Summary of Rules

Course Staff

In these notes, we provide a self-contained summary of the rules of a few of the logics that we have looked at recently.

1 Modal Natural Deduction

The modal logic that we have studied in class uses three forms of judgment, in which Δ is a multiset of $\overline{A\ \textit{valid}}$ and Γ is a multiset of $\overline{A\ \textit{true}}$:

(truth)	$\Delta; \Gamma \vdash A\ \textit{true}$
(validity)	$\Delta \vdash A\ \textit{valid}$
(possibility)	$\Delta; \Gamma \vdash A\ \textit{poss}$

Rules of weakening and contraction are admissible for both contexts; exchange is silent because we treat the contexts as multisets. We omit the rules for introduction and elimination of the non-modal connectives, noting that these always target the form of judgment $\Delta; \Gamma \vdash A\ \textit{true}$ and leave Δ unchanged throughout.

Structural

$$\frac{\Delta; \cdot \vdash A\ \textit{true}}{\Delta \vdash A\ \textit{valid}}\ \mathsf{valid} \qquad \frac{\Delta; \Gamma \vdash A\ \textit{true}}{\Delta; \Gamma \vdash A\ \textit{poss}}\ \mathsf{poss} \qquad \frac{}{\Delta, A\ \textit{valid}; \Gamma \vdash A\ \textit{true}}\ \mathsf{hypv}$$

$$\frac{}{\Delta; \Gamma, A\ \textit{true} \vdash A\ \textit{true}}\ \mathsf{hyp}$$

Modalities

$$\frac{\Delta \vdash A\ \textit{valid}}{\Delta; \Gamma \vdash \Box A\ \textit{true}}\ \Box\mathsf{I} \qquad \frac{\Delta; \Gamma \vdash \Box A\ \textit{true} \quad \Delta, A\ \textit{valid}; \Gamma \vdash C\ \textit{true}}{\Delta; \Gamma \vdash C\ \textit{true}}\ \Box\mathsf{E_t}$$

$$\frac{\Delta; \Gamma \vdash \Box A\ \textit{true} \quad \Delta, A\ \textit{valid}; \Gamma \vdash C\ \textit{poss}}{\Delta; \Gamma \vdash C\ \textit{poss}}\ \Box\mathsf{E_p}$$

$$\frac{\Delta; \Gamma \vdash A\ \textit{poss}}{\Delta; \Gamma \vdash \Diamond A\ \textit{true}}\ \Diamond\mathsf{I} \qquad \frac{\Delta; \Gamma \vdash \Diamond A\ \textit{true} \quad \Delta; A\ \textit{true} \vdash C\ \textit{poss}}{\Delta; \Gamma \vdash C\ \textit{poss}}\ \Diamond\mathsf{E}$$

2 Lax Natural Deduction

Lax natural deduction involves two forms of judgment, in which Γ is a multiset of $\overline{A\ \mathit{true}}$:

$$
\begin{array}{ll}
\textit{(truth)} & \Gamma \vdash A\ \mathit{true} \\
\textit{(lax truth)} & \Gamma \vdash A\ \mathit{lax}
\end{array}
$$

Rules of weakening and contraction are admissible; exchange is silent because we treat the contexts as multisets.

Structural

$$
\frac{}{\Gamma, A\ \mathit{true} \vdash A\ \mathit{true}}\ \mathsf{hyp}
\qquad
\frac{\Gamma \vdash A\ \mathit{true}}{\Gamma \vdash A\ \mathit{lax}}\ \mathsf{lax}
$$

Modalities

$$
\frac{\Gamma \vdash A\ \mathit{lax}}{\Gamma \vdash \bigcirc A\ \mathit{true}}\ \bigcirc\mathsf{I}
\qquad
\frac{\Gamma \vdash \bigcirc A\ \mathit{true} \quad \Gamma, A\ \mathit{true} \vdash C\ \mathit{lax}}{\Gamma \vdash C\ \mathit{lax}}\ \bigcirc\mathsf{E}
$$

