Applications of DFS, BFS

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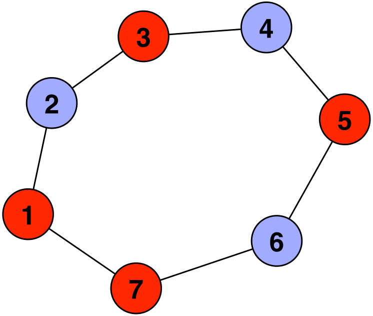
Based on/Reading: Chapter 3 of Kleinberg & Tardos

An Application of BFS

Testing Bipartiteness

Problem Determine if a graph G is bipartite.

Bipartite graphs can't contain odd cycles:



Bipartite Testing

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- Do a BFS starting from some node s.
- Color even levels "blue" and odd levels "red."
- Check each edge to see if any edge has both endpoints the same color.

Proof of Correctness for Bipartite Testing

One of two cases happen:

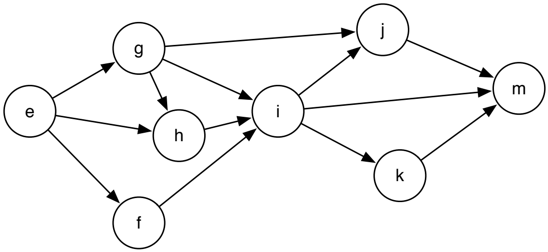
1. There is no edge of G between two nodes of the same layer. In this case, *every* edge just connects two nodes in adjacent layers. But adjacent layers are oppositely colored, so G must be bipartite.

2. There is an edge of G joining two nodes x and y of the same layer L_j. Let $z \in L_i$ be the least common ancestor of x and y in the BFS tree T.
   $z - x - y - z$ is a cycle of length $2(j - i) + 1$, which is odd, so G is not bipartite.

An Application of DFS

DAGs

- A directed, acyclic graph (DAG) is a graph that contains no directed cycles. (After leaving any node u you can never get back to u by following edges along the arrows.)
- DAGs are very useful in modeling project dependencies: Task i has to be done before task j and k which have to be done before m.



Topological Sort

Given a DAG D representing dependencies, how do you order the jobs so that when a job is started, all its dependencies are done?

Topological Sort. Given a DAG $D = (V, E)$, find a mapping f from V to $\{1, \ldots, |V|\}$), so that for every edge $(u, v) \in E$, $f(u) < f(v)$.

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How can we turn this into an algorithm?

Topological Sort Algorithm

Topological sort:

1. Let $i = 1$
2. Find a node u with no incoming edges, and let $f(u) = i$
3. Delete u from the graph
4. Increment i

Implementation: Maintain

- `Income[w]` = number of incoming edges for node w
- a list S of nodes that currently have no incoming edges.

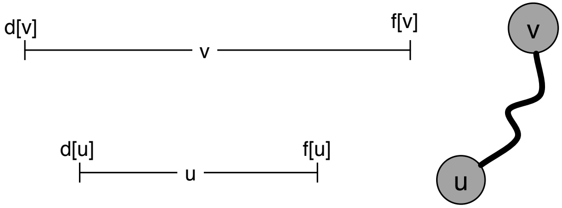
When we delete a node u, we decrement `Income[w]` for all neighbors w of u. If `Income[w]` becomes 0, we add w to S.

Discovery and Finishing Times

DFS can be used to associate 2 numbers with each node of a graph G:

- discovery time: $d[u] =$ the time at which u is first visited
- finishing time: $f[u] =$ the time at which all u and all its neighbors have been visited.

Clearly $d[u] \leq f[u]$.



Non-DFS-Trees Edges of a DAG

Let (u, v) be an edge of a DAG D. What can we say about the relationship between $f[u]$ and $f[v]$?

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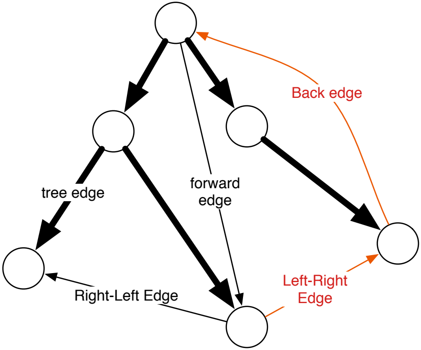

Back edges and Left-Right edges cannot occur

$\implies f[v] < f[u]$ if $(u, v) \in D$.

Topological Sort Via Finishing Times

Every edge (u, v) in a DAG has $f[v] < f[u]$.

If we list nodes from largest $f[u]$ to smallest $f[u]$ then every edge goes from left to right.

Exactly a topological sort.

So: as each node is finished, add it to the front of a linked list.