[|\mathcal{F}| - |A_f^M| - |A_m^M|\right] + \left|[A_f^M \cup A_m^M] \cap [A_f^{M^*} \cup A_s^{M^*}]\right|.$$

Rearranging, we get:

$$\left|[A_f^M \cup A_m^M] \cap A_s^{M^*}\right| = \left|[A_s^M \cup A_l^M] \cap A_f^{M^*}\right| - \left[|\mathcal{F}| - |A_f^M| - |A_m^M|\right].$$

Hence $\Delta(M^*, M) = \left[|\mathcal{F}| - |A_f^M| - |A_m^M|\right] + |A_m^M \cap A_f^{M^*}| + |A_l^M \cap A_s^{M^*}|$. The theorem follows immediately, since $|A_m^M \cap A_f^{M^*}|$ and $|A_l^M \cap A_s^{M^*}|$ are both non-negative, while $|\mathcal{F}| - |A_f^M| - |A_m^M| \geq 0$ by Lemma 4. ∎

Given $\langle G, M_0 \rangle$ and some popular matching $M^*$ of $G$, we do not need Lemma 5 to determine if $M^*$ is more popular than $M_0$ - instead, we can just count the number of applicants that prefer one matching to the other. Suppose, however, that $M^*$ is not more popular than $M_0$ so that $|A_f^{M_0}| + |A_m^{M_0}| = |\mathcal{F}|$, $|A_l^{M_0} \cap A_s^{M^*}| = 0$, and $|A_m^{M_0} \cap A_f^{M^*}| = 0$. Our aim is to use Lemma 5 as a guide in finding a popular matching that is more popular than $M_0$, or proving that no such matching exists.

First we remark that $|A_f^{M_0}| + |A_m^{M_0}| = |\mathcal{F}|$, for otherwise, *any* popular matching, including $M^*$, is more popular than $M_0$ by Lemma 5. It follows that $A_m^{M_0} \neq \emptyset$ or $A_l^{M_0} \neq \emptyset$, for otherwise, $M_0$ has signature $(|\mathcal{F}|, 0, |\mathcal{A}| - |\mathcal{F}|, 0)$, contradicting the assumption from the previous section that $M_0$ is not popular.

Suppose $A_m^{M_0} \neq \emptyset$, so that there is an applicant $a \in A_m^{M_0} \cap A_s^{M^*}$. By definition, such an applicant is even/unreachable. If there is a popular matching that pairs $a$ with a post in $f(a)$, then it must be more popular than $M_0$ by condition (ii) of Lemma 5. In order to test if there exists such a popular matching, we proceed in the following way.

Let $G'$ be the subgraph of $G$ defined in Figure 2 after step 3. So, $G'$ contains all edges between applicants and their $f$-posts and $s$-posts, except those between nodes in $\mathcal{O}$ and nodes in $\mathcal{O} \cup \mathcal{U}$. Now, modify $G'$ and $M^*$ by removing all edges between this particular applicant $a$ and posts in $s(a)$. Call the resulting structures $G'_a$ and $M^*_a$ respectively.

**Lemma 6** *There exists a popular matching which pairs a with some post in $f(a)$ if and only if $G'_a$ admits an augmenting path with respect to $M^*_a$.*

**Proof:** Suppose $G'_a$ admits an augmenting path $Q_a$ with respect to $M^*_a$. Since $M^*$ is popular, the only unmatched applicant in $M^*_a$ is $a$, and so $M^*_a \oplus Q_a$ matches $a$ with some post in $f(a)$. We want to claim that $M^*_a \oplus Q_a$ is popular. First note that its signature is of the form $(k, 0, |\mathcal{A}| - k, 0)$ for some $k \geq 0$, since it is a matching in a subgraph of $G'$, and $G'$ only contains edges between applicants and their $f$-posts and $s$-posts. Now, $M^*_a \oplus Q_a$ matches all posts in $\mathcal{F}$, since every post matched in $M^*_a$ is also matched in $M^*_a \oplus Q_a$. Recall that posts in $\mathcal{F}$ are incident in $G'$ to rank-1 edges only, and furthermore, odd posts in $\mathcal{F}$ are only adjacent to even applicants, while unreachable posts in $\mathcal{F}$ are only adjacent to unreachable applicants. Hence, $k = |\mathcal{F}|$, and so by Lemma 3, $M^*_a \oplus Q_a$ is popular, since its signature is $(|\mathcal{F}|, 0, |\mathcal{A}| - |\mathcal{F}|, 0)$.

