

# Constructive Logic (15-317), Fall 2019

## Assignment 6: Admissibility and Derivability

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Due: Friday, October 16, 2020, 11:59 pm

This assignment must be submitted electronically via Gradescope. Submit your homework as a pdf containing your written solutions.

### 1 Admissibility and Derivability

**Task 1** (24 points). In the following question you'll be considering whether a given rule is *derivable*, *admissible*, or neither. It's important to understand the difference. Given a set of inference rules, a rule is *derivable* if the conclusion of the rule can be *derived* from its premises using only the other rules. That is, a derivable rule could be thought of as a definition that stands for the use of other rules. You saw an example on Homework 5: the derived rules for negation in sequent calculus. Admissibility is a weaker claim. A rule is *admissible* if the conclusion holds whenever the premises hold. All derivable rules are admissible, but not all admissible rules are derivable. It may be impossible to reach the conclusion of an admissible rule from the premises using only other rules. You saw an example of an admissible rule like this in class: the Cut rule for sequent calculus. The admissibility of Cut can only be proven by induction on sequent calculus derivations. An important practical consequence of the distinction is this: Derivable rules remain derivable even when one adds new primitive rules to the system. However, admissible rules can be lost when one adds new primitive rules to the system.

For each of the following rules (with  $A, B, C$  atomic) in the cut-free sequent calculus, indicate which are derivable, admissible or neither. If a rule is derivable, you must supply the derivation; if it is not derivable but is admissible, you must include a proof that it is admissible.<sup>1</sup> If it is neither admissible or derivable, please just indicate why you believe this (but no rigorous proof is required).

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<sup>1</sup>You may use lemmas that we have proved in class, including the admissibility of cut.

$$\frac{\Gamma \Rightarrow A \quad \Gamma, B \Rightarrow \perp}{\Gamma \Rightarrow A \wedge \neg B} \quad (1)$$

$$\frac{\Gamma, A \Rightarrow B \vee C}{\Gamma, A \wedge \neg B \Rightarrow C} \quad (2)$$

$$\frac{\Gamma, B \Rightarrow A}{\Gamma \Rightarrow B \supset (C \wedge A)} \quad (3)$$

$$\frac{}{\Gamma, A \vee (B \supset (C \wedge B)) \Rightarrow A \vee (B \supset (C \wedge B))} \quad (4)$$

## 2 Admissibility of Cut

**Task 2** (16 points). Extend the proof of the admissibility of cut from class by filling in the following inductive case. A valid proof should include a detailed english explanation—not only notation.

**Case:**  $\mathcal{D}$  ends in  $\forall R_2$  and  $\mathcal{E}$  ends in  $\forall L$ , where  $\forall L$  is applied on the principal formula of the cut.

## 3 Practicing Sequent Calculus

**Task 3** (15 pts). Derive each of the following judgments in the cut-free sequent calculus, with  $A, B, C$  atomic.

a.  $\cdot \Rightarrow A \wedge (B \vee C) \supset (A \wedge B) \vee (A \wedge C)$

b.  $\cdot \Rightarrow (A \supset B) \supset (\neg B \supset \neg A)$

c.  $\cdot \Rightarrow (A \vee \neg A) \supset \neg\neg A \supset A$