Homework 2: Trees and Parallel Reduction
15-814: Types and Programming Languages
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Out: 9/26/13
Due: 10/3/13 (12 PM)

In this assignment, we will consider the Church encoding of binary trees, and prove a lemma used in lecture to prove the Church–Rosser theorem. Please submit your work by the start of lecture on the due date, as a PDF to cangiuli@cs.cmu.edu. Include the phrase “15-814 Homework 2” in the subject line of your email.

1 Church trees

In lecture, we saw that λ-calculus can express natural numbers via Church encodings. Recall that a number is either zero or succ(M) for some number M. Furthermore, we can define the test function ifz(M, N1, N2) satisfying the specification:

\[
\begin{align*}
\text{ifz}(\text{zero}, N_1, N_2) & \equiv \beta N_1 \\
\text{ifz}(\text{succ}(M), N_1, N_2) & \equiv \beta N_2
\end{align*}
\]

The Church encoding of these constructs is:

\[
\begin{align*}
\text{zero} &= \lambda z.\lambda s.\text{z} \\
\text{succ}(M) &= \lambda z.\lambda s.(M \, z \, s) \\
\text{ifz}(M, N_1, N_2) &= M \, N_1 \, (\lambda x. N_2)
\end{align*}
\]

Your task is to construct the Church encoding of binary trees. For our purposes, a binary tree is either an unlabeled leaf, or a node with two distinguished subtrees. You must define:

\[
\begin{align*}
\text{leaf} & \\
\text{branch}(M_1, M_2) & \\
\text{rec}_{\text{tree}}(M, N_1, N_2)
\end{align*}
\]

where the \( \text{rec}_{\text{tree}} \) is a recursor which “walks through the tree,” satisfying the specification:

\[
\begin{align*}
\text{rec}_{\text{tree}}(\text{leaf}, N_1, N_2) & \equiv \beta N_1 \\
\text{rec}_{\text{tree}}(\text{branch}(M_1, M_2), N_1, N_2) & \equiv \beta N_2(\text{rec}_{\text{tree}}(M_1, N_1, N_2))(\text{rec}_{\text{tree}}(M_2, N_1, N_2))
\end{align*}
\]

Task 1. Implement these constructs in \( \lambda \) terms.

Task 2. Write a \( \lambda \) term \( M \) which computes the height of a tree (given in the encoding you defined above). To be precise, for any tree \( T \), \( MT \) must \( \beta \)-reduce to a Church numeral \( N \) where \( N \) encodes the height of the tree \( T \).

Please make your code easy to follow. You may use \text{plus} and \text{pred} without defining them.

You may define the height of the one-node tree to be either one or zero, but write down which.

(Hint) Use \text{rec}_{\text{tree}}.

(Hint) As in real-life programming, it may be helpful to implement some subproblems, give them names, and use them later.
2 Parallel reduction

Recall the definitions of $\beta$-reduction:

\[
\begin{align*}
M & \rightarrow_\beta M' \\
MN & \rightarrow_\beta M'N \quad \text{(ap)}
\end{align*}
\]

and parallel reduction:

\[
\begin{align*}
M & \rightarrow_{\|} M \\
\lambda x.M & \rightarrow_{\|} (\lambda x.M') \quad \text{(abs)}
\end{align*}
\]

(Unlike on the previous homework, we include the (\text{abs}) rule so that $\rightarrow_{\|}$ reduces under $\lambda$-abstractions.)

We define the Kleene closure of these relations in the standard way, yielding the relations $\rightarrow^*_{\beta}$ and $\rightarrow^*_{\|}$. Note that the $\rightarrow^*_{\beta}$ relation automatically satisfies the following compatibility properties, which you may use without proof.

\textbf{Lemma 2.1.} If $M \rightarrow^*_{\beta} M'$ and $M' \rightarrow^*_{\beta} M''$ then $M \rightarrow^*_{\beta} M''$.

\textbf{Lemma 2.2.} If $M \rightarrow^*_{\|} M'$ then for any $M''$ we have $M M'' \rightarrow^*_{\|} M' M''$.

\textbf{Lemma 2.3.} If $M \rightarrow^*_{\|} M'$ then for any $M''$ we have $M'' M \rightarrow^*_{\|} M'' M'$.

\textbf{Lemma 2.4.} If $M \rightarrow^*_{\|} M'$ then for any $x$ we have $\lambda x.M \rightarrow^*_{\|} \lambda x.M'$.

In lecture, we proved that $\rightarrow_{\beta}$ satisfies the Church–Rosser theorem if $\rightarrow^*_{\beta}$ has the diamond property, and that $\rightarrow^*_{\|}$ has the diamond property. Thus, to prove the Church–Rosser theorem, it suffices to show that $\rightarrow^*_{\|}$ and $\rightarrow^*_{\beta}$ coincide. Your task is to prove one direction of this.

\textbf{Task 3.} Prove that if $M \rightarrow^*_{\|} N$, then $M \rightarrow^*_{\beta} N$.

(Hint) Start by writing out the rules for $\rightarrow^*_{\|}$ and $\rightarrow^*_{\beta}$.