Notes on **g4ip**

Course Staff

Syntax

Let P, Q, R stand for an atomic proposition and let A, B, C stand for an arbitrary proposition. Let us fix the following additional restricted classes of propositions and contexts:

$$\begin{array}{llll} A^+ & ::= & P \mid \bot \mid A \vee B & \text{(non-invertible on the right)} \\ A^- & ::= & P \mid P \supset B \mid (A_1 \supset A_2) \supset B & \text{(non-invertible on the left)} \\ \Gamma^- & ::= & \cdot \mid \Gamma^-, A^- & \\ \Omega & ::= & \cdot \mid \Omega, A & \end{array}$$

By convention, Ω is treated as an *ordered* context, whereas Γ^- is treated as unordered.

What's going on here? Why are some propositions marked with $+$ and others with $-$? One of the central ideas of structural proof theory is to classify propositions according to whether their left or right rules are non-invertible (synchronous). A proposition with synchronous right rules is called *positive*, and always has asynchronous left rules; a proposition with synchronous left rules is called *negative*, and always has asynchronous right rules. The duality (positive, negative) is called *polarity*.

The fly in the ointment is that in intuitionistic/constructive logic, propositions like $\wedge, \top$ have asynchronous rules on both sides, so in the proper sense they are neither negative nor positive.

Judgments

For the sake of a clean presentation, we are extending the inversion calculus with a new form of judgment to apply synchronous (non-invertible) rules; these include the right rules of right-synchronous connectives and the left rules of left-synchronous connectives. The main innovation of **g4ip** is to render the single left rule of the implication connective (whose premises are *not* smaller than its conclusion) into several compound rules whose premises do decrease in size.

Our forms of judgment are as follows:

$$\begin{array}{ll} \Gamma^-; \Omega \xrightarrow{\mathsf{g4ip}}_{\mathsf{R}} C & \text{Decompose } C \text{ on the right} \\ \Gamma^-; \Omega \xrightarrow{\mathsf{g4ip}}_{\mathsf{L}} C^+ & \text{Decompose } \Omega \text{ on the left} \\ \Gamma^- \xrightarrow{\mathsf{g4ip}} C^+ & \text{Apply non-invertible rules} \end{array}$$

If you *implement* this version of sequent calculus as a functional program, you would define mutually recursive functions for each of the above forms of judgment; the restricted classes A^+, A^- which we have introduced above are going to be essential for designing a version of the inversion calculus which is sufficiently deterministic, and will guide your implementation in a crucial way. When in doubt, check that in each you have maintained the invariants specified by the form of judgment.

Rules

The rules of our version of **g4ip** are divided roughly into the inversion phases (right and left) and the search phase.

Inversion Phase

We begin in the right inversion phase, where we apply invertible (asynchronous) right rules as much as we can.

$$\frac{\Gamma^-; \Omega \xrightarrow{\mathsf{g4ip}}_{\mathsf{R}} A \quad \Gamma^-; \Omega \xrightarrow{\mathsf{g4ip}}_{\mathsf{R}} B}{\Gamma^-; \Omega \xrightarrow{\mathsf{g4ip}}_{\mathsf{R}} A \wedge B} \wedge\mathsf{R} \qquad \frac{\Gamma^-; \Omega, A \xrightarrow{\mathsf{g4ip}}_{\mathsf{R}} B}{\Gamma^-; \Omega \xrightarrow{\mathsf{g4ip}}_{\mathsf{R}} A \supset B} \supset\mathsf{R}$$

$$\frac{}{\Gamma^-; \Omega \xrightarrow{\mathsf{g4ip}}_{\mathsf{R}} \top} \top\mathsf{R}$$

When you are performing right inversion with a positive proposition or an atomic proposition on the right, then you must switch to left inversion (since positive propositions don't have invertible right rules!).

$$\frac{\Gamma^-; \Omega \xrightarrow{\mathsf{g4ip}}_{\mathsf{L}} C^+}{\Gamma^-; \Omega \xrightarrow{\mathsf{g4ip}}_{\mathsf{R}} C^+} \mathsf{LR}_+$$

