Constructive Logic (15-317), Fall 2018
Assignment 1: Say hi to logic!

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Due: Friday, September 7, 2018, 11:59 pm

Welcome to 15-317, Fall 2018 edition! In this homework assignment, you will practice some basic principles you'll need for the rest of the course. As a special exception to the usual rules, for this homework assignment, you may collaborate with other students in the class on the answers to all of the questions as long as you do your write-up individually.

We STRONGLY SUGGEST that you typeset this homework assignment in LaTeX so that you learn how to typeset your proofs now, while the problems are easier and you are perhaps less busy. You can find the code handout for this assignment on autolab; you can use it as a starting point.

This assignment must be submitted electronically on autolab. The link to the course's autolab page can be found on the course homepage once it becomes available. Submit your homework as a `tar` archive containing one file: `hw1.pdf` (your written solutions).

One and one is one[1]

Prove the following formulas using the inference rules given in lecture (be sure to name each rule when you use it).

Task 1 (1 point)**.**

$$(A \wedge (A \supset B)) \supset B$$

Task 2 (2 points)**.**

$$(A \wedge ((A \wedge A) \supset B)) \supset B$$

Task 3 (2 points)**.**

$$(A \wedge (A \supset B)) \supset (B \wedge B)$$

Doing constructive mathematics

An integer a is said to *divide* an integer b, written $a \mid b$, if there exists an integer k such that $b = ak$. We write $a \nmid b$ for $\neg(a \mid b)$.

Task 4 (1 point)**.** Give an (informal) constructive proof of the following proposition: for all integers a, b, and c, if $a \mid b$ and $b \mid c$, then $a \mid c$.

Task 5 (1 point)**.** Give an (informal) non-constructive proof of the following proposition: for all integers a, b, and c, one of the following is true:

- $a \nmid b$ or $b \nmid c$;
- $a \mid c$.

Hint: Use the proposition from the previous task.

[1] For the appropriate definition of "and".

Task 6 (1 point)**.** In the previous task, you likely used the following instance of the law of the excluded middle

$$a \mid b \vee a \nmid b.$$

Though constructivism rejects the law of the excluded middle as a general axiom, it will still be the case that for some specific propositions A we have

$$(A \vee \neg A) \; \textit{true}.$$

In particular, this will be the case whenever A is decidable, that is, whenever there exists an effective procedure for deciding whether or not A is true. Give a brief and informal constructive justification for concluding $a \mid b$ when $\neg(a \nmid b)$.

The *Fundamental Theorem of Arithmetic* states that every integer greater than 1 either is prime or factors as the product of primes numbers, and moreover, that this factorisation is unique up to reordering of factors.

Task 7 (1 point)**.** Is the following proof that $3 \nmid 10$ constructive? Justify your answer.

> *Proof.* Assume to the contrary that $3 \mid 10$. Then there exists a k such that $10 = 3k$. By the fundamental theorem of arithmetic, k has some unique prime factorisation $k = \prod_{i=1}^{n} p_i$. So 10 factors into primes as $10 = 3 \prod_{i=1}^{n} p_i$. But we also know that 10 factors into primes as $10 = 2 \times 5$. The existence of two distinct prime factorisations for 10 contradicts the uniqueness guaranteed by the fundamental theorem of arithmetic. We thus conclude that $3 \nmid 10$. □