Constructive Logic (15-317), Fall 2019
Assignment 3: Proof Terms, Verification and Quantification

Instructor: Karl Crary
TAs: Avery Cowan, David Kahn, Siva Somayyajula

Due: Submit to Gradescope by Friday, September 20, 2019, 11:59 pm

1 Hype for *hyps*

Consider the following notations for a proofs of $A \wedge B \supset B \wedge A$.

Floating Hypothesis Notation

$$\dfrac{\dfrac{\dfrac{\dfrac{}{A\ \mathit{true}}\,x \qquad \dfrac{}{B\ \mathit{true}}\,y}{A \wedge B\ \mathit{true}}\wedge I}{B \supset (A \wedge B)\ \mathit{true}}\supset I_y}{A \supset (B \supset (A \wedge B))\ \mathit{true}}\supset I_x$$

$$\dfrac{\dfrac{\dfrac{\dfrac{}{A\ \mathit{true}, B\ \mathit{true} \vdash A\ \mathit{true}}\,\mathit{hyp} \qquad \dfrac{}{A\ \mathit{true}, B\ \mathit{true} \vdash B\ \mathit{true}}\,\mathit{hyp}}{A\ \mathit{true}, B\ \mathit{true} \vdash A \wedge B\ \mathit{true}}\wedge I}{A\ \mathit{true} \vdash B \supset (A \wedge B)\ \mathit{true}}\supset I}{\vdash A \supset (B \supset (A \wedge B))\ \mathit{true}}\supset I$$

Context Notation

The Γ notation of hypotheses can be created by slightly modifying the rules of Natural Deduction to carry a context.

$$\dfrac{\Gamma \vdash A\ \mathit{true} \quad \Gamma \vdash B\ \mathit{true}}{\Gamma \vdash A \wedge B\ \mathit{true}}\wedge I \qquad \dfrac{\Gamma \vdash A \wedge B\ \mathit{true}}{\Gamma \vdash A\ \mathit{true}}\wedge E_1 \qquad \dfrac{\Gamma \vdash A \wedge B\ \mathit{true}}{\Gamma \vdash B\ \mathit{true}}\wedge E_2$$

$$\dfrac{\Gamma \vdash A\ \mathit{true}}{\Gamma \vdash A \vee B\ \mathit{true}}\vee I_1 \qquad \dfrac{\Gamma \vdash B\ \mathit{true}}{\Gamma \vdash A \vee B\ \mathit{true}}\vee I_2 \qquad \dfrac{\Gamma \vdash A \vee B\ \mathit{true} \quad \Gamma, A\ \mathit{true} \vdash C\ \mathit{true} \quad \Gamma, B\ \mathit{true} \vdash C\ \mathit{true}}{\Gamma \vdash C\ \mathit{true}}\vee E$$

$$\dfrac{\Gamma, A\ \mathit{true} \vdash B\ \mathit{true}}{\Gamma \vdash A \supset B\ \mathit{true}}\supset I \qquad \dfrac{\Gamma \vdash A \supset B\ \mathit{true} \quad \Gamma \vdash A\ \mathit{true}}{\Gamma \vdash B\ \mathit{true}}\supset E$$

$$\dfrac{}{\Gamma \vdash \top\ \mathit{true}}\top I \qquad \dfrac{\Gamma \vdash \bot\ \mathit{true}}{\Gamma \vdash C\ \mathit{true}}\bot E$$

Finally, the hypothesis rule, which allows you to conclude a hypothesis.

$$\dfrac{J \in \Gamma}{\Gamma \vdash J}\,\mathit{hyp}$$

Task 1 (5 points)**.** Consider this proof (LaTeX) and write the same proof using context notation.

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{(A \supset A) \wedge (A \supset A)\ \mathit{true}}\,u}{A \supset A\ \mathit{true}}\wedge E_2 \qquad \dfrac{\dfrac{\dfrac{}{(A \supset A) \wedge (A \supset A)\ \mathit{true}}\,u}{A \supset A\ \mathit{true}}\wedge E_1 \quad \dfrac{}{A\ \mathit{true}}\,w}{A\ \mathit{true}}\supset E}{A\ \mathit{true}}\supset E}{(A \supset A)\ \mathit{true}}\supset I_w}{(A \supset A) \wedge (A \supset A) \supset (A \supset A)\ \mathit{true}}\supset I_u$$

2 All the things you can do with a $\Diamond$

Consider the $\Diamond$ connective.

$$\dfrac{\begin{array}{c}\overline{A\ \mathit{true}}^{\,u}\\ \vdots\\ B\ \mathit{true}\end{array}}{\Diamond(A,B,C)\ \mathit{true}}\ \Diamond I_1^u \qquad \dfrac{\begin{array}{c}\overline{A\ \mathit{true}}^{\,u}\\ \vdots\\ C\ \mathit{true}\end{array}}{\Diamond(A,B,C)\ \mathit{true}}\ \Diamond I_2^u \qquad \dfrac{\Diamond(A,B,C)\ \mathit{true} \quad A\ \mathit{true} \quad \begin{array}{c}\overline{B\ \mathit{true}}^{\,u}\\ \vdots\\ D\ \mathit{true}\end{array} \quad \begin{array}{c}\overline{C\ \mathit{true}}^{\,v}\\ \vdots\\ D\ \mathit{true}\end{array}}{D\ \mathit{true}}\ \Diamond E^{u,v}$$

Task 2 (4 points)**.** Invent a proof term assignment for the rules.

Task 3 (4 points)**.** Show all the local reduction(s) and expansion(s) for these rules (proving local soundness and completeness) in proof term notation. Be sure to indicate which are reductions and which are expansions.

3 Verifications

Recall the $\Diamond$ connective from Section 2:

Task 4 (9 points)**.** Give rules for forming the judgments that $\Diamond(A,B,C)$ has a verification and that $\Diamond(A,B,C)$ can be used.

Task 5 (5 points)**.** On assignment 2, you showed

$$(\neg A \wedge B) \supset ((A \supset B) \supset (\neg A \supset \neg B)) \supset \bot\ \mathit{true}.$$

Give a verification for this proposition.

Task 6 (20 points)**.** For each of the following propositions, give a verification and its corresponding proof term. (The proof term for a verifications-and-uses derivation is the same as the proof term for the corresponding natural-deduction derivation.)

a. $\bot \supset \top$

b. $\bot \supset \top$ **(Do not use the same verification/proof term as part a. Use a new one.)**

c. $(A \supset B) \supset (\neg B \supset \neg A)$

d. $(A \supset B) \supset (B \supset C) \supset (A \supset C)$