Constructive Logic (15-317), Fall 2019
Assignment 7: Classical Logic

Instructor: Karl Crary
TAs: David M Kahn, Siva Somayyajula, Avery Cowan

Due: Friday, October 18, 11:59 pm

Submit your homework via GradeScope as a file named **hw7.pdf**.

1 A New Constructive Logic: Classical Logic

Intuitionistic logic is based on the idea that the fundamental mathematical activity is to *affirm* the truth of something using evidence. Classical logic should be understood as a different, *dialectical* model of mathematical activity, in which one party tries to affirm and the other party tries to deny. Whereas the central duality of intuitionistic natural deduction was between the *introduction* and *elimination* rules for truth $A\ true$, in classical natural deduction, each proposition is explained through the interaction between rules for affirmation A 👍 (i.e. $A\ true$) and rules for denial A 👎 (i.e. $A\ false$), a contest governed by the nullary form of judgment # (contradiction).[1]

To be precise, each connective comes equipped with introduction rules for *both* affirmation and denial; classical negation $\neg A$ implements the involutive "change of perspective" between player (affirmation) and opponent (denial).

Conjunction

$$\frac{A\,\text{👍} \quad B\,\text{👍}}{A \wedge B\,\text{👍}}\ \wedge\text{👍} \qquad \frac{A\,\text{👎}}{A \wedge B\,\text{👎}}\ \wedge\text{👎}_1 \qquad \frac{B\,\text{👎}}{A \wedge B\,\text{👎}}\ \wedge\text{👎}_2$$

Disjunction

$$\frac{A\,\text{👍}}{A \vee B\,\text{👍}}\ \vee\text{👍}_1 \qquad \frac{B\,\text{👍}}{A \vee B\,\text{👍}}\ \vee\text{👍}_2 \qquad \frac{A\,\text{👎} \quad B\,\text{👎}}{A \vee B\,\text{👎}}\ \vee\text{👎}$$

[1] The symbols A 👍 and A 👎 can be written using the provided macros; if you have trouble using these symbols, it is also acceptable to write $A\ true$ and $A\ false$, as from lecture.

Implication

$$\dfrac{\begin{array}{c}\overline{A\ 👍}^{\,u}\\ \vdots\\ B\ 👍\end{array}}{A \supset B\ 👍}\ {\supset}👍^{u} \qquad\qquad \dfrac{A\ 👍 \quad B\ 👎}{A \supset B\ 👎}\ {\supset}👎$$

Units

$$\dfrac{}{\top\ 👍}\ \top👍 \qquad\qquad \dfrac{}{\bot\ 👎}\ \bot👎$$

Negation

$$\dfrac{A\ 👎}{\neg A\ 👍}\ \neg👍 \qquad\qquad \dfrac{A\ 👍}{\neg A\ 👎}\ \neg👎$$

In classical natural deduction, affirmation and denial compete with each other in a *formal contradiction*, a nullary judgment written $\boxed{\#}$. The rules for contradictions are as follows:

Contradiction

$$\dfrac{A\ 👍 \quad A\ 👎}{\#}\ \mathsf{link} \qquad \dfrac{\begin{array}{c}\overline{A\ 👍}^{\,u}\\ \vdots\\ \#\end{array}}{A\ 👎}\ \#👎^{u} \qquad \dfrac{\begin{array}{c}\overline{A\ 👎}^{\,u}\\ \vdots\\ \#\end{array}}{A\ 👍}\ \#👍^{u}$$

Using the rules #👍 and #👎, all the usual "elimination rules" for truth can be *derived* in classical natural deduction.

Task 1 (14 pts)**.** Recall the introduction and elimination rules for the universal quantifier in intuitionistic natural deduction:

$$\dfrac{\begin{array}{c}[z:\tau]\\ \vdots\\ A(z)\ \mathit{true}\end{array}}{\forall x:\tau.\ A(x)\ \mathit{true}}\ \forall\mathsf{I}^{z} \qquad\qquad \dfrac{t:\tau \quad \forall x:\tau.\ A(x)\ \mathit{true}}{A(t)\ \mathit{true}}\ \forall\mathsf{E}$$

Now it's your turn: *invent* affirmation and denial rules ∀👍, ∀👎 for the universal quantifier, as an extension to the classical natural deduction calculus which we have seen so far.

Task 2 (14 pts)**.** Recall the introduction and elimination rules for the existential quantifier in intuitionistic natural deduction:

$$\frac{t : \tau \quad A(t)\ \mathit{true}}{\exists x : \tau.\ A(x)\ \mathit{true}}\ \exists\mathsf{I} \qquad\qquad \frac{\exists x : \tau.\ A(x)\ \mathit{true} \qquad \begin{array}{c} [z : \tau] \quad \overline{A(z)\ \mathit{true}}^{\,u} \\ \vdots \\ C\ \mathit{true} \end{array}}{C\ \mathit{true}}\ \exists\mathsf{E}^{z,u}$$

As in the previous task, invent affirmation and denial rules $\exists$👍, $\exists$👎 for the existential quantifier.

Task 3 (22 pts)**.** Using the rules you invented in the previous tasks, show that the following *elimination* rules for the universal and the extistential quantifier are derivable.

$$\frac{t : \tau \quad \forall x : \tau.\ C(x)\ \text{👍}}{C(t)\ \text{👍}}\ \forall\text{👍}\mathsf{E} \qquad\qquad \frac{\exists x : \tau.\ A(x)\ \text{👍} \qquad \begin{array}{c} [z : \tau] \quad \overline{A(z)\ \text{👍}}^{\,u} \\ \vdots \\ C\ \text{👍} \end{array}}{C\ \text{👍}}\ \exists\text{👍}\mathsf{E}^{z,u}$$