Midterm II
Solutions

15-317: Constructive Logic

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Instructions

- This exam is closed book and closed Internet. A two-sided sheet of handwritten notes is permitted. The last page of the exam recaps some rules you may find useful.
- There are four problems. Not all problems are the same size or difficulty, so it may help to read through the whole exam first. You have ninety minutes to complete the exam.
- When writing proofs, remember to label each inference with the rule used and any variables or parameters discharged (e.g., $\supset I^u$).
- You may find it helpful to construct your proofs on scratch paper (such as the back of a page) before writing it clearly in the space provided.
- Good luck!

	Problem 1	Problem 2	Problem 3	Problem 4	**Total**
Score					
Max	30	30	30	30	120

1 Theorem Proving (30 points)

Below are sequent-calculus rules for an unusual logical connective written $A \smiley B$.

$$\frac{\Gamma \longrightarrow A \quad \Gamma, B \longrightarrow \bot}{\Gamma \longrightarrow A \smiley B}\,\smiley R \qquad \frac{\Gamma, A \smiley B, A \longrightarrow C}{\Gamma, A \smiley B \longrightarrow C}\,\smiley L1 \qquad \frac{\Gamma, A \smiley B, A \longrightarrow B}{\Gamma, A \smiley B \longrightarrow C}\,\smiley L2$$

Which of these rules are invertible, and which are non-invertible? For any invertible rules, give a proof of invertibility. (You may use weakening, cut, and/or generalized initial sequent.)

Solution. $\smiley R$ is invertible, since the following two rules are admissible:

$$\frac{\Gamma \longrightarrow A \smiley B}{\Gamma \longrightarrow A} \qquad \frac{\Gamma \longrightarrow A \smiley B}{\Gamma, B \longrightarrow \bot}$$

Proof of the former:

$$\dfrac{\Gamma \longrightarrow A \smiley B \qquad \dfrac{\dfrac{}{\Gamma, A \smiley B, A \longrightarrow A}\,\text{g.i.s.}}{\Gamma, A \smiley B \longrightarrow A}\,\smiley L1}{\Gamma \longrightarrow A}\,\text{cut}$$

Proof of the latter:

$$\dfrac{\dfrac{\Gamma \longrightarrow A \smiley B}{\Gamma, B \longrightarrow A \smiley B}\,\text{weak} \qquad \dfrac{\dfrac{}{\Gamma, B, A \smiley B, A \longrightarrow B}\,\text{g.i.s.}}{\Gamma, B, A \smiley B \longrightarrow \bot}\,\smiley L2}{\Gamma, B \longrightarrow \bot}\,\text{cut}$$

$\smiley L1$ is invertible, since the following rule is admissible:

$$\frac{\Gamma, A \smiley B \longrightarrow C}{\Gamma, A \smiley B, A \longrightarrow C}$$

Proof:

$$\frac{\Gamma, A \smiley B \longrightarrow C}{\Gamma, A \smiley B, A \longrightarrow C}\,\text{weak}$$

$\smiley L2$ is not invertible.

□

2 Classical Logic (30 points)

We may adapt the sequent calculus to classical logic by adding multiple conclusions to the right-hand-side of the judgement. The classical sequent

$$A_1, \ldots A_m \longrightarrow B_1, \ldots B_m$$

means, roughly, that the conjunction of A_1 through A_m implies the *disjunction* of B_1 through B_n. More precisely, it means that we may draw a contradiction if we assume that A_1 through A_m are true, and B_1 through B_n are false.

Below are three rules from the classical sequent calculus:

$$\frac{A \text{ atomic}}{\Gamma, A \longrightarrow A, \Delta}\text{init} \qquad \frac{\Gamma, A \longrightarrow B, \Delta}{\Gamma \longrightarrow A \supset B, \Delta}{\supset}R \qquad \frac{\Gamma, A \supset B \longrightarrow A, \Delta \quad \Gamma, A \supset B, B \longrightarrow \Delta}{\Gamma, A \supset B \longrightarrow \Delta}{\supset}L$$

In this problem you will prove the soundness of this fragment of the classical sequent calculus.

Definition If $\Gamma = A_1, \ldots, A_m$, define $\Gamma\ \mathsf{true} = A_1\ \mathsf{true}, \ldots, A_m\ \mathsf{true}$. Similarly, if $\Delta = B_1, \ldots, B_n$, define $\Delta\ \mathsf{false} = B_1\ \mathsf{false}, \ldots, B_n\ \mathsf{false}$.

Theorem (Soundness) If $\Gamma \longrightarrow \Delta$ then $\Gamma\ \mathsf{true}, \Delta\ \mathsf{false} \vdash \#$.

Prove the theorem for the cases shown above. (You may use weakening.)

