Midterm I
Solutions

15-317: Constructive Logic

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Instructions

- This exam is closed book and closed Internet. A two-sided sheet of handwritten notes is permitted. The last page of the exam recaps some rules you may find useful.
- There are five problems. Not all problems are the same size or difficulty, so it may help to read through the whole exam first. You have ninety minutes to complete the exam.
- When writing proofs, remember to label each inference with the rule used and any variables or parameters discharged (e.g., $\supset I^u$).
- You may find it helpful to construct your proofs on scratch paper (such as the back of a page) before writing it clearly in the space provided.
- Good luck!

	Problem 1	Problem 2	Problem 3	Problem 4	Problem 5	**Total**
Score						
Max	30	20	30	30	10	120

1 Sequent Calculus (30 points)

In the sequent calculus, the initial sequent rule can only be used when the proposition in the conclusion is atomic. However, a generalized principle is nevertheless valid:

Theorem (Generalized initial sequent): $\Gamma, A \longrightarrow A$ is derivable, for all Γ and A.

Prove the generalized initial sequent theorem. (*Hint*: the proof is similar to the proof in verifications and uses that $A \downarrow$ implies $A \uparrow$.) You may omit the cases for $\wedge$ and $\top$.

Solution. By induction on the structure of the proposition A.

Case: A is atomic.

$$\frac{A \ atomic}{\Gamma, A \longrightarrow A}\ \mathsf{init}$$

Case: $A = \bot$

$$\frac{}{\Gamma, \bot \longrightarrow \bot}\ \bot L$$

Case: $A = B \vee C$

By the inductive hypothesis on B, we have $\Gamma, B \vee C, B \longrightarrow B$.
By the inductive hypothesis on C, we have $\Gamma, B \vee C, C \longrightarrow C$.

$$\frac{\dfrac{\Gamma, B \vee C, B \longrightarrow B}{\Gamma, B \vee C, B \longrightarrow B \vee C}\ \vee R1 \qquad \dfrac{\Gamma, B \vee C, C \longrightarrow C}{\Gamma, B \vee C, C \longrightarrow B \vee C}\ \vee R2}{\Gamma, B \vee C \longrightarrow B \vee C}\ \vee L$$

Case: $A = B \supset C$

By the inductive hypothesis on B, we have $\Gamma, B \supset C, B \longrightarrow B$.
By the inductive hypothesis on C, we have $\Gamma, B \supset C, B, C \longrightarrow C$.

$$\frac{\dfrac{\Gamma, B \supset C, B \longrightarrow B \quad \Gamma, B \supset C, B, C \longrightarrow C}{\Gamma, B \supset C, B \longrightarrow C}\ \supset L}{\Gamma, B \supset C \longrightarrow B \supset C}\ \supset R$$

□

2 Proof Terms (20 points)

Consider the following derivation of A true:

$$
\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\begin{array}{c}[A\lor A\ \mathsf{true}]^x\\ \vdots\\ A\lor A\ \mathsf{true}\end{array} \quad \begin{array}{c}[A\ \mathsf{true}]^y\\ \vdots\\ A\ \mathsf{true}\end{array} \quad \begin{array}{c}[A\ \mathsf{true}]^z\\ \vdots\\ A\ \mathsf{true}\end{array}}{A\ \mathsf{true}}\lor E^{y,z}}{(A\lor A)\supset A\ \mathsf{true}}\supset I^x \quad \begin{array}{c}\mathcal{D}\\ B\ \mathsf{true}\end{array}}{((A\lor A)\supset A)\land B\ \mathsf{true}}\land I}{(A\lor A)\supset A\ \mathsf{true}}\land E1 \quad \dfrac{\begin{array}{c}\mathcal{E}\\ A\ \mathsf{true}\end{array}}{A\lor A\ \mathsf{true}}\lor I1}{A\ \mathsf{true}}\supset E
$$

where $\mathcal{D}$ is a derivation of B true, and $\mathcal{E}$ is a derivation of A true.

