# Shortest Paths in a Graph

Slides by Carl Kingsford

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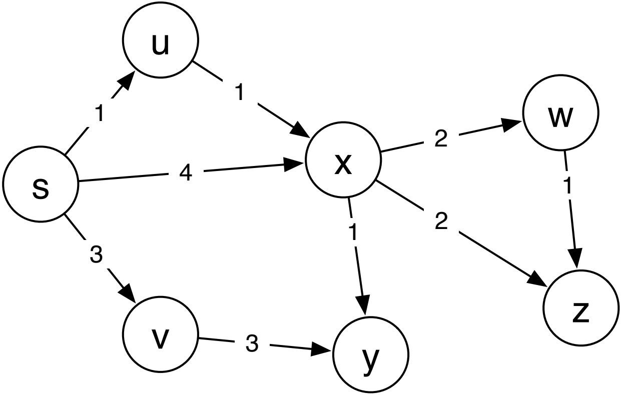
Based on/Reading: Chapter 4.5 of Kleinberg & Tardos

# Shortest Paths in a Weighted, Directed Graph

Given a directed graph $G$ with lengths $\ell_e > 0$ on each edge $e$:

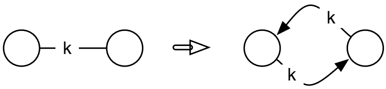



**Goal:** Find the shortest path from a given node $s$ to every other node in the graph.

# Shortest Paths

Shortest Paths. Given directed graph $G$ with $n$ nodes, and non-negative lengths on each edge, find the $n$ shortest paths from a given node $s$ to each $v_i$.

- Dijkstra's algorithm (1959) solves this problem.
- If we have an undirected graph, we can replace each undirected edge by 2 directed edges:




- If all the edge lengths are $= 1$, how can we solve this?

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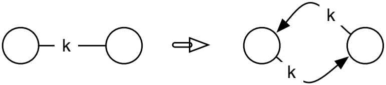
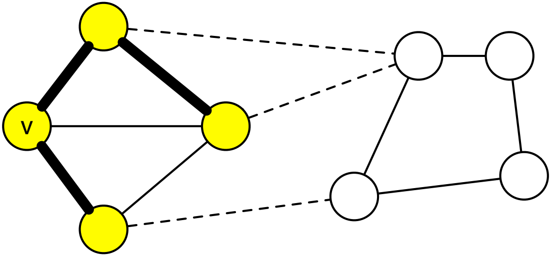



- If all the edge lengths are $= 1$, how can we solve this? BFS

# General Tree Growing

Dijkstra's algorithm is just a special case of tree growing:

- Let $T$ be the current tree $T$, and
- Maintain a list of frontier edges: the set of edges of $G$ that have one endpoint in $T$ and one endpoint not in $T$:




- Repeatedly choose a frontier edge (somehow) and add it to $T$.

## Tree Growing

```
TreeGrowing(graph G, vertex v, func nextEdge):
    T = (v,∅)
    S = set of edges incident to v
    While S is not empty:
        e = nextEdge(G, S)
        T = T + e                // add edge e to T
        S = updateFrontier(G, S, e)
    return T
```

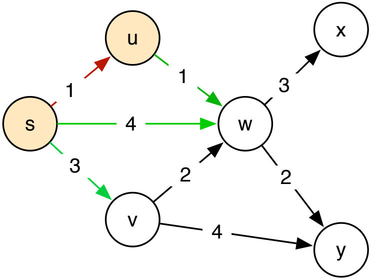
- The function `nextEdge(G, S)` returns a frontier edge from $S$.
- `updateFrontier(G, S, e)` returns the new frontier after we add edge $e$ to T.

# `nextEdge` for Shortest Path

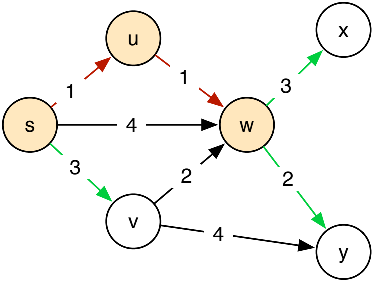
- Let $u$ be some node that we've already visited (it will be in $S$).
- Let $d(u)$ be the length of the path found for node $u \in S$.
- `nextEdge`: return the frontier edge $(u, v)$ for which $d(u) + \mathrm{length}(u, v)$ is minimized.

# Example

$d[s] = 0$; $d[u] = 1$
(green gives frontier)




$d[w] = 2$



# Analysis

**Theorem.** *Let $T$ be the set of nodes explored at some point during the algorithm. For each $u \in T$, the path to $u$ found by Dijkstra's algorithm is the shortest.*

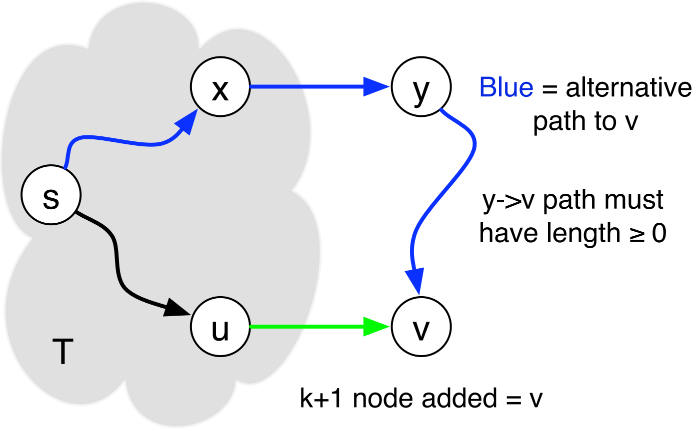
Proof. By induction on the size of $T$. **Base case**: When $|T| = 1$, the only node in $T$ is $s$, for which we've obviously found the shortest path.

**Induction Hypothesis**: Assume theorem is true when $|T| \leq k$.

Let $v$ be the $k+1$ node added using edge $(u, v)$.

Let $P_v$ be the path chosen by Dijkstra's to $v$ and let $P$ be any other path from $s$ to $v$.
Then we have the situation on the next slide. □

# Proof, cont.




The path to $v$ chosen by Dijkstra's is of length $\leq$ the alternative blue path.

# Shortest Paths $\implies$ Tree

**Theorem.** *There is some optimal set of shortest paths from source $s$ such that their union forms a tree.*

*Proof.* Dijkstra's algorithm is correct and produces a tree.

$\square$

# Implementation of Dijkstra

```
for u ∈ V do dist[u] ← ∞
H ← MakeHeap( )
u ← s # (s is an arbitrary start vertex)
while u ≠ null do
    for v ∈ Neighbors(u) do
        # If the distance is smaller than before, we have to update
        if dist[u] + d(u,v) < dist[v] then
            dist[v] ← dist[v] + d(u,v)
            if v ∉ H then
                Insert(H, v, dist[v])
            else
                ReduceKey(H, v, dist[v]) # Sift up for new key
            parent[v] ← u
    u ← DeleteMin(H)
return parent
```

# Running time of Dijkstra

Same as Prim's MST algorithm:

- Every edge is processed in the **for** loop at most once.
- In response to that processing, we may either
  1. do nothing; $O(1)$,
  2. insert a item into the heap of at most $|V|$ items; $O(\log|V|)$, or
  3. reduce the key of an item in a heap of at most $|V|$ items; $O(\log|V|)$
- Total time is therefore: $O(|E|\log|V|)$.