Solution. By induction on the derivation of $\Gamma \longrightarrow \Delta$.

Case:

$$\frac{A \text{ atomic}}{\Gamma, A \longrightarrow A, \Delta}\text{init}$$

Then we have

$$\frac{\dfrac{}{\Gamma\ \mathsf{true}, A\ \mathsf{true}, \Delta\ \mathsf{false}, A\ \mathsf{false} \vdash A\ \mathsf{true}}\mathsf{hyp}T \quad \dfrac{}{\Gamma\ \mathsf{true}, A\ \mathsf{true}, \Delta\ \mathsf{false}, A\ \mathsf{false} \vdash A\ \mathsf{false}}\mathsf{hyp}F}{\Gamma\ \mathsf{true}, A\ \mathsf{true}, \Delta\ \mathsf{false}, A\ \mathsf{false} \vdash \#}\#$$

Case:

$$\frac{\Gamma, A \longrightarrow B, \Delta}{\Gamma \longrightarrow A \supset B, \Delta}{\supset}R$$

By the induction hypothesis, we have a derivation of $\Gamma\ \mathsf{true}, A\ \mathsf{true}, B\ \mathsf{false}, \Delta\ \mathsf{false} \vdash \#$, which we weaken to $\Gamma\ \mathsf{true}, A\ \mathsf{true}, A \supset B\ \mathsf{false}, B\ \mathsf{false}, \Delta\ \mathsf{false} \vdash \#$. Then

$$\frac{\dfrac{\dfrac{\Gamma\ \mathsf{true}, A\ \mathsf{true}, A \supset B\ \mathsf{false}, B\ \mathsf{false}, \Delta\ \mathsf{false} \vdash \#}{\Gamma\ \mathsf{true}, A\ \mathsf{true}, A \supset B\ \mathsf{false}, \Delta\ \mathsf{false} \vdash B\ \mathsf{true}}\mathsf{PBC}T}{\Gamma\ \mathsf{true}, A \supset B\ \mathsf{false}, \Delta\ \mathsf{false} \vdash A \supset B\ \mathsf{true}}{\supset}T \quad \dfrac{}{\Gamma\ \mathsf{true}, A \supset B\ \mathsf{false}, \Delta\ \mathsf{false} \vdash A \supset B\ \mathsf{false}}\mathsf{hyp}F}{\Gamma\ \mathsf{true}, A \supset B\ \mathsf{false}, \Delta\ \mathsf{false} \vdash \#}\#$$

Case:

$$\frac{\Gamma, A \supset B \longrightarrow A, \Delta \quad \Gamma, A \supset B, B \longrightarrow \Delta}{\Gamma, A \supset B \longrightarrow \Delta} \supset L$$

By the induction hypothesis, we have a derivation of $\Gamma\ \mathsf{true}, A \supset B\ \mathsf{true}, A\ \mathsf{false}, \Delta\ \mathsf{false} \vdash \#$ and a derivation of $\Gamma\ \mathsf{true}, A \supset B\ \mathsf{true}, B\ \mathsf{true}, \Delta\ \mathsf{false} \vdash \#$. Then

$$\frac{\dfrac{}{\Gamma\ \mathsf{true}, A \supset B\ \mathsf{true}, \Delta\ \mathsf{false} \vdash A \supset B\ \mathsf{true}}\ \mathsf{hyp}T \quad \dfrac{\mathcal{D} \quad \mathcal{E}}{\Gamma\ \mathsf{true}, A \supset B\ \mathsf{true}, \Delta\ \mathsf{false} \vdash A \supset B\ \mathsf{false}} \supset F}{\Gamma\ \mathsf{true}, A \supset B\ \mathsf{true}, \Delta\ \mathsf{false} \vdash \#} \#$$

where

$$\mathcal{D} = \frac{\Gamma\ \mathsf{true}, A \supset B\ \mathsf{true}, A\ \mathsf{false}, \Delta\ \mathsf{false} \vdash \#}{\Gamma\ \mathsf{true}, A \supset B\ \mathsf{true}, \Delta\ \mathsf{false} \vdash A\ \mathsf{true}}\ \mathsf{PBC}T$$

and

$$\mathcal{E} = \frac{\Gamma\ \mathsf{true}, A \supset B\ \mathsf{true}, B\ \mathsf{true}, \Delta\ \mathsf{false} \vdash \#}{\Gamma\ \mathsf{true}, A \supset B\ \mathsf{true}, \Delta\ \mathsf{false} \vdash B\ \mathsf{false}}\ \mathsf{PBC}F$$

□

3 Prolog (30 points)

Using Prolog, implement a relation `isort(L1, L2)` that, when given a list as `L1`, sorts the list using **insertion sort**, and returns the sorted list as `L2`. (Hint: you will need a helper relation.)

