

# Midterm I Exam

15-317 Constructive Logic  
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Name:

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## Instructions

- This exam is closed-book, closed-notes.
- You have 80 minutes to complete the exam.
- There are 5 problems.

|       |                      |         |               |                              |           |       |
|-------|----------------------|---------|---------------|------------------------------|-----------|-------|
|       | Natural<br>Deduction | Harmony | Verifications | Quantifiers<br>in Arithmetic | Induction |       |
|       | Prob 1               | Prob 2  | Prob 3        | Prob 4                       | Prob 5    | Total |
| Score |                      |         |               |                              |           |       |
| Max   | 30                   | 30      | 30            | 25                           | 35        | 150   |

# 1 Natural Deduction (30 pts)

**Task 1** (15 pts). The following purported proof (where we have abbreviated the judgment  $P$  true by just writing  $P$ ) is fatally flawed. Circle every incorrect rule application and unjustified hypothesis, leaving the correct ones unmarked.

$$\begin{array}{c}
 \frac{}{(A \wedge B) \supset C}^u \quad \frac{\frac{}{A}^{\bar{v}} \quad \frac{}{B}^{\bar{w}}}{A \wedge B} \wedge I}{(A \wedge B) \supset C} \supset E \quad \frac{}{(A \wedge B) \supset C}^u \quad \frac{\frac{}{A}^{\bar{v}} \quad \frac{}{B}^{\bar{w}}}{A \wedge B} \wedge I}{(A \wedge B) \supset C} \supset E \\
 \frac{\frac{}{C}}{A \supset C} \supset I^v \quad \frac{\frac{}{C}}{B \supset C} \supset I^w}{(A \supset C) \wedge (B \supset C)} \wedge I \\
 \frac{(A \supset C) \wedge (B \supset C)}{((A \wedge B) \supset C) \supset ((A \supset C) \wedge (B \supset C))} \supset I^u
 \end{array}$$

**Task 2** (5 pts). A logic is *inconsistent* if (circle all that apply)

- (a) it frequently changes its mind about truth
- (b)  $A \vee \neg A$  true for every proposition  $A$
- (c)  $A \wedge \neg A$  true for some proposition  $A$
- (d)  $A$  true for every proposition  $A$

**Task 3** (10 pts). Pick some propositions  $A$ ,  $B$ , and  $C$  which demonstrate that if  $((A \wedge B) \supset C) \supset ((A \supset C) \wedge (B \supset C))$  were true for all propositions  $A$ ,  $B$ , and  $C$ , then intuitionistic logic would be inconsistent. You do not need to prove the inconsistency.

$A$  = \_\_\_\_\_

$B$  = \_\_\_\_\_

$C$  = \_\_\_\_\_

## 2 Harmony (30 pts)

In this problem we consider a new judgment  $A \text{ false}$  which expresses that  $A$  has a *refutation*. The defining property we use is that  $A \text{ false}$  must *contradict*  $A \text{ true}$  in the sense that

$$\frac{A \text{ false} \quad A \text{ true}}{\#} \text{ contra}_A$$

where “#” is our judgment of contradiction. We index the rule of contradiction with the proposition  $A$  that has both a refutation and a proof.

Each connective should now have zero or more *refutation rules*. For example,  $A \vee B \text{ false}$  if  $A \text{ false}$  and  $B \text{ false}$ , and  $\neg A \text{ false}$  if  $A \text{ true}$ . Written as rules (where  $R$  stands for *refutation*):

$$\frac{A \text{ false} \quad B \text{ false}}{A \vee B \text{ false}} \vee R \qquad \frac{A \text{ true}}{\neg A \text{ false}} \neg R$$

For the refutation rule(s) for a connective to be *sound* with respect to the introduction rule(s) for the connective, we have to show that any contradiction at the proposition that is introduced can already be obtained from its constituents. For example, the following reductions (written as  $\Rightarrow_C$  to distinguish them from the usual local reductions) witness the soundness of  $\vee R$  because we match up any possible refutation of  $A \vee B$  (only one,  $\vee R$ ) with every possible introduction of  $A \vee B$  (two, namely  $\vee I_1$  and  $\vee I_2$ ).

$$\begin{array}{c} \frac{\frac{\mathcal{D} \quad \mathcal{E}}{A \text{ false} \quad B \text{ false}} \vee R \quad \frac{\mathcal{F}}{A \text{ true}} \vee I_1}{\frac{A \vee B \text{ false}}{\#} \text{ contra}_{A \vee B}} \Rightarrow_C \frac{\frac{\mathcal{D} \quad \mathcal{F}}{A \text{ false} \quad A \text{ true}}}{\#} \text{ contra}_A \\[2ex] \frac{\frac{\mathcal{D} \quad \mathcal{E}}{A \text{ false} \quad B \text{ false}} \vee R \quad \frac{\mathcal{F}}{B \text{ true}} \vee I_2}{\frac{A \vee B \text{ false}}{\#} \text{ contra}_{A \vee B}} \Rightarrow_C \frac{\frac{\mathcal{E} \quad \mathcal{F}}{B \text{ false} \quad B \text{ true}}}{\#} \text{ contra}_B \end{array}$$

**Task 1** (10 pts). Give refutation rule(s) for  $A \wedge B$  *false*.

**Task 2** (10 pts). Give refutation rule(s) for  $A \supset B$  *false*.

**Task 3** (10 pts). Show the soundness of your refutation rule(s) from Task 2.

### 3 Verifications (30 pts)

**Task 1** (15 pts). Complete the following partial verification by writing in the missing propositions, judgments ( $\uparrow$  and  $\downarrow$ ), and inference rule names.

