Constructive Logic (15-317), Fall 2020
Assignment 5: Sequent Calculus

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Due: Friday, October 9, 2019, 11:59 pm

1 Sequent Proofs

Task 1 (5 points)**.** Provide a proof of the following proposition in the sequent calculus. You may silently drop antecedents you no longer need in the remainder of the sequent proof, but beware of dropping antecedents too early!

$$\Longrightarrow (A \supset (B \supset C)) \supset ((A \supset B) \supset (A \supset C))$$

2 Negation

Task 2 (2 points)**.** Show that the following two rules for negation are *derived rules* in the sequent calculus under the usual definition of $\neg A \triangleq A \supset F$.

$$\frac{\Gamma, A \Longrightarrow F}{\Gamma \Longrightarrow \neg A}\ \neg R \qquad \frac{\Gamma, \neg A \Longrightarrow A}{\Gamma, \neg A \Longrightarrow C}\ \neg L$$

Task 3 (10 points)**.** The rules for negation in Task 2 are not only derived rules, but they are also sufficient if we treat $\neg A$ as a primitive rather than as defined. For each of the following propositions, supply a proof in the sequent calculus using $\neg R$ and $\neg L$ or indicate that no proof exists. Again, you may silently drop antecedents you no longer need in the rest of the proof, but beware of dropping antecedents too early!

a. $\Longrightarrow \neg\neg(A \vee \neg A)$

b. $\Longrightarrow \neg\neg((\neg\neg A) \supset A)$

3 Applications of Cut

The central theorem of structural proof theory is the closure of sequent calculus under the principle of *cut*; the statement of cut depends on the logic, but for our purposes it can be stated as follows.

Theorem 1 (Cut)**.** If $\Gamma \Longrightarrow A$ and $\Gamma, A \Longrightarrow C$ then $\Gamma \Longrightarrow C$.

This theorem can be used to prove many difficult properties about a proof system, including consistency, constructivity, and others. In mathematics, the same technique is also used to establish difficult coherence theorems for higher-dimensional structures.

Another theorem about intuitionistic sequent calculus is closure under the principle of *weakening*, stated as follows.

Theorem 2 (Weakening)**.** If $\Gamma \Longrightarrow C$ then $\Gamma, A \Longrightarrow C$.

Task 4 (10 points)**.** Using Theorem 1 and/or Theorem 2, prove:

> *If* $\Gamma, A \wedge B \Longrightarrow C$ *then* $\Gamma, A, B \Longrightarrow C$.

In particular, you should not use any induction in your argument.