Constructive Logic (15-317), Fall 2020
Assignment 12: Ordered and Modal Logic

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Due: Friday, December 11, 11:59 pm

Submit your homework via Gradescope. Also, according to CMU academic policy, **no late days are allowed for this homework**.

1 Cut Admissibility for Ordered Logic

In class, we realized a natural deduction presentation of ordered logic. In this homework, we turn to sequent calculus as a formal system for ordered logic. Hence, we introduce right and left rules for each connective, instead of introduction and elimination rules, respectively.

Just like propositional logic, we can prove a cut admissibility theorem for ordered logic (or linear logic). For your convenience, the cut theorem statement is given below.

Theorem 1. If $\Delta \stackrel{\mathcal{D}}{\Longrightarrow} A$ and $\Delta_L, A, \Delta_R \stackrel{\mathcal{E}}{\Longrightarrow} C$, then $\Delta_L, \Delta, \Delta_R \Longrightarrow C$.

Please note that the context is an ordered list, *exchange*, *weakening* and *contraction* are *disallowed*. So, please pay careful attention to the order of propositions in the context.

Task 1 (20 pts)**.** Prove the following principal cases of cut admissibility for ordered logic.

1. Case $(\rightarrowtail)$: $\mathcal{D}$ ends in $\rightarrowtail R$ and $\mathcal{E}$ ends in $\rightarrowtail L$, where $\rightarrowtail L$ is applied on the principal formula of the cut.
2. Case $(\bullet)$: $\mathcal{D}$ ends in $\bullet R$ and $\mathcal{E}$ ends in $\bullet L$, where $\bullet L$ is applied on the principal formula of the cut.

For reference, the relevant rules for ordered logic are provided below.

$$\frac{A, \Delta \Longrightarrow B}{\Delta \Longrightarrow A \rightarrowtail B} \rightarrowtail R \qquad \frac{\Delta \Longrightarrow A \qquad \Delta_L, B, \Delta_R \Longrightarrow C}{\Delta_L, \Delta, (A \rightarrowtail B), \Delta_R \Longrightarrow C} \rightarrowtail L$$

$$\frac{\Delta_1 \Longrightarrow A \qquad \Delta_2 \Longrightarrow B}{\Delta_1, \Delta_2 \Longrightarrow A \bullet B} \bullet R \qquad \frac{\Delta_L, A, B, \Delta_R \Longrightarrow C}{\Delta_L, (A \bullet B), \Delta_R \Longrightarrow C} \bullet L$$

2 Fun with Modal Logic

Task 2 (30 pts)**.** Prove the truth of the following propositions in modal logic, or state they do not hold.

1. $\Box \top$
2. $\Box A \supset \neg \Diamond \neg A$
3. $\Diamond A \supset \neg \Box \neg A$
4. $(\Diamond A \wedge \Diamond B) \supset \Diamond (A \wedge B)$
5. $(\Box A \wedge \Box B) \supset \Box (A \wedge B)$
6. $\Diamond (\Box A \wedge \Diamond (A \supset B)) \supset \Diamond B$