Conversely, suppose that $G'_a$ admits no augmenting path with respect to $M^*_a$. Then, $M^*_a$ is a maximum matching in $G'_a$, which, since $a$ is unmatched in $M^*_a$, means that there is no $\mathcal{A}$-perfect matching in $G'_a$. But by Lemma 2(b), every popular matching is an $\mathcal{A}$-perfect matching in $G'$. Hence, every popular matching must contain an edge in $G' \setminus G'_a = \{(a, p) : p \in s(a)\}$. ∎

We now make use of the previous lemma. Begin by looking for an augmenting path in $G'_a$ with respect to $M^*_a$ by using depth-first search to construct the Hungarian tree $T_a$ rooted at $a$ (see Figure 3). If we find such a path $Q_a$, then by Lemma 5, $\langle M_0, M^*_a \oplus Q_a \rangle$ is a length-1 voting path to a popular matching. Otherwise, we repeat this process with some other $a' \in A^M_m \cap A^{M^*}_s$, if any.

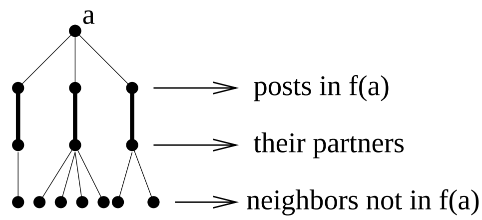


Figure 3: Example of a Hungarian tree $T_a$ rooted at $a$, with matched edges bold

Suppose we are successful in finding an augmenting path $Q_{a'}$ such that $M^*_{a'} \oplus Q_{a'}$ is popular. We claim that $Q_{a'}$ is edge disjoint from the set of edges in $T_a$, where $a \neq a'$.

For suppose otherwise. Then let $e$ be the *first* edge in $Q_{a'}$ that is also in $T_a$. Now, $G'_a$ admits an alternating path from $a$ through $e$. However, since $Q_a$ does not exist, this path cannot be extended in $G'_a$ to end in some unmatched post. Hence, $Q_{a'}$ must contain an alternating path from $e$ through the edge matching $a$ with $M^*(a)$ (which is missing in $G'_a$). But $M^*(a)$ is unmatched in $G'_a$, and hence if we join the two alternating paths above, we get an augmenting path $Q_a$ (from $a$ to $M^*(a)$ through the edge $e$) in $G'_a$. This gives the required contradiction.

Since we only need to examine each edge a constant number of times, it is clear that we can determine in linear time if there is a popular matching that pairs some applicant in $A^{M_0}_m \cap A^{M^*}_s$ with one of its $f$-posts.

If there is no such applicant, we repeat this procedure with applicants $a \in A^{M_0}_l \cap A^{M^*}_f$, who must by definition be even/unreachable. Here, though our aim is to find a popular matching that satisfies Lemma 5(iii) by pairing $a$ with a post in $s(a)$. It follows that for this to occur, $a$ must be even, since by Lemma 1 and 2(i), every unreachable applicant is matched by any popular matching to a post in $f(a)$.

Suppose we find such a matching $M^*_a \oplus Q_a$. Since $a$ is even, $M^*(a)$ is odd, and so any popular matching must match $M^*(a)$ along a rank-1 edge to an even applicant. However, $M^*(a)$ may be unmatched in $M^*_a \oplus Q_a$, as we removed all edges between $a$ and posts in $f(a)$ from $G'$ (including $(a, M^*(a))$). Hence we may need to augment $M^*_a \oplus Q_a$ in $G_1$. But every odd node $M^*(a)$ is adjacent to at least one other even applicant along a rank-1 edge, namely its predecessor in the odd length alternating path from a vertex unmatched w.r.t. $M^* \cap E_1$ to $M^*(a)$ in $G_1$). Hence, such an augmentation always exists. And it is easy to see that here too we only examine each edge a constant number of times.

