Homework 2: Trees and Parallel Reduction

15-814: Types and Programming Languages
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Out: 9/26/13
Due: 10/3/13 (12 PM)

In this assignment, we will consider the Church encoding of binary trees, and prove a lemma used in lecture to prove the Church–Rosser theorem. Please submit your work by the start of lecture on the due date, as a PDF to `cangiuli@cs.cmu.edu`. Include the phrase "15-814 Homework 2" in the subject line of your email.

1 Church trees

In lecture, we saw that λ-calculus can express natural numbers via Church encodings. Recall that a number is either $\texttt{zero}$ or $\texttt{succ}(M)$ for some number M. Furthermore, we can define the test function $\texttt{ifz}(M, N_1, N_2)$ satisfying the specification:

$$\texttt{ifz}(\texttt{zero}, N_1, N_2) \equiv_\beta N_1$$
$$\texttt{ifz}(\texttt{succ}(M), N_1, N_2) \equiv_\beta N_2$$

The Church encoding of these constructs is:

$$\texttt{zero} = \lambda z.\lambda s.z$$
$$\texttt{succ}(M) = \lambda z.\lambda s.s\,(M\,z\,s)$$
$$\texttt{ifz}(M, N_1, N_2) = M\,N_1\,(\lambda x.N_2)$$

Your task is to construct the Church encoding of binary trees. For our purposes, a binary tree is either an unlabeled leaf, or a node with two distinguished subtrees. You must define:

$$\texttt{leaf}$$
$$\texttt{branch}(M_1, M_2)$$
$$\texttt{rec}_{\texttt{tree}}(M, N_1, N_2)$$

where the $\texttt{rec}_{\texttt{tree}}$ is a recursor which "walks through the tree," satisfying the specification:

$$\texttt{rec}_{\texttt{tree}}(\texttt{leaf}, N_1, N_2) \equiv_\beta N_1$$
$$\texttt{rec}_{\texttt{tree}}(\texttt{branch}(M_1, M_2), N_1, N_2) \equiv_\beta N_2\,(\texttt{rec}_{\texttt{tree}}(M_1, N_1, N_2))\,(\texttt{rec}_{\texttt{tree}}(M_2, N_1, N_2))$$

Task 1. *Implement these constructs in λ terms.*

Task 2. *Write a λ term M which computes the height of a tree (given in the encoding you defined above). To be precise, for any tree T, $M\,T$ must β-reduce to a Church numeral N where N encodes the height of the tree T.*

Please make your code easy to follow. ☺ You may use `plus` *and* `pred` *without defining them. You may define the height of the one-node tree to be either one or zero, but write down which.*
(Hint) *Use* $\texttt{rec}_{\texttt{tree}}$.
(Hint) *As in real-life programming, it may be helpful to implement some subproblems, give them names, and use them later.*

2 Parallel reduction

Recall the definitions of β-reduction:

$$\frac{M \to_\beta M'}{M\,N \to_\beta M'\,N}\ (\beta\mathsf{ap}_1) \qquad \frac{N \to_\beta N'}{M\,N \to_\beta M\,N'}\ (\beta\mathsf{ap}_2)$$

$$\frac{M \to_\beta M'}{(\lambda x.M) \to_\beta (\lambda x.M')}\ (\beta\mathsf{abs}) \qquad \frac{}{(\lambda x.M)\,N \to_\beta [N/x]M}\ (\beta)$$

and parallel reduction:

$$\frac{}{M \to_\parallel M}\ (\parallel\mathsf{refl}) \qquad \frac{M \to_\parallel M'}{(\lambda x.M) \to_\parallel (\lambda x.M')}\ (\parallel\mathsf{abs})$$

$$\frac{M \to_\parallel M' \quad N \to_\parallel N'}{M\,N \to_\parallel M'\,N'}\ (\parallel\mathsf{ap}) \qquad \frac{M \to_\parallel M' \quad N \to_\parallel N'}{(\lambda x.M)\,N \to_\parallel [N'/x]M'}\ (\parallel\beta)$$

(Unlike on the previous homework, we include the ($\beta\mathsf{abs}$) rule so that $\to_\beta$ reduces under λ-abstractions.)

We define the Kleene closure of these relations in the standard way, yielding the relations $\to_\beta^*$ and $\to_\parallel^*$. Note that the $\to_\beta^*$ relation automatically satisfies the following compatibility properties, which you may use without proof.

Lemma 2.1. *If $M \to_\beta^* M'$ and $M' \to_\beta^* M''$ then $M \to_\beta^* M''$.*

Lemma 2.2. *If $M \to_\beta^* M'$ then for any M'' we have $M\,M'' \to_\beta^* M'\,M''$.*

Lemma 2.3. *If $M \to_\beta^* M'$ then for any M'' we have $M''\,M \to_\beta^* M''\,M'$.*

Lemma 2.4. *If $M \to_\beta^* M'$ then for any x we have $\lambda x.M \to_\beta^* \lambda x.M'$.*

In lecture, we proved that $\to_\beta$ satisfies the Church–Rosser theorem if $\to_\beta^*$ has the diamond property, and that $\to_\parallel^*$ has the diamond property. Thus, to prove the Church–Rosser theorem, it suffices to show that $\to_\beta^*$ and $\to_\parallel^*$ coincide. Your task is to prove one direction of this.

Task 3. *Prove that if $M \to_\parallel^* N$, then $M \to_\beta^* N$.*
(Hint) *Start by writing out the rules for $\to_\beta^*$ and $\to_\parallel^*$.*