Double-Negation Translation (12 points)

Unlike in constructive logic, which has only a judgment for truth, classical logic has judgments for truth, falsity, and contradiction. Classical logic also has a primitive notion of negation ($\neg A$), whereas constructive logic simply defines $\neg A$ to be $A \supset \bot$.

The Gödel–Gentzen double–negation translation takes a classical proposition $A$ to a constructive proposition $A^*$, and is defined inductively on the structure of $A$ as follows:

- $\top^* = \top$
- $\bot^* = \bot$
- $(A \land B)^* = A^* \land B^*$
- $(A \lor B)^* = A^* \lor B^*$
- $(A \supset B)^* = \neg(\neg A^* \lor B^*)$
- $(\neg A)^* = \neg A^*$
- $P^* = \neg \neg P$ where $P$ atomic

(Note: On the left, $\neg$ is the primitive classical notion of negation; on the right, $\neg$ is an abbreviation for $\neg \supset \bot$.)

Task 1 (12 points). Prove that, for any classical proposition $A$,

$\vdash \neg A^* \supset A^* \text{ true}$

is derivable (constructively). You need only show the cases for:
2 Embedding Classical Logic (28 points)

The Gödel–Gentzen double–negation translation allows us to embed classical logic into constructive logic in the following sense:

Theorem 1.

1. If \( \Gamma \vdash C \) true, then \( \Gamma^* \vdash A^* \) true.
2. If \( \Gamma \vdash C \) false, then \( \Gamma^* \vdash \neg A^* \) true.
3. If \( \Gamma \vdash C \) #, then \( \Gamma^* \vdash \bot \) true.

In the above theorem, \( \Gamma^* \) is the result of applying the double–negation translation to each proposition in the context; \( \vdash_C \) indicates a classical derivation, using the rules listed at the end of this assignment; and \( \vdash \) indicates a constructive derivation. In other words, this theorem states that any classically–derivable proposition has a constructively–derivable counterpart, given by the translation.

Task 2 (28 points). Prove Theorem 1, showing the following cases:

- \( T \)
- atomic propositions
- \( \top \)
- \( \lor \)

You may use the result stated in Task 1. (Hint: You will also need weakening.)
A  Classical rules

Γ,x : A true ⊢C M : B true ⊃T

Γ ⊢C λx : A.M : A ⊃ B true

Γ ⊢C M : A true ⊢C (M,N) : A ∧ B true ∧T

Γ ⊢C N : B true

Γ ⊢C T

Γ ⊢C ⊢C CM : A true

Γ ⊢C C N : B true ∧T

Γ ⊢C C ⟨M,N⟩ : A ∧ B true ∧T

Γ ⊢C C ¬K : ¬A true ¬T

Γ ⊢C C ⊢C K : A false

Γ ⊢C C L : B false

Γ ⊢C C ⊢C • : ⊥ false ⊥F

Γ ⊢C C ⊢C K : A false

Γ ⊢C C π1.K : A ∧ B false ∧F1

Γ ⊢C C π2.K : A ∧ B false ∧F2

Γ ⊢C C ⊢C K : A false ¬T

Γ ⊢C C ⊢C M : A true ¬F

Γ ⊢C C ⊢C +K : # #

Γ,x : A true ⊢C x : A true hypT

Γ,x : A false ⊢C x : A false hypF

Γ,u : A false ⊢C E : # PBCT

Γ,u : A true ⊢C u : A true.E : A true PBCF