Problem 1: Write the proof term that corresponds to the derivation above. Use $M : B$ as the proof term for $\mathcal{D}$, and $N : A$ as the proof term for $\mathcal{E}$.

Solution. $(\pi_1 \langle \lambda x{:}A \lor A.\mathsf{case}(x, y.y, z.z), M\rangle)\ (\mathsf{in}_1\ N)$ □

Problem 2: Give a step-by-step reduction of your proof term to normal form. (That is, apply reduction rules until no more reductions apply.)

Solution.

$$
\begin{array}{rl}
 & (\pi_1 \langle \lambda x{:}A \lor A.\mathsf{case}(x, y.y, z.z), M\rangle)\ (\mathsf{in}_1\ N) \\
\longrightarrow_R & (\lambda x{:}A \lor A.\mathsf{case}(x, y.y, z.z))\ (\mathsf{in}_1\ N) \\
\longrightarrow_R & [(\mathsf{in}_1\ N)/x]\mathsf{case}(x, y.y, z.z) \\
= & \mathsf{case}(\mathsf{in}_1\ N, y.y, z.z) \\
\longrightarrow_R & [N/y]y \\
= & N
\end{array}
$$

□

3 Unprovability (30 points)

Show that that following judgement is not derivable:

$$(A \supset B) \vee (B \supset A) \text{ true}$$

Solution. $(A \supset B) \vee (B \supset A)$ true is derivable in natural deduction if and only if $\cdot \longrightarrow (A \supset B) \vee (B \supset A)$ is derivable in sequent calculus. Thus, it suffices to show it is not derivable in sequent calculus.

The only rules which may conclude this sequent are $\vee R1$ and $\vee R2$. If $\vee R1$ is applied first, then we must prove $\cdot \longrightarrow A \supset B$; then only $\supset R$ applies, and we must prove $A \longrightarrow B$. No rule applies here for arbitrary propositions A, B, so we conclude there is no proof.

$$\dfrac{\dfrac{A \longrightarrow B}{\cdot \longrightarrow A \supset B}\supset R}{\cdot \longrightarrow (A \supset B) \vee (B \supset A)}\vee R1$$

The case where $\vee R2$ is applied first is symmetric.

$$\dfrac{\dfrac{B \longrightarrow A}{\cdot \longrightarrow B \supset A}\supset R}{\cdot \longrightarrow (A \supset B) \vee (B \supset A)}\vee R2$$

□

4 Natural Numbers (30 points)

Recall our rules defining the type of natural numbers:

$$\frac{}{\mathsf{0} : \mathsf{nat}}\ \mathsf{nat}I_0 \qquad \frac{n : \mathsf{nat}}{\mathsf{s}(n) : \mathsf{nat}}\ natI_{\mathsf{s}} \qquad \frac{n : \mathsf{nat} \quad C(0)\ \mathsf{true} \quad \begin{array}{c} \overline{x : \mathsf{nat}}^{\,x} \quad \overline{C(x)\ \mathsf{true}}^{\,u} \\ \vdots \\ C(\mathsf{s}(x))\ \mathsf{true} \end{array}}{C(n)\ \mathsf{true}}\ \mathsf{nat}E^{x,u}$$

Consider also the following rules which define a $\mathsf{sum}(l, n, m)$ relation, which holds when $l + n = m$:

$$\frac{}{\mathsf{sum}(0, n, n)\ \mathsf{true}}\ \mathsf{sum}I_0 \qquad \frac{\mathsf{sum}(l, n, m)\ \mathsf{true}}{\mathsf{sum}(\mathsf{s}(l), n, \mathsf{s}(m))\ \mathsf{true}}\ \mathsf{sum}I_{\mathsf{s}}$$

Prove:

$$\forall x{:}\mathsf{nat}.\ \mathsf{sum}(x, 0, x)\ \mathsf{true}$$

Solution.