Solution.

```
isort([], []).
isort([X | L1], L2) :-
  isort(L1, M),
  insert(X, M, L2).

% insert(X, L1, L2)
% L1 and L2 are sorted lists; L2 contains the elements of L1, plus X.

insert(X, [], [X]).
insert(X, [Y | L], [X, Y | L]) :-
  X =< Y.
insert(X, [Y | L1], [Y | L2]) :-
  X > Y,
  insert(X, L1, L2).
```

□

4 Higher-Order Logic Programming (30 points)

Below is a fragment of natural deduction, encoded in Elf. The code is similar, but not identical, to the code you saw in class.

```
pr       : type.

tt       : pr.
ff       : pr.
atom     : pr.
and      : pr -> pr -> pr.
imp      : pr -> pr -> pr.

true     : pr -> type.

true/tt  : true tt.

true/and : true (and P Q)
            <- true P
            <- true Q.

true/imp : true (imp P Q)
            <- (true P -> true Q).

true/ffe : true C
            <- true ff.
```

Recall that the rules for proof search are as follows:

- Find a rule whose conclusion can unify with the current goal, trying first rules in the dynamic pool (from most recent to least recent), then rules in the static pool (from first to last).
- Recursively prove the rule's subgoals, in order from first to last.
- When backtracking (either because a subproof failed, or because you are searching for additional solutions), return to the most recent choice point and try the next applicable rule.

[Part 4, continued.]

Below are several queries. For each query, give the first three (3) solutions for `P` generated by Elf, in the order in which Elf generates them. (If Elf generates the same solution more than once, you should list it each time.)

1. `true P`

Solution.

1. `tt`
2. `and tt tt`
3. `and tt (and tt tt)`

□

2. `true (imp P atom)`

Solution.

1. `atom`
2. `ff`
3. `ff`

□

3. `true (imp atom P)`

Solution.

1. `atom`
2. `tt`
3. `and atom atom`

□

4. `true (imp P (imp P atom))`

Solution.

1. `atom`
2. `atom`
3. `ff`

□

A Classical Logic rules

$$\frac{\Gamma, A\ \textsf{true} \vdash B\ \textsf{true}}{\Gamma \vdash A \supset B\ \textsf{true}} \supset T \qquad \frac{\Gamma \vdash A\ \textsf{true} \quad \Gamma \vdash B\ \textsf{true}}{\Gamma \vdash A \wedge B\ \textsf{true}} \wedge T \qquad \frac{}{\Gamma \vdash \top\ \textsf{true}} \top T$$

$$\frac{\Gamma \vdash A\ \textsf{true}}{\Gamma \vdash A \vee B\ \textsf{true}} \vee T1 \qquad \frac{\Gamma \vdash B\ \textsf{true}}{\Gamma \vdash A \vee B\ \textsf{true}} \vee T2 \qquad \frac{\Gamma \vdash A\ \textsf{false}}{\Gamma \vdash \neg A\ \textsf{true}} \neg T$$

$$\frac{\Gamma \vdash A\ \textsf{true} \quad \Gamma \vdash B\ \textsf{false}}{\Gamma \vdash A \supset B\ \textsf{false}} \supset F \qquad \frac{\Gamma \vdash A\ \textsf{false} \quad \Gamma \vdash B\ \textsf{false}}{\Gamma \vdash A \vee B\ \textsf{false}} \vee F \qquad \frac{}{\Gamma \vdash \bot\ \textsf{false}} \bot F$$

$$\frac{\Gamma \vdash A\ \textsf{false}}{\Gamma \vdash A \wedge B\ \textsf{false}} \wedge F1 \qquad \frac{\Gamma \vdash B\ \textsf{false}}{\Gamma \vdash A \wedge B\ \textsf{false}} \wedge F2 \qquad \frac{\Gamma \vdash A\ \textsf{true}}{\Gamma \vdash \neg A\ \textsf{false}} \neg F$$

$$\frac{\Gamma \vdash A\ \textsf{true} \quad \Gamma \vdash A\ \textsf{false}}{\Gamma \vdash \#} \# \qquad \frac{}{\Gamma, A\ \textsf{true} \vdash A\ \textsf{true}} \textsf{hyp}T \qquad \frac{}{\Gamma, A\ \textsf{false} \vdash A\ \textsf{false}} \textsf{hyp}F$$

$$\frac{\Gamma, A\ \textsf{false} \vdash \#}{\Gamma \vdash A\ \textsf{true}} \textsf{PBC}T \qquad \frac{\Gamma, A\ \textsf{true} \vdash \#}{\Gamma \vdash A\ \textsf{false}} \textsf{PBC}F$$