3 Ordered Natural Deduction

Ordered natural deduction uses a single form of judgment, where Ω is an *ordered list* of assumptions $\overline{A\ \mathit{true}}$:

$$
\textit{(ordered truth)} \qquad \Omega \Vdash A\ \mathit{true}
$$

Structural

$$
\frac{}{A\ \mathit{true} \Vdash A\ \mathit{true}}\ \mathsf{hyp}
$$

Additive disjunction

$$
\frac{\Omega \Vdash \mathbf{0}\ \mathit{true}}{\Omega_1, \Omega, \Omega_2 \Vdash C\ \mathit{true}}\ \mathsf{0E}
$$

$$
\frac{\Omega \Vdash A\ \mathit{true}}{\Omega \Vdash A \oplus B\ \mathit{true}}\ \oplus\mathsf{I}_1
\qquad
\frac{\Omega \Vdash B\ \mathit{true}}{\Omega \Vdash A \oplus B\ \mathit{true}}\ \oplus\mathsf{I}_2
$$

$$
\frac{\Omega \Vdash A \oplus B\ \mathit{true} \quad \Omega_1, A\ \mathit{true}, \Omega_2 \Vdash C\ \mathit{true} \quad \Omega_1, B\ \mathit{true}, \Omega_2 \Vdash C\ \mathit{true}}{\Omega_1, \Omega, \Omega_2 \Vdash C\ \mathit{true}}\ \oplus\mathsf{E}
$$

Additive conjunction

$$
\frac{}{\Omega \Vdash \top\ \mathit{true}}\ \top\mathsf{I}
$$

$$
\frac{\Omega \Vdash A\ \mathit{true} \quad \Omega \Vdash B\ \mathit{true}}{\Omega \Vdash A \,\&\, B\ \mathit{true}}\ \&\mathsf{I}
\qquad
\frac{\Omega \Vdash A \,\&\, B\ \mathit{true}}{\Omega \Vdash A\ \mathit{true}}\ \&\mathsf{E}_1
\qquad
\frac{\Omega \Vdash A \,\&\, B\ \mathit{true}}{\Omega \Vdash B\ \mathit{true}}\ \&\mathsf{E}_2
$$

Multiplicative conjunction

$$\frac{}{\cdot \Vdash \mathbf{1}\ \mathit{true}}\ \mathbf{1}\mathsf{I} \qquad \frac{\Omega \Vdash \mathbf{1}\ \mathit{true} \quad \Omega_1, \Omega_2 \Vdash C\ \mathit{true}}{\Omega_1, \Omega, \Omega_2 \Vdash C\ \mathit{true}}\ \mathbf{1}\mathsf{E}$$

$$\frac{\Omega_1 \Vdash A\ \mathit{true} \quad \Omega_2 \Vdash B\ \mathit{true}}{\Omega_1, \Omega_2 \Vdash A \bullet B\ \mathit{true}}\ \bullet\mathsf{I}$$

$$\frac{\Omega \Vdash A \bullet B\ \mathit{true} \quad \Omega_1, A\ \mathit{true}, B\ \mathit{true}, \Omega_2 \Vdash C\ \mathit{true}}{\Omega_1, \Omega, \Omega_2 \Vdash C\ \mathit{true}}\ \bullet\mathsf{E}$$

The following connective, called “twist”, was not covered in class; but we include it to complete the adjoint duality induced by the two versions of implication, $\rightarrowtail$ and $\twoheadrightarrow$.

$$\frac{\Omega_1 \Vdash B\ \mathit{true} \quad \Omega_2 \Vdash A\ \mathit{true}}{\Omega_1, \Omega_2 \Vdash A \circ B\ \mathit{true}}\ \circ\mathsf{I}$$

$$\frac{\Omega \Vdash A \circ B\ \mathit{true} \quad \Omega_1, B\ \mathit{true}, A\ \mathit{true}, \Omega_2 \Vdash C\ \mathit{true}}{\Omega_1, \Omega, \Omega_2 \Vdash C\ \mathit{true}}\ \circ\mathsf{E}$$