By Lemmas 5 and 6, it is clear that the above algorithm correctly finds a length-1 voting path from $M_0$ to some popular matching, or proves that no such path exists. Additionally, given a popular matching $M^*$, we have just shown that the algorithm runs in linear time.

## 4 Length-2 voting paths

In this section, we show that, given any popular matching $M^*$ of $G$, the problem of finding a length-2 voting path from $M_0$ to some popular matching can be solved in linear time. We will assume that $M_0$ admits no shorter such voting path. In particular, this means that $M^*$ is not more popular than $M_0$, so that $A^{M_0}_m \neq \emptyset$, or $A^{M_0}_l \neq \emptyset$.

Suppose that $A^{M_0}_m \neq \emptyset$. Let $a \in A^{M_0}_m$ and let $T_a$ be the Hungarian tree associated with $a$, as described in Section 3. In the following lemma, we give a sufficient condition for the existence of a length-2 voting path from $M_0$ to $M^*$.

**Lemma 7** *Suppose there exists an applicant $a' \in T_a$ such that $M_0(a') \notin s(a')$ and $M^*(a') \in s(a')$. Then there exists a length-2 voting path from $M_0$ to $M^*$.*

**Proof:** Our goal is to find a matching $M_1$ such that (i) $M_1$ is more popular than $M_0$, and (ii) $a' \in A_l^{M_1}$. This last condition guarantees that $M^*$ is more popular than $M_1$ by Lemma 5(iii), and hence we get the length-2 voting path $\langle M_0, M_1, M^* \rangle$.

Before discussing how we construct $M_1$, we first need to show that $a' \in A_f^{M_0} \cup A_m^{M_0}$: It is clear that $a' \notin A_l^{M_0}$, for otherwise $a' \in A_l^{M_0} \cap A_s^{M^*}$ and $M^*$ is more popular than $M_0$ by Lemma 5(iii) - a contradiction. Suppose then that $a' \in A_s^{M_0}$. By definition we have that $M_0(a') \notin s(a')$, and so it must be the case that $M_0(a')$ is odd/unreachable and belongs to $\mathcal{F}$. But $M_0$ matches all posts in $\mathcal{F}$ to applicants in $A_f^{M_0} \cup A_m^{M_0}$, for otherwise $M^*$ is more popular than $M_0$ by Lemma 5(i) - a contradiction. Hence, $a' \in A_f^{M_0} \cup A_m^{M_0}$.

Now we need to show that $G'_a$ contains no edge between $a'$ and $l(a')$: For suppose this is not the case. Then since $l(a')$ is strictly the least preferred post of $a'$, we have that $s(a') = \{l(a')\}$. By definition, no other applicant is adjacent to $l(a')$, and so $l(a')$ is a leaf node in $T_a$ with parent $a'$. It follows from the construction of $T_a$ that $l(a')$ is unmatched in $M_a^*$, and hence $M_a^*$ admits an augmenting path from $a$ through $a'$ to $l(a')$. This contradicts our assumption that $M_0$ admits no length-1 voting path to a popular matching. Hence, $G'_a$ contains no edge between $a'$ and $l(a')$.

Finally, we describe how to construct $M_1$. Add an edge between $a'$ and $l(a')$ to $G'_a$. From the argument above, we have that $G'_a$ admits an augmenting path $Q_a$ from $a$ through $a'$ and ending in $l(a')$. Let $M_1 = M_a^* \oplus Q_a$. The signature of $M_1$ is $(|\mathcal{F}|, 0, |\mathcal{A}| - |\mathcal{F}| - 1, 1)$, by an argument similar to the one used in the proof of Lemma 6, except that here we have one applicant $a' \in A_l^{M_1}$. It remains to show that $M_1$ is more popular than $M_0$.