The left inversion phase is where we decompose Ω as much as we can. The first few rules are familiar:

$$\frac{\Gamma^-; \Omega, A, B \xrightarrow{\mathsf{g4ip}}_{\mathsf{L}} C^+}{\Gamma^-; \Omega, A \wedge B \xrightarrow{\mathsf{g4ip}}_{\mathsf{L}} C^+} \wedge\mathsf{L} \qquad \frac{\Gamma^-; \Omega, A \xrightarrow{\mathsf{g4ip}}_{\mathsf{L}} C^+ \quad \Gamma^-; \Omega, B \xrightarrow{\mathsf{g4ip}}_{\mathsf{L}} C^+}{\Gamma^-; \Omega, A \vee B \xrightarrow{\mathsf{g4ip}}_{\mathsf{L}} C^+} \vee\mathsf{L}$$

$$\frac{}{\Gamma^-; \Omega, \bot \xrightarrow{\mathsf{g4ip}}_{\mathsf{L}} C^+} \bot\mathsf{L} \qquad \frac{\Gamma^-; \Omega \xrightarrow{\mathsf{g4ip}}_{\mathsf{L}} C^+}{\Gamma^-; \Omega, \top \xrightarrow{\mathsf{g4ip}}_{\mathsf{L}} C^+} \top\mathsf{L}$$

Now we come to some compound left-rules for implication, which are the thing that **g4ip** added to our existing inversion calculus. These are not *all* the left rules for implication: two of them are waiting for us in the *search phase*.

$$\frac{\Gamma^-; \Omega, B \xrightarrow{\mathsf{g4ip}}_{\mathsf{L}} C^+}{\Gamma^-; \Omega, \top \supset B \xrightarrow{\mathsf{g4ip}}_{\mathsf{L}} C^+}\ \top{\supset}\mathsf{L} \qquad \frac{\Gamma^-; \Omega, A_1 \supset A_2 \supset B \xrightarrow{\mathsf{g4ip}}_{\mathsf{L}} C^+}{\Gamma^-; \Omega, (A_1 \wedge A_2) \supset B \xrightarrow{\mathsf{g4ip}}_{\mathsf{L}} C^+}\ \wedge{\supset}\mathsf{L}$$

$$\frac{\Gamma^-; \Omega, A_1 \supset B, A_2 \supset B \xrightarrow{\mathsf{g4ip}}_{\mathsf{L}} C^+}{\Gamma^-; \Omega, (A_1 \vee A_2) \supset B \xrightarrow{\mathsf{g4ip}}_{\mathsf{L}} C^+}\ \vee{\supset}\mathsf{L} \qquad \frac{\Gamma^-; \Omega \xrightarrow{\mathsf{g4ip}}_{\mathsf{L}} C^+}{\Gamma^-; \Omega, \bot \supset B \xrightarrow{\mathsf{g4ip}}_{\mathsf{L}} C^+}\ \bot{\supset}\mathsf{L}$$

If we hit a negative proposition or an atom in Ω, then we cannot invert it, and we simply move it to the negative context Γ^-.

$$\frac{\Gamma^-, A^-; \Omega \xrightarrow{\mathsf{g4ip}}_{\mathsf{L}} C^+}{\Gamma^-; \Omega, A^- \xrightarrow{\mathsf{g4ip}}_{\mathsf{L}} C^+}\ \mathsf{shift}$$

If we run out of things to invert, then we switch to the search phase!

$$\frac{\Gamma^- \xrightarrow{\mathsf{g4ip}} C^+}{\Gamma^-; \cdot \xrightarrow{\mathsf{g4ip}}_{\mathsf{L}} C^+}\ \mathsf{search}$$

Search phase

The search phase is all about applying synchronous (non-invertible) rules. This is the only source of non-determinism in the proof search algorithm, since one order of rule applications may fail whereas another may succeed. In this presentation, we treat the init rule as non-invertible, whereas in class we have made it invertible using a side condition.

