Constructive Logic (15-317), Fall 2012
Assignment 4: Sequent Calculus and Natural Numbers

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Out: Thursday, September 27, 2012
Due: Thursday, October 4, 2012 (before class)

In this assignment, you will prove metatheorems about sequent calculus, and propositions about natural numbers.

1 Contraction (20 points)

The contraction lemma for sequent calculus states that two equal propositions on the left of a sequent can be replaced by a single instance:

Lemma 1 (Contraction)**.** If $\Delta, A, A \longrightarrow D$, then $\Delta, A \longrightarrow D$.

We can prove contraction by structural induction on the derivation $\Delta, A, A \longrightarrow D$, in a similar fashion to the proofs of the substitution and weakening lemmas for natural deduction.

Task 1 (20 points)**.** Prove contraction for the $\supset, \wedge$ fragment of sequent calculus. That is, show the cases of the proof where the last rule to be applied is:

- init
- $\supset R$
- $\supset L$
- $\wedge R$
- $\wedge L_1$

You don't need to show the $\wedge L_2$ case, because it is very similar to the $\wedge L_1$ case.

2 Derivability and Admissibility (10 points)

Task 2 (10 points)**.** For each of the following sequent calculus rules, state whether or not it is derivable, and whether or not it is admissible. Explain.

$$\frac{\Delta, A, A \longrightarrow D}{\Delta, A \longrightarrow D}\ (1) \qquad \frac{A\ \textit{atomic}}{\Delta, A \wedge B \longrightarrow A}\ (2)$$

$$\frac{}{\Delta, A, A \supset B \longrightarrow B}\ (3) \qquad \frac{\Delta \longrightarrow A \quad \Delta, A \longrightarrow C}{\Delta \longrightarrow C}\ (4)$$

3 Natural numbers (10 points)

In recitation, we saw how to define the natural numbers as a data type, and prove theorems quantified over them. We also defined the even and odd predicates on natural numbers:

$$\frac{}{\mathsf{even}(0)} \qquad \frac{n : \mathsf{nat} \quad \mathsf{odd}(n)\ \textit{true}}{\mathsf{even}(\mathsf{s}(n))\ \textit{true}} \qquad \frac{n : \mathsf{nat} \quad \mathsf{even}(n)\ \textit{true}}{\mathsf{odd}(\mathsf{s}(n))\ \textit{true}}$$

Task 3 (5 points)**.** Derive the following judgment:

$$\forall n : \mathsf{nat}.(\mathsf{even}(n) \supset \mathsf{even}(\mathsf{s}(\mathsf{s}(n))))\ \textit{true}$$

Task 4 (5 points)**.** Derive the following judgment:

$$\forall n : \mathsf{nat}.(\mathsf{even}(n) \vee \mathsf{even}(\mathsf{s}(n)))\ \textit{true}$$