Implication

$$\frac{\Omega, A\ \mathit{true} \Vdash B\ \mathit{true}}{\Omega \Vdash A \twoheadrightarrow B\ \mathit{true}}\ {\twoheadrightarrow}\mathsf{I} \qquad \frac{\Omega_1 \Vdash A \twoheadrightarrow B\ \mathit{true} \quad \Omega_2 \Vdash A\ \mathit{true}}{\Omega_1, \Omega_2 \Vdash B\ \mathit{true}}\ {\twoheadrightarrow}\mathsf{E}$$

$$\frac{A\ \mathit{true}, \Omega \Vdash B\ \mathit{true}}{\Omega \Vdash A \rightarrowtail B\ \mathit{true}}\ {\rightarrowtail}\mathsf{I} \qquad \frac{\Omega_1 \Vdash A\ \mathit{true} \quad \Omega_2 \Vdash A \rightarrowtail B\ \mathit{true}}{\Omega_1, \Omega_2 \Vdash B\ \mathit{true}}\ {\rightarrowtail}\mathsf{E}$$

4 Linear Natural Deduction

Ordered natural deduction uses a single form of judgment, where Γ is a multiset of assumptions $\overline{A\ \mathit{true}}$:

$$\textit{(linear truth)} \qquad \Gamma \Vdash A\ \mathit{true}$$

The rule of exchange is silent, because we have treated the context as a multiset.

Structural

$$\frac{}{A\ \mathit{true} \Vdash A\ \mathit{true}}\ \mathsf{hyp}$$

Additive disjunction

$$\frac{\Gamma \Vdash \mathbf{0}\ \mathit{true}}{\Gamma, \Gamma' \Vdash C\ \mathit{true}}\ \mathsf{0E}$$

$$\frac{\Gamma \Vdash A\ \mathit{true}}{\Gamma \Vdash A \oplus B\ \mathit{true}}\ \oplus\mathsf{I}_1 \qquad \frac{\Gamma \Vdash B\ \mathit{true}}{\Gamma \Vdash A \oplus B\ \mathit{true}}\ \oplus\mathsf{I}_2$$

$$\frac{\Gamma \Vdash A \oplus B\ \mathit{true} \quad \Gamma', A\ \mathit{true} \Vdash C\ \mathit{true} \quad \Gamma', B\ \mathit{true} \Vdash C\ \mathit{true}}{\Gamma, \Gamma' \Vdash C\ \mathit{true}}\ \oplus\mathsf{E}$$

Additive conjunction

$$\frac{}{\Gamma \Vdash \top\ \mathit{true}}\ \top\mathsf{I}$$

$$\frac{\Gamma \Vdash A\ \mathit{true} \quad \Gamma \Vdash B\ \mathit{true}}{\Gamma \Vdash A \,\&\, B\ \mathit{true}}\ \&\mathsf{I} \qquad \frac{\Gamma \Vdash A \,\&\, B\ \mathit{true}}{\Gamma \Vdash A\ \mathit{true}}\ \&\mathsf{E}_1 \qquad \frac{\Gamma \Vdash A \,\&\, B\ \mathit{true}}{\Gamma \Vdash B\ \mathit{true}}\ \&\mathsf{E}_2$$

Multiplicative conjunction

$$\frac{}{\cdot \Vdash \mathbf{1}\ \mathit{true}}\ \mathbf{1}\mathsf{I} \qquad \frac{\Gamma \Vdash \mathbf{1}\ \mathit{true} \quad \Gamma' \Vdash C\ \mathit{true}}{\Gamma, \Gamma' \Vdash C\ \mathit{true}}\ \mathbf{1}\mathsf{E}$$

$$\frac{\Gamma_1 \Vdash A\ \mathit{true} \quad \Gamma_2 \Vdash B\ \mathit{true}}{\Gamma_1, \Gamma_2 \Vdash A \otimes B\ \mathit{true}}\ \otimes\mathsf{I}$$

$$\frac{\Gamma \Vdash A \otimes B\ \mathit{true} \quad \Gamma', A\ \mathit{true}, B\ \mathit{true} \Vdash C\ \mathit{true}}{\Gamma, \Gamma' \Vdash C\ \mathit{true}}\ \otimes\mathsf{E}$$

