Constructive Logic (15-317), Fall 2020
Assignment 7: Classical Logic

Instructor: Karl Crary
TAs: Avery Cowan, Matthew McQuaid, Long Pham, Ariel Uy

Due: Friday, October 23, 11:59 pm

This assignment must be submitted electronically via Gradescope. Submit your homework as a pdf containing your written solutions.

1 A New Constructive Logic: Classical Logic

Intuitionistic logic is based on the idea that the fundamental mathematical activity is to *affirm* the truth of something using evidence. Classical logic should be understood as a different, *dialectical* model of mathematical activity, in which one party tries to affirm and the other party tries to deny. Whereas the central duality of intuitionistic natural deduction was between the *introduction* and *elimination* rules for truth $A\ true$, in classical natural deduction, each proposition is explained through the interaction between rules for affirmation $A\ \text{👍}$ (i.e. $A\ true$) and rules for denial $A\ \text{👎}$ (i.e. $A\ false$), a contest governed by the nullary form of judgment $\#$ (contradiction).[1]

To be precise, each connective comes equipped with introduction rules for *both* affirmation and denial; classical negation $\neg A$ implements the involutive "change of perspective" between player (affirmation) and opponent (denial).

Conjunction

$$\frac{A\ \text{👍} \quad B\ \text{👍}}{A \wedge B\ \text{👍}}\ \wedge\text{👍} \qquad \frac{A\ \text{👎}}{A \wedge B\ \text{👎}}\ \wedge\text{👎}_1 \qquad \frac{B\ \text{👎}}{A \wedge B\ \text{👎}}\ \wedge\text{👎}_2$$

Disjunction

$$\frac{A\ \text{👍}}{A \vee B\ \text{👍}}\ \vee\text{👍}_1 \qquad \frac{B\ \text{👍}}{A \vee B\ \text{👍}}\ \vee\text{👍}_2 \qquad \frac{A\ \text{👎} \quad B\ \text{👎}}{A \vee B\ \text{👎}}\ \vee\text{👎}$$

[1] The symbols $A\ \text{👍}$ and $A\ \text{👎}$ can be written using the provided macros; if you have trouble using these symbols, it is also acceptable to write $A\ true$ and $A\ false$, as from lecture.

Implication

$$\dfrac{\begin{array}{c}\overline{A\ 👍}^{\,u}\\ \vdots\\ B\ 👍\end{array}}{A \supset B\ 👍}\ {\supset}👍^{u} \qquad\qquad \dfrac{A\ 👍 \quad B\ 👎}{A \supset B\ 👎}\ {\supset}👎$$

Units

$$\dfrac{}{T\ 👍}\ T👍 \qquad\qquad \dfrac{}{F\ 👎}\ F👎$$

Negation

$$\dfrac{A\ 👎}{\neg A\ 👍}\ \neg👍 \qquad\qquad \dfrac{A\ 👍}{\neg A\ 👎}\ \neg👎$$

In classical natural deduction, affirmation and denial compete with each other in a *formal contradiction*, a nullary judgment written $\boxed{\#}$. The rules for contradictions are as follows:

Contradiction

$$\dfrac{A\ 👍 \quad A\ 👎}{\#}\ \mathsf{link} \qquad \dfrac{\begin{array}{c}\overline{A\ 👍}^{\,u}\\ \vdots\\ \#\end{array}}{A\ 👎}\ 👎\#^{u} \qquad \dfrac{\begin{array}{c}\overline{A\ 👎}^{\,u}\\ \vdots\\ \#\end{array}}{A\ 👍}\ 👍\#^{u}$$

Using the rules #👍 and #👎, all the usual "elimination rules" for truth can be *derived* in classical natural deduction.

Task 1 (14 pts)**.** Recall the introduction and elimination rules for the universal quantifier in intuitionistic natural deduction:

$$\dfrac{\begin{array}{c}[z : \tau]\\ \vdots\\ A(z)\ \mathit{true}\end{array}}{\forall x : \tau.\ A(x)\ \mathit{true}}\ \forall\mathsf{I}^{z} \qquad\qquad \dfrac{t : \tau \quad \forall x : \tau.\ A(x)\ \mathit{true}}{A(t)\ \mathit{true}}\ \forall\mathsf{E}$$

Now it's your turn: *invent* affirmation and denial rules ∀👍, ∀👎 for the universal quantifier, as an extension to the classical natural deduction calculus which we have seen so far.

Task 2 (14 pts)**.** Recall the introduction and elimination rules for the existential quantifier in intuitionistic natural deduction:

$$\dfrac{t:\tau \quad A(t)\ \mathit{true}}{\exists x:\tau.\,A(x)\ \mathit{true}}\ \exists\mathsf{I} \qquad\qquad \dfrac{\exists x:\tau.\,A(x)\ \mathit{true} \qquad \begin{array}{c}[z:\tau]\quad \overline{A(z)\ \mathit{true}}^{\,u}\\ \vdots \\ C\ \mathit{true}\end{array}}{C\ \mathit{true}}\ \exists\mathsf{E}^{z,u}$$

As in the previous task, invent affirmation and denial rules $\exists$👍, $\exists$👎 for the existential quantifier.

Task 3 (22 pts)**.** Using the rules you invented in the previous tasks, show that the following *elimination* rules for the universal and the extistential quantifier are derivable.

$$\dfrac{t:\tau \quad \forall x:\tau.\,C(x)\ \text{👍}}{C(t)\ \text{👍}}\ \forall\text{👍}\mathsf{E} \qquad\qquad \dfrac{\exists x:\tau.\,A(x)\ \text{👍} \qquad \begin{array}{c}[z:\tau]\quad \overline{A(z)\ \text{👍}}^{\,u}\\ \vdots \\ C\ \text{👍}\end{array}}{C\ \text{👍}}\ \exists\text{👍}\mathsf{E}^{z,u}$$