Midterm I Sample Questions

15-317 Constructive Logic
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Instructions

- These sample questions are closed-book, closed-notes.
- You have 80 minutes to complete the sample questions.
- There are only 4 problems!

	Natural Deduction	Harmony	Verifications	Arithmetic		
	Prob 1	Prob 2	Prob 3	Prob 4	Prob 5	Total
Score						
Max	25	40	25	30	0	120

1 Natural Deduction (25 pts)

Recall that we defined $\neg A \triangleq A \supset \bot$ and derived rules $\neg I$ and $\neg E$:

$$\dfrac{\begin{array}{c}\dfrac{}{A\ \mathit{true}}\,u \\ \vdots \\ \bot\ \mathit{true}\end{array}}{\neg A\ \mathit{true}}\,\neg I^u \qquad\qquad \dfrac{\neg A\ \mathit{true} \quad A\ \mathit{true}}{\bot\ \mathit{true}}\,\neg E$$

Task 1 (10 pts). The following purported intuitionistic proof has one or more fatal flaws. Circle every incorrect rule application or unjustified hypothesis, leaving the correct ones unmarked.

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{(\neg B) \supset (\neg A)}\,u \quad \dfrac{}{\neg B}\,w}{\neg A}\,\supset E \quad \dfrac{}{A}\,v}{\bot}\,\neg E}{B}\,\bot E^w}{A \supset B}\,\supset I^v}{((\neg B) \supset (\neg A)) \supset (A \supset B)}\,\supset I^u$$

Task 2 (15 pts). Circle each possible premise H such that

$$\dfrac{H}{((\neg B) \supset (\neg A)) \supset (A \supset B)}\,\textsf{contrapos}$$

would be a correct derived rule of inference. You do not need to show any derivations.

(a) A

(b) B

(c) $A \vee \neg A$

(d) $B \vee \neg B$

(d) $(\neg\neg A) \supset A$

(f) $(\neg\neg B) \supset B$

2 Harmony (40 pts)

Task 1 (5 pts). Local soundness, as witnessed by (circle the correct answer)

(a) local reductions

(b) local expansions

(c) local weather

establishes that (circle the correct answer)

(a) the elimination rule(s) are not too weak

(b) the elimination rule(s) are not too strong

(c) the elimination rule(s) are not too strung out

Consider a new logical operator $\heartsuit A$ with the following introduction and elimination rules.

$$\frac{A\ \mathit{true}}{\heartsuit A\ \mathit{true}}\ \heartsuit I_1 \qquad \frac{}{\heartsuit A\ \mathit{true}}\ \heartsuit I_2 \qquad \frac{\heartsuit A\ \mathit{true} \quad C\ \mathit{true} \quad \begin{array}{c}\overline{A\ \mathit{true}}\ u \\ \vdots \\ C\ \mathit{true}\end{array}}{C\ \mathit{true}}\ \heartsuit E^u$$

In the elimination rules, the scope of the hypothesis labeled u is the proof of the third premise.

Task 2 (20 pts). Provide a sufficient set of witnesses (reductions or expansions) to demonstrate local soundness.

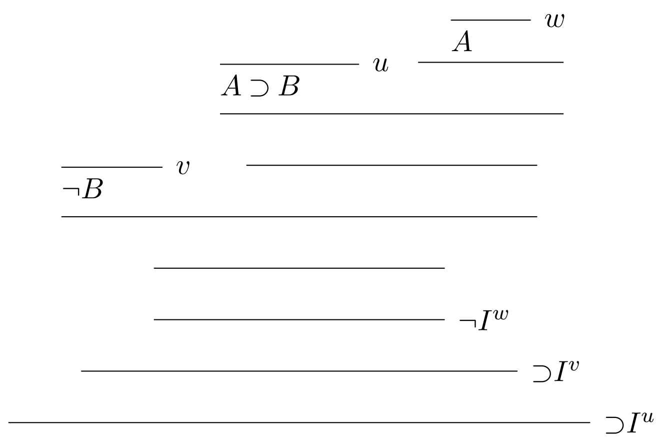
Task 3 (10 pts). Are the local reductions or expansions from Task 2 uniquely determined? State either "*unique*" or show one alternative to the reductions given in Task 2.

Task 4 (5 pts). Could we have defined $\heartsuit A$ inside the logic by a notational definition? State "*none*" or provide an alternative definition.

$\heartsuit A \triangleq$ ____________________

3 Verifications (25 pts)

Task 1 (25 pts). Complete the following partial verification by writing in the missing propositions, judgments (↑ and ↓), and inference rule names.



4 Arithmetic (30 pts)

In this problem we consider an alternative specification and implementation of the predecessor function in Heyting arithmetic. Because in Heyting arithmetic equality and all quantifiers range over natural numbers, we omit ":nat" and presuppose that propositions involving equality are well-formed.

Recall the rules for equality:

$$\frac{}{0 = 0 \; true} =I_{00} \qquad \frac{x = y \; true}{\mathsf{s}\,x = \mathsf{s}\,y \; true} =I_{ss}$$

$$\text{no rule } E_{00} \qquad \frac{0 = \mathsf{s}\,x \; true}{C \; true} =E_{0s} \qquad \frac{\mathsf{s}\,x = 0 \; true}{C \; true} =E_{s0} \qquad \frac{\mathsf{s}\,x = \mathsf{s}\,y \; true}{x = y \; true} =E_{ss}$$

In lecture we proved a useful derived rule of inference

$$\frac{}{x = x \; true} \,\mathsf{refl}$$

which you may use freely in this problem.

Theorem. $\forall x. (\neg\, x = 0) \supset \exists y. x = \mathsf{s}\, y$

Task 1 (20 pts).
Complete the following proof skeleton.

Proof: By mathematical induction on x.

Base: $x = 0$. To show: ____.
Assume ____________________.
By rule __________ we have ____________________.
Therefore ____________________ by rules ____________________.

Step: Assume ____.
To show: ____.
Assume ____________________.
It remains to show ______________________________.
This follows by ____
____.

□

Task 2 (10 pts).

Now assume we have a proof term refl : $x = x$ for an arbitrary natural number x. With this, complete the following program which represents the proof of you gave above, including the reasoning about equality. If you need additional proof terms for rules concerning equality, please show the rules and annotate them with proof terms as needed.

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fun pred 0 =

  | pred (s x) =
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