Implication

$$\frac{\Gamma, A\ \mathit{true} \Vdash B\ \mathit{true}}{\Gamma \Vdash A \multimap B\ \mathit{true}}\ \multimap\mathsf{I} \qquad \frac{\Gamma \Vdash A \multimap B\ \mathit{true} \quad \Gamma' \Vdash A\ \mathit{true}}{\Gamma, \Gamma' \Vdash B\ \mathit{true}}\ \multimap\mathsf{E}$$

5 Focused Sequent Calculus

Focused sequent calculus has several class of proposition[1] and context. We summarize them below, noting that Ω^+ ranges over ordered lists, whereas Γ ranges over multisets:

(negative)	A^-	$::=$	$a^- \mid \top^- \mid A^- \wedge^- B^- \mid A^+ \supset B^- \mid \uparrow A^+$
(positive)	A^+	$::=$	$a^+ \mid \top^+ \mid \bot \mid A^+ \wedge^+ B^+ \mid A^+ \vee B^+ \mid \downarrow A^-$
(inv. antecedents)	Ω^+	$::=$	$\cdot \mid \Omega^+, A^+$
(stable antecedents)	Γ	$::=$	$\cdot \mid \Gamma, A^- \mid \Gamma, a^+$
(stable succedent)	ρ	$::=$	$A^+ \mid a^-$

[1] We are writing a^-, a^+ for negative and positive atoms respectively. You may also see these written as P^-, P^+.

Focused sequent calculus features several forms of judgment:

$$\begin{array}{ll}
\textit{(right inversion)} & \Gamma; \Omega^+ \xrightarrow{\mathsf{R}} A^- \\
\textit{(left inversion)} & \Gamma; \Omega^+ \xrightarrow{\mathsf{L}} \rho \\
\textit{(stable)} & \Gamma \longrightarrow \rho \\
\textit{(right focus)} & \Gamma \longrightarrow [A^+] \\
\textit{(left focus)} & \Gamma, [A^-] \longrightarrow \rho
\end{array}$$

Right inversion

$$\frac{\Gamma; \Omega^+ \xrightarrow{\mathsf{L}} a^-}{\Gamma; \Omega^+ \xrightarrow{\mathsf{R}} a^-}\ a\mathsf{R} \qquad \frac{\Gamma; \Omega^+ \xrightarrow{\mathsf{L}} A^+}{\Gamma; \Omega^+ \xrightarrow{\mathsf{R}} {\uparrow}A^+}\ {\uparrow}\mathsf{R} \qquad \frac{\Gamma; \Omega^+, A^+ \xrightarrow{\mathsf{R}} B^-}{\Gamma; \Omega^+ \xrightarrow{\mathsf{R}} A^+ \supset B^-}\ {\supset}\mathsf{R}$$

$$\frac{\Gamma; \Omega^+ \xrightarrow{\mathsf{R}} A^- \quad \Gamma; \Omega^+ \xrightarrow{\mathsf{R}} B^-}{\Gamma; \Omega^+ \xrightarrow{\mathsf{R}} A^- \wedge^- B^-}\ \wedge^-\mathsf{R} \qquad \frac{}{\Gamma; \Omega^+ \xrightarrow{\mathsf{R}} \top^-}\ \top^-\mathsf{R}$$