$$
\begin{aligned}
\Delta(M_1, M_0) &= \left| [A_m^{M_0} \cup A_s^{M_0} \cup A_l^{M_0}] \cap A_f^{M_1} \right| + \left| A_l^{M_0} \cap A_s^{M_1} \right| - \left| [A_f^{M_0} \cup A_m^{M_0}] \cap [A_s^{M_1} \cup A_l^{M_1}] \right| \\
&\qquad \{\textit{NB: } |A_s^{M_0} \cap A_l^{M_1}| = 0, \textit{ since } a' \notin A_s^{M_0}\} \\
&\geq \left| [A_m^{M_0} \cup A_s^{M_0} \cup A_l^{M_0}] \cap A_f^{M_1} \right| - \left| [A_f^{M_0} \cup A_m^{M_0}] \cap [A_s^{M_1} \cup A_l^{M_1}] \right| \\
&= \left| [A_f^{M_0} \cup A_m^{M_0} \cup A_s^{M_0} \cup A_l^{M_0}] \cap A_f^{M_1} \right| - \left| A_f^{M_0} \cap [A_f^{M_1} \cup A_s^{M_1} \cup A_l^{M_1}] \right| - \left| A_m^{M_0} \cap [A_s^{M_1} \cup A_l^{M_1}] \right| \\
&= |\mathcal{F}| - |A_f^{M_0}| - \left| A_m^{M_0} \cap [A_s^{M_1} \cup A_l^{M_1}] \right| \qquad \{\textit{since } |A_f^{M_1}| = |\mathcal{F}|\} \\
&= |A_m^{M_0}| - \left| A_m^{M_0} \cap [A_s^{M_1} \cup A_l^{M_1}] \right| \qquad \{\textit{by Lemmas 4 and 5(i)}\} \\
&= \left| A_m^{M_0} \cap A_f^{M_1} \right| > 0 \qquad \{\textit{since } a \in A_m^{M_0} \cap A_f^{M_1}\}
\end{aligned}
$$

∎

It is clear that in linear time, we can check if there exists an applicant $a' \in T_a$ such that $M_0(a') \notin s(a')$ and $M^*(a') \in s(a')$, and if so, we can construct the matching $M_1$.

Suppose there is no such applicant in $T_a$. Our aim then is to find a different popular matching in which such an applicant exists. We will find such a matching by searching for a particular type of alternating cycle $\mathcal{C}$ in $G'_a$ with respect to $M^*$. First though, we make some observations about $T_a$ and $G'_a$.

By construction, posts in $T_a$ are discovered along unmatched edges. Also, no post $p$ is a leaf node in $T_a$, since then $p$ would be unmatched in $M_a^*$, and $M_a^*$ would admit an augmenting path, which contradicts our assumption that $M_0$ admits no length-1 voting path to a popular matching. Therefore, posts in $T_a$ have degree 2.

By construction, an applicant $a'' \in T_a \setminus \{a\}$ is discovered along a matched edge. If $a''$ is even or unreachable, then $a''$ is incident to at least one unmatched child edge in $G'_a$, since $f(a'')$ and $s(a'')$ are non-empty and disjoint. If $a''$ is odd, then $a''$ must also be incident to at least one unmatched child edge in $G'_a$ - since $a''$ is odd, $G_1$ admits an odd-length augmenting path from an unmatched vertex to $a''$, and the last edge $e(a'')$ on this path is unmatched in $M^*$.

Since every node in $T_a \setminus \{a\}$ is incident to at least one matching edge and one non-matching edge (w.r.t. the matching $M^*$) in $G'_a$, we can build the following alternating path $Q$. Begin with an edge $(a, p)$ where $p \in f(a)$. Let the successor of any post $p$ in $Q$ be its matched partner $M^*(p)$. The successor of any even/unreachable applicant $a''$ is any post in $f(a'')$ if $M^*(a'') \in s(a'')$, or any post in $s(a'')$ if $M^*(a'') \in f(a'')$. Finally, the successor of any odd applicant $a''$ is the post incident to $e(a'')$. Since $|T_a|$ is finite, at some point this alternating path must form a cycle $\mathcal{C}$ by adding a post that is already in the path. This procedure clearly takes linear time.

