Constructive Logic (15-317), Fall 2018
Assignment 3: Proof Terms, Verification and Quantification

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Due: Friday, September 21, 2018, 11:59 pm

This assignment must be submitted electronically at autolab. Submit your homework as a `tar` archive containing the following files:

- `hw3.pdf` (your written solutions); and
- `hw3_8a.tut`, `hw3_8b.tut`, `hw3_8c.tut` (your Tutch solutions for task 8)

1 All the things you can do with a $\Diamond$

Consider the $\Diamond$ connective.

$$\dfrac{\begin{array}{c}\overline{A\ \mathit{true}}^{\,u}\\ \vdots\\ B\ \mathit{true}\end{array}}{\Diamond(A,B,C)\ \mathit{true}}\ \Diamond I_1^u \qquad \dfrac{\begin{array}{c}\overline{A\ \mathit{true}}^{\,u}\\ \vdots\\ C\ \mathit{true}\end{array}}{\Diamond(A,B,C)\ \mathit{true}}\ \Diamond I_2^u \qquad \dfrac{\Diamond(A,B,C)\ \mathit{true} \quad A\ \mathit{true} \quad \begin{array}{c}\overline{B\ \mathit{true}}^{\,u}\\ \vdots\\ D\ \mathit{true}\end{array} \quad \begin{array}{c}\overline{C\ \mathit{true}}^{\,v}\\ \vdots\\ D\ \mathit{true}\end{array}}{D\ \mathit{true}}\ \Diamond E^{u,v}$$

Task 1 (4 points)**.** Invent a proof term assignment for the rules.

Task 2 (4 points)**.** Show all the local reduction(s) and expansion(s) for these rules (proving local soundness and completeness) in proof term notation. Be sure to indicate which are reductions and which are expansions.

2 Verifications

Recall the $\Diamond$ connective from Section 1:

Task 3 (9 points)**.** Give rules for forming the judgments that $\Diamond(A,B,C)$ has a verification and that $\Diamond(A,B,C)$ can be used.

Task 4 (5 points)**.** On assignment 2, you showed

$$(\neg A \wedge B) \supset ((A \supset B) \supset (\neg A \supset \neg B)) \supset \bot\ \mathit{true}.$$

Give a verification for this proposition.

Task 5 (14 points)**.** For each of the following propositions, give a verification and its corresponding proof term:

a. $\bot \supset \top$

b. $(A \supset B) \supset (B \supset C) \supset (A \supset C)$

3 Quantification

It is important to note that quantification extends as far to the right as syntactically possible. For example, the proposition $\exists x : \tau.A(x) \supset \forall x : \tau.A(x)$ should be interpreted as $\exists x : \tau.(A(x) \supset \forall x : \tau.A(x))$ and not as $(\exists x : \tau.A(x)) \supset (\forall x : \tau.A(x))$. Tutch implements the same convention.

3.1 Distributivity properties

In class, we saw that universal quantification distributes over conjuction, that is,

$$(\forall x : \tau.A(x) \wedge B(x)) \equiv (\forall x : \tau.A(x)) \wedge \forall x : \tau.B(x) \ \textit{true}.$$

In this section, we will explore various other distributivity properties.

Task 6 (10 points)**.** Dually, existential quantification distributes over disjunction, that is,

$$(\exists x : \tau.A(x) \vee B(x)) \equiv (\exists x : \tau.A(x)) \vee \exists x : \tau.B(x) \ \textit{true}.$$

In this task, you will show this equivalence by giving a natural deduction proof of each of the following directions:

a. $(\exists x : \tau.A(x) \vee B(x)) \supset (\exists x : \tau.A(x)) \vee \exists x : \tau.B(x)$ *true*

b. $(\exists x : \tau.A(x)) \vee (\exists x : \tau.B(x)) \supset \exists x : \tau.A(x) \vee B(x)$ *true*

3.2 Constructive and classical quantification

Task 7 (9 points)**.** For each of the following judgments, give a constructive natural deduction proof and the corresponding proof term if it is constructively valid. If it is not constructively valid, state this. *N.B. The following judgments are all classically valid.*

a. $(\neg\forall x : \tau.\neg A(x)) \supset \exists x : \tau.A(x)$ *true*

b. $(\exists x : \tau.A(x)) \supset \neg\forall x : \tau.\neg A(x)$ *true*

3.3 Tutch, Quantified

Tutch uses the concrete syntax `?x:t.A(x)` and `!x:t.A(x)` for $\exists x : \tau.A(x)$ and $\forall x : \tau.A(x)$, respectively. We encourage you to review the scoping rules for quantifiers described at the beginning of Section 3 of this assignment before starting this portion of the assignment. Please see the Tutch manual for more information on how to use quantifiers in Tutch.

Task 8 (6 points)**.** Prove each of the following propositions using Tutch. Place the proof for part a (and only the proof for part a) in `hw3_8a.tut`, ..., and the proof for part c (and only the proof for part c) in `hw3_8c.tut`.

a.
```
proof apply : (!x:t.A(x) => B(x))
            => (!x:t.A(x)) => (!x:t.B(x));
```

b.
```
proof instance : (!x:t.A(x)) & (?y:t.B(y)) => ?z:t.A(z);
```

c.
```
proof frobenius : (R & ?x:t.Q(x)) <=> ?x:t.(R & Q(x));
```