Constructive Logic (15-317), Fall 2012
Assignment 2: Proof Terms and Substitution

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Out: Friday, September 14, 2012
Due: Thursday, September 20, 2012 (before class)

In this assignment, you will write proof terms and demonstrate your understanding of substitution. The Tutch portion of your work (Section 1) should be submitted electronically using the command

```
$ /afs/andrew/course/15/317/bin/submit -r hw02 <files...>
```

from any Andrew server. You may check the status of your submission by running the command

```
$ /afs/andrew/course/15/317/bin/status hw02
```

If you have trouble running either of these commands, email Carlo.

The written portion of your work (Sections 2, 3, and 4) should be submitted at the beginning of class. If you are familiar with LaTeX, you are encouraged to use this document as a template for typesetting your solutions, but you may alternatively write your solutions *neatly* by hand.

1 Tutch Proofs (15 points)

To give a proof term in Tutch, declare it with `term` rather than `proof`:

```
term andComm : A & B => B & A =
  fn u => (snd u, fst u);
```

For more examples, see Chapter 4 of the *Tutch User's Guide*. The proof terms are very similar to the ones given in lecture (even more similar to ML) and are summarized in Section A.2.1 of the *Guide*.

Task 1 (15 pts)**.** Prove the following theorems using Tutch:

```
term absurdity : (A & ~A) => B;
term demorgan : ~(A | B) => ~A & ~B;
term uncurry : (A => B => C) => (A & B => C);
term lemDne : (A | ~A) => (~~A => A);
term poe : (A | B) => ~A => B;
```

On Andrew machines, you can check your progress against the requirements file `/afs/andrew/course/15/317/req/hw02.req` by running the command

```
$ /afs/andrew/course/15/317/bin/tutch -r hw02 <files...>
```

2 Substitution (6 points)

Recall the definition of substitution given in lecture. To see how they operate on particular terms, we can expand the definition iteratively until no substitutions need be applied. Here is an example:

$$\begin{array}{rl} & [(\lambda x{:}A.\, x)/f]\ (f\ z) \\ = & ([(\lambda x{:}A.\, x)/f]\ f)\ ([\lambda x{:}A.\, x/f]\ z) \\ = & (\lambda x{:}A.\, x)\ z \end{array}$$

Task 2 (6 points)**.** Show the step-by-step unpacking of the following substitutions, taking care to avoid variable capture:

1. $[(f\ y)/x]\ (\lambda f{:}A \supset B.\ (f\ x))$
2. $[(f\ y)/x]\ (x\ (\lambda x{:}A.\ y))$
3. $[(f\ y)/x]\ (x\ (\lambda y{:}A.\ (f\ y\ x)))$

3 Verification and Uses (9 points)

In this set of tasks, we will investigate how replacing the *true* judgment with the $\uparrow$ and $\downarrow$ judgments affects the structure of proofs. Assume A and B are atomic propositions.

Task 3 (3 points)**.** Give *two* different natural deduction proofs of $(A \supset B) \supset (A \supset B)$ *true*. How many natural deduction proofs of this judgment exist? Explain.

Task 4 (3 points)**.** Give a natural deduction proof of $A \supset A$ *true*. How many natural deduction proofs of this judgment exist? Explain.

Task 5 (3 points)**.** Give a proof of $A \supset A \uparrow$. How many proofs of this judgment exist? Explain.

4 &er's Game (10 points)

4.1 Uses and verifications

Recall the & connective defined in Homework 1:

$$\frac{A\ true \quad B\ true}{A \mathbin{\&} B\ true}\ \&I \qquad \frac{A \mathbin{\&} B\ true \quad \begin{matrix}[A\ true] \\ \vdots \\ C\ true\end{matrix}}{C\ true}\ \&E_1 \qquad \frac{A \mathbin{\&} B\ true \quad \begin{matrix}[B\ true] \\ \vdots \\ C\ true\end{matrix}}{C\ true}\ \&E_2$$

Task 6 (3 points)**.** Give use and verification rules corresponding to these introduction and elimination rules. There may be more than one right answer.

4.2 Proof terms

Consider the following proof term assignment for the & connective:

$$\frac{M : A \quad N : B}{\langle\langle M, N\rangle\rangle : A \mathbin{\&} B}\ \&I \qquad \frac{M : A \mathbin{\&} B \quad \begin{matrix}[x : A] \\ \vdots \\ N : C\end{matrix}}{p_1(M, x.N) : C}\ \&E_1 \qquad \frac{M : A \mathbin{\&} B \quad \begin{matrix}[x : B] \\ \vdots \\ N : C\end{matrix}}{p_2(M, x.N) : C}\ \&E_2$$

Task 7 (4 points)**.** Show all the local reduction(s) and expansion(s) for these rules in proof term notation. Be sure to indicate which are reductions and which are expansions. You do not need to include derivations for the proof terms.

Task 8 (3 points)**.** Expand the definition of substitution given in lecture to the & proof terms.