$$\frac{\dfrac{\dfrac{}{n : \mathsf{nat}}\ n \qquad \dfrac{}{\mathsf{sum}(0,0,0)\ \mathsf{true}}\ \mathsf{sum}I_0 \qquad \dfrac{\dfrac{}{\mathsf{sum}(y, 0, y)\ \mathsf{true}}\ u}{\mathsf{sum}(\mathsf{s}(y), 0, \mathsf{s}(y))\ \mathsf{true}}\ \mathsf{sum}I_{\mathsf{s}}}{\mathsf{sum}(n, 0, n)\ \mathsf{true}}\ \mathsf{nat}E^{y,u}}{\forall x{:}\mathsf{nat}.\ \mathsf{sum}(x, 0, x)\ \mathsf{true}}\ \forall I^n$$

□

5 Mistakes Were Made (10 points)

Consider the following purported proof:

$$
\dfrac{\dfrac{\dfrac{}{(A \wedge B) \vee C\ \mathsf{true}}\,x \qquad \dfrac{\dfrac{\dfrac{\dfrac{}{A \wedge B\ \mathsf{true}}\,p}{A\ \mathsf{true}}\,\wedge E1}{A \vee C\ \mathsf{true}}\,\vee I1 \quad \dfrac{\dfrac{}{A \wedge B\ \mathsf{true}}\,p}{B\ \mathsf{true}}\,\wedge E2}{(A \vee C) \wedge B\ \mathsf{true}}\,\wedge I \qquad \dfrac{\dfrac{\dfrac{}{C\ \mathsf{true}}\,q}{A \vee C\ \mathsf{true}}\,\vee I2 \quad \dfrac{\dfrac{}{A \wedge B\ \mathsf{true}}\,p}{B\ \mathsf{true}}\,\wedge E2}{(A \vee C) \wedge B\ \mathsf{true}}\,\wedge I}{(A \vee C) \wedge B\ \mathsf{true}}\,\vee E^{p,q}}{((A \wedge B) \vee C) \supset ((A \vee C) \wedge B)\ \mathsf{true}}\,\supset I^x
$$

This proof is incorrect. Circle the rule(s) which were applied incorrectly. Explain.

Solution. The $\vee E^{p,q}$ rule applied to $(A \wedge B) \vee C$ allows us to use an assumption p of $A \wedge B$ to prove some proposition, and separately, an assumption q of C to prove the same proposition. In the above proof, we use the p assumption in the q branch, where it is out of scope. □

A Sequent Calculus rules

$$\frac{A \text{ atomic}}{\Gamma, A \longrightarrow A}\ \text{init} \qquad \frac{\Gamma \longrightarrow A \quad \Gamma \longrightarrow B}{\Gamma \longrightarrow A \wedge B}\ \wedge R \qquad \frac{\Gamma, A \longrightarrow B}{\Gamma \longrightarrow A \supset B}\ \supset R$$

$$\frac{\Gamma \longrightarrow A}{\Gamma \longrightarrow A \vee B}\ \vee R1 \qquad \frac{\Gamma \longrightarrow B}{\Gamma \longrightarrow A \vee B}\ \vee R2 \qquad \frac{}{\Gamma \longrightarrow \top}\ \top R$$

$$\frac{\Gamma, A \wedge B, A \longrightarrow C}{\Gamma, A \wedge B \longrightarrow C}\ \wedge L1 \qquad \frac{\Gamma, A \wedge B, B \longrightarrow C}{\Gamma, A \wedge B \longrightarrow C}\ \wedge L2 \qquad \frac{\Gamma, A \supset B \longrightarrow A \quad \Gamma, A \supset B, B \longrightarrow C}{\Gamma, A \supset B \longrightarrow C}\ \supset L$$

$$\frac{\Gamma, A \vee B, A \longrightarrow C \quad \Gamma, A \vee B, B \longrightarrow C}{\Gamma, A \vee B \longrightarrow C}\ \vee L \qquad \frac{}{\Gamma, \bot \longrightarrow C}\ \bot L$$