Constructive Logic (15-317), Fall 2020
Assignment 4: Quantification and Arithmetic

Instructor: Karl Crary
TAs: Avery Cowan, Matthew McQuaid, Long Pham, Ariel Uy

Due: Submit to Gradescope by Friday, October 2, 2020, 11:59 pm

1 Quantification

It is important to note that quantification extends as far to the right as syntactically possible. For example, the proposition $\exists x : \tau . A(x) \supset \forall x : \tau . A(x)$ should be interpreted as $\exists x : \tau . (A(x) \supset \forall x : \tau . A(x))$ and not as $(\exists x : \tau . A(x)) \supset (\forall x : \tau . A(x))$. Tutch implements the same convention.

1.1 Distributivity properties

In class, we saw that universal quantification distributes over conjuction, that is,

$$(\forall x : \tau . A(x) \wedge B(x)) \equiv (\forall x : \tau . A(x)) \wedge \forall x : \tau . B(x) \textit{ true}.$$

In this section, we will explore various other distributivity properties.

Task 1 (10 points)**.** Dually, existential quantification distributes over disjunction, that is,

$$(\exists x : \tau . A(x) \vee B(x)) \equiv (\exists x : \tau . A(x)) \vee \exists x : \tau . B(x) \textit{ true}.$$

In this task, you will show this equivalence by giving a natural deduction proof of each of the following directions:

a. $(\exists x : \tau . A(x) \vee B(x)) \supset (\exists x : \tau . A(x)) \vee \exists x : \tau . B(x)$ *true*

b. $(\exists x : \tau . A(x)) \vee (\exists x : \tau . B(x)) \supset \exists x : \tau . A(x) \vee B(x)$ *true*

1.2 Constructive and classical quantification

Task 2 (9 points)**.** For each of the following judgments, give a constructive natural deduction proof and the corresponding proof term if it is constructively valid. If it is not constructively valid, state this. *N.B. The following judgments are all classically valid.*

a. $(\neg \forall x : \tau . \neg A(x)) \supset \exists x : \tau . A(x)$ *true*

b. $(\exists x : \tau . A(x)) \supset \neg \forall x : \tau . \neg A(x)$ *true*

1.3 Tutch, Quantified

Tutch uses the concrete syntax `?x:t.A(x)` and `!x:t.A(x)` for $\exists x : \tau.A(x)$ and $\forall x : \tau.A(x)$, respectively. We encourage you to review the scoping rules for quantifiers described at the beginning of Section 1 of this assignment before starting this portion of the assignment. Please see the Tutch manual for more information on how to use quantifiers in Tutch.

Task 3 (6 points)**.** Prove each of the following propositions using Tutch. Place the proof for part a (and only the proof for part a) in `hw4_8a.tut`, ..., and the proof for part c (and only the proof for part c) in `hw4_8c.tut`.

a.
```
proof apply : (!x:t.A(x) => B(x))
              => (!x:t.A(x)) => (!x:t.B(x));
```

b.
```
proof instance : (!x:t.A(x)) & (?y:t.B(y)) => ?z:t.A(z);
```

c.
```
proof frobenius : (R & ?x:t.Q(x)) <=> ?x:t.(R & Q(x));
```

2 Heyting Arithmetic

First, we'll review the system of Heyting arithmetic that we discussed in class.

2.1 Natural Numbers

First, here are the rules for the judgment $n : \mathsf{nat}$.

$$\frac{}{0 : \mathsf{nat}}\ \mathsf{nat}I_0 \qquad \frac{x : \mathsf{nat}}{S(x) : \mathsf{nat}}\ \mathsf{nat}I_S$$

$$\frac{x : \mathsf{nat} \quad C[0]\ \mathit{true} \quad \begin{array}{c} \overline{y : \mathsf{nat}} \quad \overline{C[y]\ \mathit{true}} \\ \vdots \\ C[S(y)]\ \mathit{true} \end{array}}{C[x]\ \mathit{true}}\ \mathsf{nat}E^{y,u}$$

2.2 Equality

Now, we'll add a new atomic proposition, equality, and define the truth judgment at equality.

$$\frac{}{0 = 0\ \mathit{true}} =I_{00} \qquad \frac{x = y\ \mathit{true}}{S(x) = S(y)\ \mathit{true}} =I_{SS}$$

$$\frac{0 = S(x)\ \mathit{true}}{C\ \mathit{true}} =E_{0S} \qquad \frac{S(x) = 0\ \mathit{true}}{C\ \mathit{true}} =E_{S0} \qquad \frac{S(x) = S(y)\ \mathit{true}}{x = y\ \mathit{true}} =E_{SS}$$

2.3 Primitive Recursion

Lastly, we need the ability to define primitive recursive functions like addition and multiplication *within* the theory of the natural numbers so that we may reason about them in the natural deduction system with the induction principle introduced two subsections ago.

$$\frac{}{R(0; t_0; x, r.t_S) \Rightarrow_R t_0} \Rightarrow_R I_0 \qquad \frac{}{R(S(n); t_0; x, r.t_S) \Rightarrow_R [R(n; t_0; x, r.t_S)/r][n/x]t_S} \Rightarrow_R I_S$$

$$\frac{A[x]\ \mathit{true} \quad x \Rightarrow_R y}{A[y]\ \mathit{true}} \Rightarrow_R E_1 \qquad \frac{A[y]\ \mathit{true} \quad x \Rightarrow_R y}{A[x]\ \mathit{true}} \Rightarrow_R E_2$$

Task 4 (10 points)**.** Let $m + n$ be defined by $R(m; n; x, r.S(r))$. Show that 0 is the right additive identity i.e. give a derivation of $\forall m : \mathsf{nat}.\ R(m; 0; x, r.S(r)) = m$ *true*.