$$\begin{array}{c}
 \frac{\frac{\frac{}{A \supset B} \quad v \quad \frac{\overline{x}}{A \downarrow}}{} \quad \frac{\frac{}{A \supset C} \quad w \quad \frac{\overline{y}}{A \downarrow}}{} \\
 \frac{}{\quad \quad \quad} \vee I_1 \quad \frac{}{\quad \quad \quad} \vee I_2 \\
 \frac{}{\quad \quad \quad} u \quad \frac{}{\quad \quad \quad} \supset I^x \quad \frac{}{\quad \quad \quad} \supset I^y \\
 \hline
 \frac{}{\quad \quad \quad} \vee E^{v,w} \\
 \hline
 \frac{}{\quad \quad \quad} \supset I^u
 \end{array}$$

**Task 2** (5 pts). Annotate the induction rule  $\text{nat}E$  with  $\uparrow$  and  $\downarrow$  to carry over verifications to Heyting arithmetic. You do not need to annotate the judgment  $n : \text{nat}$  or  $x : \text{nat}$  since we have not refined the typing judgment in lecture.

$$\frac{\frac{n : \text{nat} \quad C(0) \quad \frac{\frac{x : \text{nat} \quad C(x) \quad u}{\vdots} \quad C(sx)}{\text{nat}E^{x,u}}}{C(n)}$$

**Task 3** (10 pts). Annotate the five equality rules of Heyting arithmetic with  $\uparrow$  and  $\downarrow$ .

$$\begin{array}{c}
 \frac{}{0 = 0} = I_{00} \quad \frac{x = y}{sx = sy} = I_{ss} \\
 \text{no rule } E_{00} \quad \frac{0 = sx}{C} = E_{0s} \quad \frac{sx = 0}{C} = E_{s0} \quad \frac{sx = sy}{x = y} = E_{ss}
 \end{array}$$

## 4 Quantifiers in Arithmetic (25 pts)

**Task 1** (25 pts). Heyting arithmetic is based on natural numbers  $n : \text{nat}$ , equality between natural numbers  $x = y$ , and the usual intuitionistic connectives including universal and existential quantification. Moreover, functions on natural numbers can be defined by primitive recursion. Many other notions are usually given as *notational definitions*. Fill in the missing notational definitions below. We have already provided some examples for you. The answers will not be unique, but you are constrained in that you can use only  $\forall, \exists, \supset, \wedge, \vee, \perp, \top$ , equality ( $=$ ), and addition ( $x + y$ ). No other predicates or functions are allowed unless explicitly stated.

(a)  $x \leq y \triangleq$  \_\_\_\_\_

For the remaining questions, you may use  $x \leq y$  for arbitrary natural numbers  $x$  and  $y$ .

(-) The bounded existential quantifier:

$$\exists x \leq n. A(x) \triangleq \exists x:\text{nat}. x \leq n \wedge A(x)$$

(b) The bounded universal quantifier:

$$\forall x \leq n. A(x) \triangleq$$

(c) Uniqueness:  $A(x)$  is true for exactly one natural number  $x$ .

$$\exists! x:\text{nat}. A(x) \triangleq$$

(-) Infinitely many:  $A(x)$  is true for infinitely many natural numbers  $x$ .

$$\exists^\infty x:\text{nat}. A(x) \triangleq \forall y:\text{nat}. \exists z:\text{nat}. y \leq z \wedge A(z)$$

(d) Finitely many:  $A(x)$  is true for finitely many natural numbers  $x$ . Note that this includes the possibility that there are *no*  $x$  satisfying  $A(x)$ .

$$\exists^{\text{fin}} x:\text{nat}. A(x) \triangleq$$

(e) Almost all:  $A(x)$  is true for all natural numbers  $x$  with finitely many exceptions.

$$\forall^{\text{alm}} x:\text{nat}. A(x) \triangleq$$

## 5 Induction (35 pts)

We define new predicates  $\text{even}(x)$  and  $\text{odd}(x)$  on natural numbers with the following introduction rules (again abbreviating  $P \text{ true}$  as  $P$ ):

$$\frac{}{\text{even}(0)} \text{even}I_0 \quad \frac{\text{odd}(x)}{\text{even}(s x)} \text{even}I_s \quad \text{no rule odd}I_0 \quad \frac{\text{even}(x)}{\text{odd}(s x)} \text{odd}I_s$$

**Task 1** (10 pts). Write eliminations rule(s) for the even and odd predicates. If there is no elimination rule for a particular form of proposition, please note this explicitly.

**Task 2** (15 pts). Fill in the details of the following proof.

**Theorem.**  $\forall x. \text{even}(x) \vee \text{odd}(x)$ .

**Proof:** By mathematical induction on  $x$ .

**Base:**  $x = 0$ . Then \_\_\_\_\_.

**Step:** Assume the ind. hyp.: \_\_\_\_\_.

To show: \_\_\_\_\_.

We complete the proof as follows:

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□

**Task 3** (10 pts). We assign proof terms as follows:

$$\frac{}{\text{ev0} : \text{even}(0)} \text{even}I_0 \quad \frac{M : \text{odd}(x)}{\text{evs } M : \text{even}(s x)} \text{even}I_s \quad \frac{M : \text{even}(x)}{\text{ods } M : \text{odd}(s x)} \text{odd}I_s$$

Complete the following function extracted from your proof. If you need additional proof terms for your elimination rules from Task 1, please state the rule and its proof term below the program.

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fun even_odd 0 =
```

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  | even_odd (s x) =
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