

# Constructive Logic (15-317), Fall 2012

## Assignment 3: Quantifiers and Metatheorems

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Out: Thursday, September 20, 2012

Due: Thursday, September 27, 2012 (before class)

In this assignment, you will prove propositions with quantifiers ( $\forall$  and  $\exists$ ), and prove metatheorems about natural deduction and sequent calculus.

The Tutch portion of your work (Section 1) should be submitted electronically using the command

```
$ /afs/andrew/course/15/317/bin/submit -r hw03 <files...>
```

from any Andrew server. You may check the status of your submission by running the command

```
$ /afs/andrew/course/15/317/bin/status hw03
```

If you have trouble running either of these commands, email Carlo.

The written portion of your work (Sections 2, 3) should be submitted at the beginning of class. If you are familiar with  $\text{\LaTeX}$ , you are encouraged to use this document as a template for typesetting your solutions, but you may alternatively write your solutions *neatly* by hand.

### 1 Tutch Proofs (15 points)

For quantifiers, you'll need a little bit of new syntax:

- $\forall x : \tau.A$  is written `!x :  $\tau$ .A`
- $\exists x : \tau.A$  is written `?x :  $\tau$ .A`
- The existential elimination rule requires a hypothetical with two assumptions. These are written by separating the assumptions with a comma. For example:

```

proof exp : (?x:t.A(x)) => ?x:t.A(x) =
begin
  [(?x:t.A(x));
   [a:t , A(a);
    ?x:t.A(x)];
   ?x:t.A(x)];
  (?x:t.A(x)) => ?x:t.A(x);
end;

```

**Task 1** (15 points). Prove the following theorems using Tutch.

```

proof weak : (!x:t. P(x) => (?y:t. P(y)));
proof dm : ~(?x:t.P(x)) => (!x:t.~P(x));
proof emp : (!x:t. P(x) => Q(x)) => (?x:t. P(x)) => (?x:t. Q(x));
proof compose : (!x:t. P(x) => Q(x)) => (!x:t. Q(x) => R(x))
=> (!x:t. P(x) => R(x));
proof deor : ((?x:t. P(x)) | (?x:t. Q(x))) => (?x:t. P(x) | Q(x));

```

On Andrew machines, you can check your progress against the requirements file `/afs/andrew/course/15/317/req/hw03.req` by running the command

```
$ /afs/andrew/course/15/317/bin/tutch -r hw03 <files...>
```

## 2 Weakening (13 points)

The weakening lemma states that typing judgments hold in the presence of additional variables. Formally:

**Lemma 1** (Weakening). If  $\Gamma \vdash M : B$  and  $x \notin \Gamma$ , then  $\Gamma, x : A \vdash M : B$ .

We can prove weakening by structural induction on the typing derivation  $\Gamma \vdash M : B$ , in a similar fashion to the proof of the substitution lemma presented in lecture.

**Task 2** (13 points). Prove weakening for the  $\supset$  fragment of our logic. That is, show the cases of the proof where the last rule to be applied is:

- The hypothesis (variable) rule
- $\supset I$
- $\supset E$

### 3 Sequent Calculus (12 points)

**Task 3** (3 points). Assuming  $A$  and  $B$  are atomic propositions, give a sequent calculus derivation of:

$$\cdot \longrightarrow (A \vee \neg A) \supset (((A \supset B) \supset A) \supset A)$$

We claimed in lecture that it is much easier to demonstrate unprovability (and related results) in sequent calculus than natural deduction.

**Task 4** (3 points). Assuming  $A$  is an atomic proposition, show there is *no* proof of:

$$\cdot \longrightarrow (\neg A) \supset A$$

**Task 5** (3 points). Assuming  $A$  and  $B$  are (distinct) atomic propositions, show there is *no* proof of:

$$\cdot \longrightarrow (A \vee B) \supset (A \wedge B)$$

**Task 6** (3 points). Assuming  $A$  is an atomic proposition, show there is *exactly one* proof of:

$$\cdot \longrightarrow A \supset A$$