It is clear that from Lemma 2 that $M^* \oplus \mathcal{C}$ is a popular matching. Now, if we can show that $M^* \oplus \mathcal{C}$ has some applicant $a' \in T_a \setminus \{a\}$ such that $M_0(a') \notin s(a')$ and $(M^* \oplus \mathcal{C})(a') \in s(a')$, then we can use $M^* \oplus \mathcal{C}$ as the popular matching in Lemma 7.

First, we prove that $\mathcal{C}$ contains at least one applicant $a'$ such that $M^*(a') \in f(a')$. Since $M^* \oplus \mathcal{C}$ matches $a'$ with $s(a')$ by construction, the final step will be to show that $M_0(a') \notin s(a')$.

**Lemma 8** *$\mathcal{C}$ contains at least one applicant $a'$ such that $M^*(a') \in f(a')$.*

**Proof:** Note that the length of $\mathcal{C}$ is at least 4 since the predecessor and successor of each applicant are always distinct. Also note that if $\mathcal{C}$ contains an even/unreachable applicant, then its predecessor/successor is an $f$-post, whose partner in $M^*$ is the required applicant. The only way that that $\mathcal{C}$ may not have contain an $f$-post is if all the applicants in $\mathcal{C}$ are odd. We show that this cannot happen.

Let $a''$ be the first odd applicant in $Q$ that is in $\mathcal{C}$. Then by construction, $\mathcal{C}$ contains a subpath from $a''$ through $e(a'')$ to some post $p$ that is unmatched in $M^* \cap E_1$. It follows that $p$ is even, since odd/unreachable posts are matched in $M^* \cap E_1$ by Lemmas 1(b) and 2(i). Since $p$ is matched in $M^*$, its partner $M^*(p)$ must be even (again by Lemmas 1(b) and 2(i)), and so $\mathcal{C}$ contains an even applicant. ∎

**Lemma 9** *Suppose there exists no applicant $a' \in T_a$ such that $M_0(a') \notin s(a')$ and $M^*(a') \in s(a')$. Then for each $a' \in T_a \setminus \{a\}$, $M_0(a') \notin s(a')$ if and only if $M^*(a') \in f(a')$.*

**Proof:** Let $a'$ be any applicant in $T_a \setminus \{a\}$ such that $M_0(a') \notin s(a')$. Then by the assumption in the statement of the lemma, we have $M^*(a') \notin s(a')$, and so $M^*(a') \in f(a) \subseteq \mathcal{F}$, since $M^*$ is popular. Hence, $|T_a \setminus \{a\} \cap [\mathcal{A} - A_s^{M_0}]|$ is at most the number of $f$-posts in $T_a$, i.e.,

$$|T_a \setminus \{a\} \cap [\mathcal{A} - A_s^{M_0}]| \leq |T_a \cap \mathcal{F}|. \tag{1}$$

Now, since there is no augmenting path in $T_a$, we have that $|T_a \setminus \{a\} \cap \mathcal{A}| = |T_a \cap \mathcal{P}|$. Partitioning $\mathcal{A}$ into $\mathcal{A} \setminus A_s^{M_0}$ and $A_s^{M_0}$ and posts in $T_a$ into $T_a \cap \mathcal{F}$ and $T_a \cap S$, where $S$ is the set of all $s$-posts, we get $|T_a \setminus \{a\} \cap [(\mathcal{A} \setminus A_s^{M_0}) \cup A_s^{M_0}]| = |T_a \cap \mathcal{F}| + |T_a \cap S|$. Note that no applicant in $A_s^{M_0}$ can be matched by $M_0$ to an odd/unreachable post, otherwise $|A_f^{M_0}| + |A_m^{M_0}| < |\mathcal{F}|$ and $M^*$ would have been more popular than $M_0$ by Lemma 5(i). Hence each applicant in $A_s^{M_0}$ has to be matched by $M_0$ to one of its most preferred even posts, that is one of its $s$-posts, so $|T_a \setminus \{a\} \cap A_s^{M_0}| \leq |T_a \cap S|$. We thus get,

$$|T_a \setminus \{a\} \cap [\mathcal{A} \setminus A_s^{M_0}]| \geq |T_a \cap \mathcal{F}|. \tag{2}$$

