Final Exam

15-317 Constructive Logic
Karl Crary

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Instructions

- This exam is closed book, closed notes, and closed internet. You may use both sides of an 8.5x11 page for notes.
- Write your solutions in the supplied blue book. Any solutions written on this exam will not be graded.
- Be sure to put your name on your blue book.
- You have 3 hours to complete the exam.
- There are 4 pages, including this one.
- Not all problems are equal difficulty. Consider reading through the entire exam before you begin.

1 Harmony (30 points)

Goofus mixed up the introduction and elimation rules for *and* and *implies*, making two new connectives:

$$\frac{A\ \mathsf{true} \quad B\ \mathsf{true}}{A \mathbin{\triangle} B\ \mathsf{true}} \triangle I \qquad \frac{A \mathbin{\triangle} B\ \mathsf{true} \quad A\ \mathsf{true}}{B\ \mathsf{true}} \triangle E$$

$$\frac{\begin{array}{c}[A\ \mathsf{true}]_u \\ \vdots \\ B\ \mathsf{true}\end{array}}{A \mathbin{\pentagon} B\ \mathsf{true}} \pentagon I^u \qquad \frac{A \mathbin{\pentagon} B\ \mathsf{true}}{B\ \mathsf{true}} \pentagon E$$

For both connectives, tell whether local soundness holds. If it holds, prove it; if it doesn't, explain briefly why not. Then do the same for local completeness.

2 Sequent Calculus (30 points)

Take a and b to be atomic propositions. Obviously $(a \vee b) \supset a$ isn't true.

1. [25 points] Prove that there exists no derivation of $\cdot \Rightarrow (a \vee b) \supset a$ in sequent calculus.
2. [5 points] Prove that there exists no derivation of $(a \vee b) \supset a\ \mathsf{true}$ in natural deduction. You may use any result we proved in class.

3 Ordered Logic and Cut (30 points)

On the last homework, you looked at sequent calculus for ordered logic. To show that ordered sequent calculus is equivalent to ordered natural deduction, one piece is to show that ordered logic's elimination rules work in ordered sequent calculus.

For this problem you will assume you already have cut admissibility and identity expansion. (You did two cases of the former on the last homework.) For ordered logic, they say:

Theorem 1 (Cut admissibility) *If $\Delta \Rightarrow A$ and $\Delta_L, A, \Delta_R \Rightarrow C$ then $\Delta_L, \Delta, \Delta_R \Rightarrow C$.*

Theorem 2 (Identity expansion) *For any A, we can derive $A \Rightarrow A$.*

Prove that the elimination rules for fuse and left-implication work in ordered sequent calculus. That is, prove:

1. [15 points] If $\Delta \Rightarrow A \bullet B$ and $\Delta_L, A, B, \Delta_R \Rightarrow C$ then $\Delta_L, \Delta, \Delta_R \Rightarrow C$.
2. [15 points] If $\Delta \Rightarrow A$ and $\Delta' \Rightarrow A \rightarrowtail B$ then $\Delta, \Delta' \Rightarrow B$.

For reference, the rules for fuse and left-implication in ordered sequent calculus are:

$$\frac{\Delta_L, A, B, \Delta_R \Rightarrow C}{\Delta_L, (A \bullet B), \Delta_R \Rightarrow C}\ \bullet L \qquad \frac{\Delta \Rightarrow A \quad \Delta' \Rightarrow B}{\Delta, \Delta' \Rightarrow A \bullet B}\ \bullet R$$

$$\frac{\Delta \Rightarrow A \quad \Delta_L, B, \Delta_R \Rightarrow C}{\Delta_L, \Delta, (A \rightarrowtail B), \Delta_R \Rightarrow C}\ {\rightarrowtail}L \qquad \frac{A, \Delta \Rightarrow B}{\Delta \Rightarrow A \rightarrowtail B}\ {\rightarrowtail}R$$

4 Prolog (20 points)

Combinatory logic is a variation on lambda calculus. Instead of variables, lambda abstractions, and application, it uses three "combinators" (called I, K, and S) and application. For simplicity, we will leave out S. (The logic without S is badly broken, but we don't need to worry about that here.) Thus, the syntax of terms is:

$$M \quad ::= \quad \mathsf{I} \mid \mathsf{K} \mid M\,M$$

The only connective in combinatory logic is implication. The type system uses a judgement $M : A$ with the rules:

$$\frac{}{\mathsf{I} : A \to A} \qquad \frac{}{\mathsf{K} : A \to B \to A} \qquad \frac{M_1 : A \to B \quad M_2 : A}{M_1\,M_2 : B}$$

1. [10 points] Write a prolog program that defines a predicate `typeof/2`, such that `typeof`(M, A) holds exactly when $M : A$ is derivable. Please use the notation `i`, `k`, and `app` for building terms (remember that in Prolog it is important that `i` and `k` be in lower case), and `arrow` for building types.

   Important: Be sure your program is written so that the query `typeof(M, X)` produces answers.

2. [10 points] Write a query to ask Prolog to find a term M such that M and $M\,M$ have the same type.

Your program should not use cut, inequality, negation, `setof`, or any other extra-logical feature of Prolog.

5 Focusing (30 points)

Peirce's law is the proposition $((A \supset B) \supset A) \supset A$. It holds in classical logic, but not in constructive logic.

Take a and b to be distinct negative atoms. We want to show that $\downarrow(\downarrow(\downarrow a \supset b) \supset a) \supset a$ has no derivation in focused logic. Since focused logic is complete with respect to sequent calculus, it follows that Peirce's law does not hold.

