Constructive Logic (15-317), Fall 2020 Assignment 7: Classical Logic

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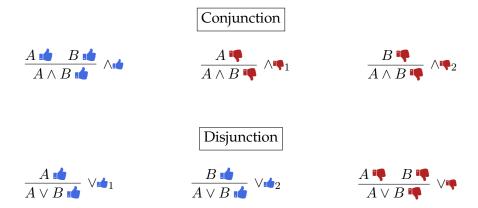
Due: Friday, October 23, 11:59 pm

This assignment must be submitted electronically via Gradescope. Submit your homework as a pdf containing your written solutions.

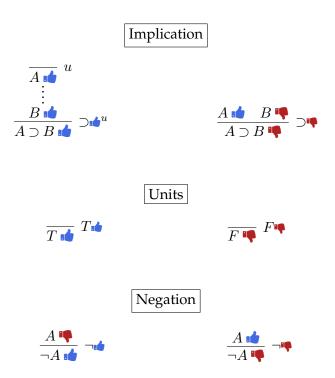
1 A New Constructive Logic: Classical Logic

Intuitionistic logic is based on the idea that the fundamental mathematical activity is to *affirm* the truth of something using evidence. Classical logic should be understood as a different, *dialectical* model of mathematical activity, in which one party tries to affirm and the other party tries to deny. Whereas the central duality of intuitionistic natural deduction was between the *introduction* and *elimination* rules for truth A true, in classical natural deduction, each proposition is explained through the interaction between rules for affirmation $A \bullet (i.e. A true)$ and rules for denial $A \bullet (i.e. A false)$, a contest governed by the nullary form of judgment # (contradiction).¹

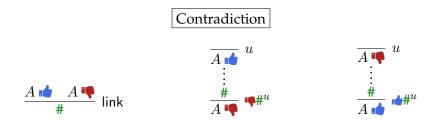
To be precise, each connective comes equipped with introduction rules for *both* affirmation and denial; classical negation $\neg A$ implements the involutive "change of perspective" between player (affirmation) and opponent (denial).



¹The symbols A = and A = can be written using the provided macros; if you have trouble using these symbols, it is also acceptable to write A true and A false, as from lecture.



In classical natural deduction, affirmation and denial compete with each other in a *formal contradiction*, a nullary judgment written **#**. The rules for contradictions are as follows:



Using the rules **#** and **#**, all the usual "elimination rules" for truth can be *derived* in classical natural deduction.

Task 1 (14 pts). Recall the introduction and elimination rules for the universal quantifier in intuitionistic natural deduction:

$$\begin{array}{c} [z:\tau] \\ \vdots \\ \hline \\ \\ \hline \\ \hline \\ \hline \\ \\ \hline \\ \\ \hline \\ \hline \\ \\ \hline \\ \hline \\ \hline \\ \\ \hline \\ \hline$$

Now it's your turn: *invent* affirmation and denial rules $\forall \mathbf{a}, \forall \mathbf{e}$ for the universal quantifier, as an extension to the classical natural deduction calculus which we have seen so far.

Task 2 (14 pts). Recall the introduction and elimination rules for the existential quantifier in intuitionistic natural deduction:

$$\begin{array}{c} [z:\tau] \quad \overline{A(z) \ true} & u \\ \vdots \\ \exists x:\tau. \ A(x) \ true \end{array} \exists \mathsf{I} \qquad \qquad \begin{array}{c} \exists x:\tau. \ A(x) \ true & C \ true \\ \hline C \ true \end{array} \exists \mathsf{E}^{z,u} \end{array}$$

As in the previous task, invent affirmation and denial rules $\exists d$, $\exists \phi$ for the existential quantifier.

Task 3 (22 pts). Using the rules you invented in the previous tasks, show that the following *elimination* rules for the universal and the extistential quantifier are derivable.

$$\begin{array}{c} [z:\tau] \quad \overline{A(z)} \stackrel{\mathfrak{l}}{•} \\ \vdots \\ \hline \\ \hline \\ C(t) \stackrel{\mathfrak{l}}{\bullet} \end{array} \quad \forall \mathfrak{l} \bullet \mathsf{E} \end{array} \qquad \begin{array}{c} [z:\tau] \quad \overline{A(z)} \stackrel{\mathfrak{l}}{\bullet} \\ \vdots \\ \hline \\ \hline \\ C \stackrel{\mathfrak{l}}{\bullet} \end{array} \quad \exists \mathfrak{l} \bullet \mathsf{E}^{z,u} \end{array}$$