Combining (1) and (2), we have $|T_a \setminus \{a\} \cap [\mathcal{A} - A_s^{M_0}]| = |T_a \cap \mathcal{F}| = |T_a \setminus \{a\} \cap A_f^{M^*}|$. That is, the number of applicants $a'$ in $T_a \setminus \{a\}$ that satisfy $M^*(a') \in f(a')$ is equal to the number of applicants in $T_a \setminus \{a\}$ that satisfy $M_0(a') \notin s(a')$. But each applicant in $T_a \setminus \{a\}$ that satisfies $M_0(a') \notin s(a')$ has to satisfy $M^*(a') \in f(a')$ by the statement of the lemma. So the equivalence follows immediately. ∎

By Lemma 8, $\mathcal{C}$ contains at least one applicant $a'$ such that $M^*(a') \in f(a')$. By Lemma 9, we have that $M_0(a') \notin s(a')$. Since $M^* \oplus \mathcal{C}$ matches $a'$ with $s(a')$, $M^* \oplus \mathcal{C}$ satisfies the sufficient condition in Lemma 7. Hence there exists a length-2 voting path from $M_0$ to the popular matching $M^* \oplus \mathcal{C}$.

As in Section 3, if $A_m^{M_0} = \emptyset$, then $A_l^{M_0}$ is non empty, and we perform an analogous procedure on the Hungarian tree associated with some $a \in A_l^{M_0}$. This finishes the proof of Theorem 1 (stated in Section 1). The overall algorithm is presented in Figure 4.

**Voting-Path**($G = (\mathcal{A} \cup \mathcal{P}, E), M_0$)
  **if** $M_0$ has signature $(|\mathcal{F}|, 0, |\mathcal{A}| - |\mathcal{F}|, 0)$ **then**
    **return** $\langle M_0 \rangle$
  Let $M^*$ be any popular matching of $G$
  **if** $M^*$ is more popular than $M_0$ **then**
    **return** $\langle M_0, M^* \rangle$
  **for each** applicant $a \in A_m^{M_0} \cap A_s^{M^*}$ and **each** even applicant $a \in A_l^{M_0} \cap A_f^{M^*}$
    Construct the Hungarian tree $T_a$ with respect to $M_a^*$, including only unmarked edges
    Mark all edges in $T_a$
    **if** $T_a$ contains an augmenting path $Q_a$ **then**
      Augment $M_a^* \oplus Q_a$ in $G_1$ from $M^*(a)$ *// applies only when $a \in A_l^{M_0} \cap A_f^{M^*}$*
      **return** $\langle M_0, M_a^* \oplus Q_a \rangle$
    Let $T_a$ be the first Hungarian tree constructed for any $a \in A_m^{M_0} \cap A_s^{M^*}$ or even $a \in A_l^{M_0} \cap A_f^{M^*}$
    **if** there exists no $a' \in T_a \setminus \{a\}$ such that $M_0(a') \notin s(a')$ and $M^*(a') \in s(a')$ **then**
      Construct $\mathcal{C}$ in $G'_a$ as described in Section 4
      Let $M^* = M^* \oplus \mathcal{C}$
      Let $T_a$ be the Hungarian tree associated with $a$ and the new $M^*$
    Add $(a', l(a'))$ to $T_a$, and find the augmenting path $Q_a$ in $T_a$
    Let $M_1 = M_a^* \oplus Q_a$
    Augment $M_1$ in $G_1$ from $M^*(a)$ *// applies only when $a \in A_l^{M_0} \cap A_f^{M^*}$*
    **return** $\langle M_0, M_1, M^* \rangle$

Figure 4: Linear-time algorithm for finding a shortest-length voting path

## 5 Conclusions and Open Problems

We considered the problem of computing a shortest-length voting path in a problem instance $\langle G, M_0 \rangle$. We showed that if $G$ admits a popular matching, then there is always a voting path of length at most 2 from $M_0$ to some popular matching. Furthermore, we showed that the problem of finding a shortest-length voting path from $M_0$ to some popular matching has the same complexity as the problem of computing a popular matching in $G$. We conclude with an open problem.