Left inversion

$$\frac{\Gamma \longrightarrow \rho}{\Gamma; \cdot \xrightarrow{\mathsf{L}} \rho}\ \mathsf{stable} \qquad \frac{\Gamma, a^+; \Omega^+ \xrightarrow{\mathsf{L}} \rho}{\Gamma; \Omega^+, a^+ \xrightarrow{\mathsf{L}} \rho}\ a\mathsf{L} \qquad \frac{\Gamma, A^-; \Omega^+ \xrightarrow{\mathsf{L}} \rho}{\Gamma; \Omega^+, {\downarrow}A^- \xrightarrow{\mathsf{L}} \rho}\ {\downarrow}\mathsf{L}$$

$$\frac{\Gamma; \Omega^+ \xrightarrow{\mathsf{L}} \rho}{\Gamma; \Omega^+, \top^+ \xrightarrow{\mathsf{L}} \rho}\ \top^+\mathsf{L} \qquad \frac{}{\Gamma; \Omega^+, \bot \xrightarrow{\mathsf{L}} \rho}\ \bot\mathsf{L}$$

$$\frac{\Gamma; \Omega^+, A^+ \xrightarrow{\mathsf{L}} \rho \quad \Gamma; \Omega^+, B^+ \xrightarrow{\mathsf{L}} \rho}{\Gamma; \Omega^+, A^+ \vee B^+ \xrightarrow{\mathsf{L}} \rho}\ \vee\mathsf{L} \qquad \frac{\Gamma; \Omega^+, A^+, B^+ \xrightarrow{\mathsf{L}} \rho}{\Gamma; \Omega^+, A^+ \wedge^+ B^+ \xrightarrow{\mathsf{L}} \rho}\ \wedge^+\mathsf{L}$$

Stable

$$\frac{\Gamma \longrightarrow [A^+]}{\Gamma \longrightarrow A^+}\ \mathsf{focR} \qquad \frac{A^- \in \Gamma \quad \Gamma, [A^-] \longrightarrow \rho}{\Gamma \longrightarrow \rho}\ \mathsf{focL}$$

Right focus

$$\frac{a^+ \in \Gamma}{\Gamma \longrightarrow [a^+]}\ \mathsf{id}^+ \qquad \frac{\Gamma; \cdot \xrightarrow{\mathsf{R}} A^-}{\Gamma \longrightarrow [{\downarrow}A^-]}\ {\downarrow}\mathsf{R} \qquad \frac{}{\Gamma \longrightarrow [\top^+]}\ \top^+\mathsf{R}$$

$$\frac{\Gamma \longrightarrow [A^+]}{\Gamma \longrightarrow [A^+ \vee B^+]}\ \vee\mathsf{R}_1 \qquad \frac{\Gamma \longrightarrow [B^+]}{\Gamma \longrightarrow [A^+ \vee B^+]}\ \vee\mathsf{R}_2$$

$$\frac{\Gamma \longrightarrow [A^+] \quad \Gamma \longrightarrow [B^+]}{\Gamma \longrightarrow [A^+ \wedge^+ B^+]}\ \wedge^+\mathsf{R}$$

Left focus

$$\frac{}{\Gamma, [a^-] \longrightarrow a^-}\ \mathsf{id}^- \qquad \frac{\Gamma; A^+ \xrightarrow{\mathsf{L}} \rho}{\Gamma, [{\uparrow}A^+] \longrightarrow \rho}\ {\uparrow}\mathsf{L} \qquad \frac{\Gamma \longrightarrow [A^+] \quad \Gamma, [B^-] \longrightarrow \rho}{\Gamma, [A^+ \supset B^-] \longrightarrow \rho}\ {\supset}\mathsf{L}$$

$$\frac{\Gamma, [A^-] \longrightarrow \rho}{\Gamma, [A^- \wedge^- B^-] \longrightarrow \rho}\ \wedge^-\mathsf{L}_1 \qquad \frac{\Gamma, [B^-] \longrightarrow \rho}{\Gamma, [A^- \wedge^- B^-] \longrightarrow \rho}\ \wedge^-\mathsf{L}_2$$

References

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