$$\frac{P \in \Gamma^-}{\Gamma^- \xrightarrow{\mathsf{g4ip}} P}\ \mathsf{init}$$

The first few rules are the familiar right rules for disjunction:

$$\frac{\Gamma^-; \cdot \xrightarrow{\mathsf{g4ip}}_{\mathsf{R}} A}{\Gamma^- \xrightarrow{\mathsf{g4ip}} A \vee B}\ \vee\mathsf{R}_1 \qquad \frac{\Gamma^-; \cdot \xrightarrow{\mathsf{g4ip}}_{\mathsf{R}} B}{\Gamma^- \xrightarrow{\mathsf{g4ip}} A \vee B}\ \vee\mathsf{R}_2$$

Finally, we hit the remaining compound left rules for implication:

$$\frac{P \in \Gamma^- \quad \Gamma^-; B \xrightarrow{\mathsf{g4ip}}_{\mathsf{L}} C^+}{\Gamma^-, P \supset B \xrightarrow{\mathsf{g4ip}} C^+}\ P{\supset}\mathsf{L}$$

$$\frac{\Gamma^-; A_2 \supset B, A_1 \xrightarrow{\mathsf{g4ip}}_{\mathsf{R}} A_2 \quad \Gamma^-; B \xrightarrow{\mathsf{g4ip}}_{\mathsf{L}} C^+}{\Gamma^-, (A_1 \supset A_2) \supset B \xrightarrow{\mathsf{g4ip}} C^+}\ {\supset}{\supset}\mathsf{L}$$

Examples

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\overline{P \in Q, P}
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\overline{Q \in Q, P}
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\overline{R \in Q, P, R}
}{Q, P, R \xrightarrow{\mathsf{g4ip}} R}\ \mathsf{init}
}{Q, P, R; \cdot \xrightarrow{\mathsf{g4ip}}_{\mathsf{L}} R}\ \mathsf{search}
}{Q, P; R \xrightarrow{\mathsf{g4ip}}_{\mathsf{L}} R}\ \mathsf{shift}
}{Q, P, Q \supset R \xrightarrow{\mathsf{g4ip}} R}\ Q{\supset}\mathsf{L}
}{Q, P, Q \supset R; \cdot \xrightarrow{\mathsf{g4ip}}_{\mathsf{L}} R}\ \mathsf{search}
}{Q, P; Q \supset R \xrightarrow{\mathsf{g4ip}}_{\mathsf{L}} R}\ \mathsf{shift}
}{Q, P, P \supset Q \supset R \xrightarrow{\mathsf{g4ip}} R}\ P{\supset}\mathsf{L}
}{Q, P, P \supset Q \supset R; \cdot \xrightarrow{\mathsf{g4ip}}_{\mathsf{L}} R}\ \mathsf{search}
}{Q, P; P \supset Q \supset R \xrightarrow{\mathsf{g4ip}}_{\mathsf{L}} R}\ \mathsf{shift}
}{Q, P; (P \wedge Q) \supset R \xrightarrow{\mathsf{g4ip}}_{\mathsf{L}} R}\ {\wedge}{\supset}\mathsf{L}
}{Q; (P \wedge Q) \supset R, P \xrightarrow{\mathsf{g4ip}}_{\mathsf{L}} R}\ \mathsf{shift}
}{\cdot; (P \wedge Q) \supset R, P, Q \xrightarrow{\mathsf{g4ip}}_{\mathsf{L}} R}\ \mathsf{shift}
}{\cdot; (P \wedge Q) \supset R, P, Q \xrightarrow{\mathsf{g4ip}}_{\mathsf{R}} R}\ \mathsf{LR}_{+}
}{\cdot; (P \wedge Q) \supset R, P \xrightarrow{\mathsf{g4ip}}_{\mathsf{R}} Q \supset R}\ {\supset}\mathsf{R}
}{\cdot; (P \wedge Q) \supset R \xrightarrow{\mathsf{g4ip}}_{\mathsf{R}} P \supset Q \supset R}\ {\supset}\mathsf{R}
}{\cdot; \cdot \xrightarrow{\mathsf{g4ip}}_{\mathsf{R}} ((P \wedge Q) \supset R) \supset P \supset Q \supset R}\ {\supset}\mathsf{R}
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