1. [15 points] Begin a derivation of $\cdot; \cdot \xrightarrow{R} \downarrow(\downarrow(\downarrow a \supset b) \supset a) \supset a$. (Since the proposition does not hold, the derivation cannot be completed.) Carry out your derivation until you reach a premise where more than one rule applies.

2. [10 points] In the one unproved premise, every rule that applies requires you (either immediately or in one step) to derive $\Gamma, [a] \longrightarrow b$, for some Γ. It is impossible to derive that, no matter what Γ is. Explain why.

3. [5 points] It is much harder to show directly that $\cdot \Rightarrow ((a \supset b) \supset a) \supset a$ is underivable in sequent calculus. Why is that?

6 Linear Logic (20 points)

1. [11 points] Write a formula in linear logic to express the following *prix fixe* menu. For $40 you get:

   - Your choice of steak, chicken, or portobello mushroom.
   - Baked potato.
   - Either broccoli or asparagus, whichever is in season.
   - Optionally, for $5 extra, a cappuccino.

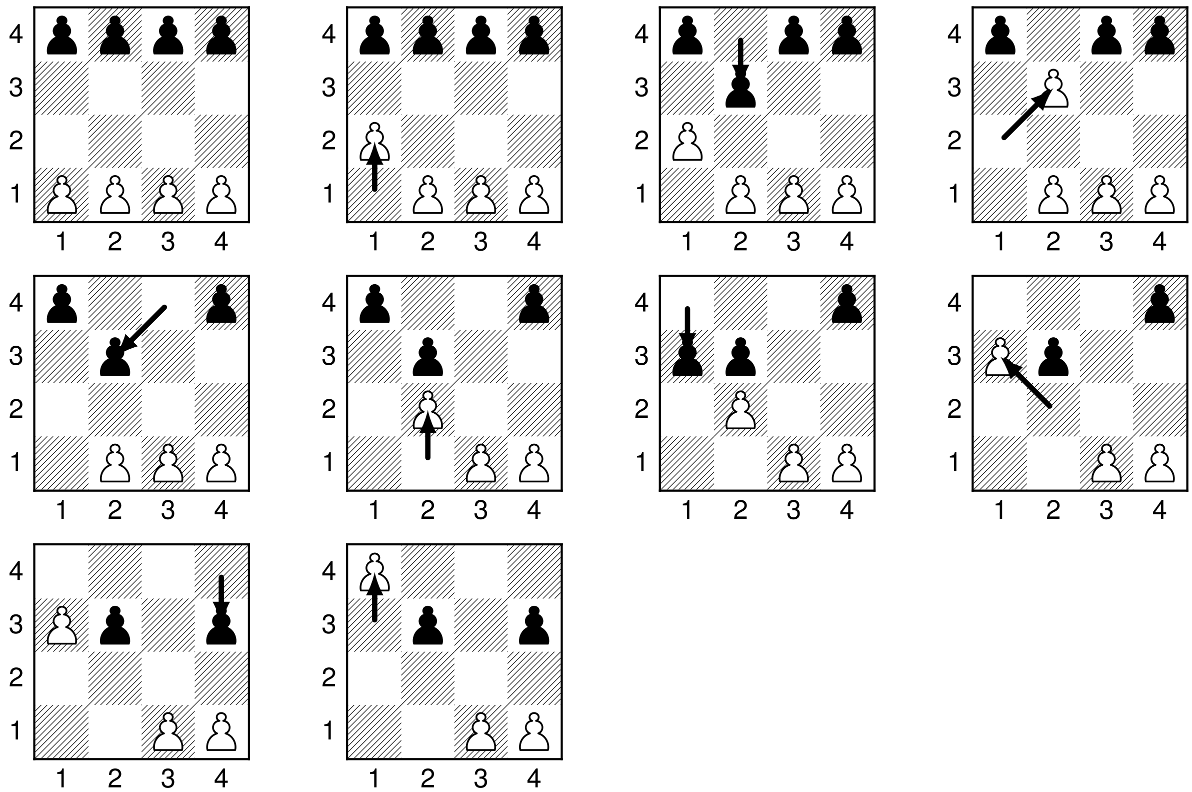
2. [9 points] Explain, in English, the meaning of the three propositions:

   - $\textsf{have-cake} \otimes \textsf{eat-cake}$
   - $\textsf{have-cake} \mathbin{\&} \textsf{eat-cake}$
   - $\textsf{have-cake} \oplus \textsf{eat-cake}$

7 More Linear Logic (20 points)

Pawn's Passing is a simple board game on a 4-by-4 chessboard. The players take turns. Each move, the player can push a pawn one square forward, or the player can capture an opposing pawn diagonally forward one square. The game ends when either player advances a pawn all the way to the other side of the board.

This is a sample game of pawn's passing:



We will express the state of the board in linear logic with three predicates: $\mathsf{white}(x, y)$, $\mathsf{black}(x, y)$, and $\mathsf{empty}(x, y)$, indicate a white pawn, a black pawn, and no pawn in square (x, y), respectively. In our coordinate system, the file (*i.e.*, x-coordinate) is given first, so the lower-right-hand corner is $(4, 1)$. You may assume that addition and subtraction are already defined.

1. [8 points] Write a formula in linear logic to express a rule for white pushing a pawn one square forward.

2. [8 points] Write a formula in linear logic to express a rule for white capturing a black pawn. (There are two such rules, only give one of them.)

3. [4 points] Write a formula in linear logic to describes states in which white wins.

Hints:

- You will not need special cases for the left and right edges of the board.
- Don't worry about game theory. It doesn't come into this problem.