Suppose we are given an instance $G$ of the popular matching problem that admits no popular matching. Rather than return "*no popular matching*", we want to return a matching that is *as popular as possible*. Since the directed graph $H$ of the *more popular than* relation has no sink, we might consider matchings that, at the very least, are members of sink components in the strongly-connected component graph of $H$. Subject to this, a most popular matching could be defined in various ways, for example one that has the smallest out-degree in $H$. However, $H$ has size exponential in $G$, and so we are interested in the complexity of finding such matchings.

## References


[1] A. Abdulkadiroğlu and T. Sönmez. *Random serial dictatorship and the core from random endowments in house allocation problems*. Econometrica, 66(3):689–701, 1998.

[2] D.J. Abraham, K. Cechlárová, D.F. Manlove, K. Mehlhorn. *Pareto-optimality in house allocation problems*. In Proceedings of ISAAC '04: the 15th Annual International Symposium on Algorithms and Computation, volume 3341 of Lecture Notes in Computer Science, pages 3-15, Springer-Verlag, 2004.

[3] D.J. Abraham, R.W. Irving, T. Kavitha, and K. Mehlhorn. *Popular Matchings*. In Proceedings of SODA '05: the 16th Annual ACM-SIAM Symposium on Discrete Algorithms, pages 424-432, 2005.

[4] Amazon.com DVD Rental: see http://www.amazon.co.uk.

[5] E. Diamantoudi, E. Miyagawa, and L. Xue. *Random paths to stability in the roommate problem*. Games and Economic Behavior, 48(1):18-28, 2004.

[6] S.P. Fekete, M. Skutella, and G.J. Woeginger. *The complexity of economic equilibria for house allocation markets*. Information Processing Letters, 88:219-223, 2003

[7] D. Gale and L.S. Shapley. *College admissions and the stability of marriage*. American Mathematical Monthly, 69:9–15, 1962.

[8] P. Gardenfors. *Match Making: assignments based on bilateral preferences*. Behavioural Sciences, 20:166–173, 1975.

[9] D. Gusfield and R.W. Irving *The Stable Marriage Problem: Structure and Algorithms*. MIT Press, 1989.

[10] A. Hylland and R. Zeckhauser. *The efficient allocation of individuals to positions*. Journal of Political Economy, 87(2):293–314, 1979.

[11] R.W. Irving. *An efficient algorithm for the "stable roommates" problem*. Journal of Algorithms, 6:577-596, 1985

[12] R.W. Irving, T. Kavitha, K. Mehlhorn, D. Michail, and K. Paluch. *Rank-maximal matchings*. In Proceedings of SODA '04: the 15th ACM-SIAM Symposium on Discrete Algorithms, pages 68–75. ACM-SIAM, 2004.

[13] D.E. Knuth. *Stable Marriage and its Relation to Other Combinatorial Problems*. CRM Proceedings and Lecture Notes, volume 10, American Mathematical Society. English translation of *Mariages Stables*, Les Presses de L'Université de Montréal, 1976

[14] H. Landau. *On dominance relations and the structure of animal societies, III: the condition for secure structure*. Bulletin of Mathematical Biophysics, 15(2):143-148, 1953

[15] M. Mahdian. *Random Popular Matchings*. In submission.

[16] J. Mestre. *Weighted Popular Matchings*. In submission.

[17] Netflix DVD Rental: see http://www.netflix.com.

[18] A.E. Roth and A. Postlewaite. *Weak versus strong domination in a market with indivisible goods*. Journal of Mathematical Economics, 4:131–137, 1977.

[19] A.E. Roth and J.H. Vande Vate. *Random Paths to Stability in Two-Sided Matching*. Econometrica, 58:1475-1480, 1990.

[20] A. Tamura. *Transformation from arbitrary matchings to stable matchings*. Journal of Combinatorial Theory, Series A, 62(2):310-323, 1993.

[21] Y. Yuan. *Residence exchange wanted: a stable residence exchange problem*. European Journal of Operational Research, 90:536–546, 1996.

[22] L. Zhou. *On a conjecture by Gale about one-sided matching problems*. Journal of Economic Theory, 52(1):123–135, 1990.