02-713 Lecture 6: Asymptotic Analysis

Notes by Carl Kingsford

1. Recall O and Ω notation

Definition 1 (O)**.** Given two real-valued functions f and g, we say $f(n) = O(g(n))$ if there exists constants $c > 0$ and $n_0 > 0$ such that

$$0 \le f(n) \le cg(n) \quad \text{for all } n \ge n_0. \tag{1}$$

Definition 2 (Ω)**.** Given two real-valued functions f and g, we say $f(n) = \Omega(g(n))$ if there exists constants $c > 0$ and $n_0 > 0$ such that

$$f(n) \ge cg(n) \ge 0 \quad \text{for all } n \ge n_0. \tag{2}$$

Definition 3 (Θ)**.** Given two real-valued functions f and g, we say $f(n) = \Theta(g(n))$ if $f(n) = O(g(n))$ and $f(n) = \Omega(g(n))$.

2. Examples and Properties

Example 1. Prove that $an^2 + bn + d = O(n^2)$ for all real a, b, d.

Solution:

$$\begin{aligned} an^2 + bn + d &\le |a|n^2 + |b|n + |d| && \text{for all } n \ge 0 && (3)\\ &\le |a|n^2 + |b|n^2 + |d|n^2 && \text{for all } n \ge 1 && (4)\\ &= (|a| + |b| + |d|)n^2. && && (5) \end{aligned}$$

Take $c = |a| + |b| + |d|$ and $n_0 = 1$. □

Example 2. Show that any degree-d polynomial $p(n) = \sum_{i=0}^{d} \alpha_i n^i$ with $\alpha_d > 0$ is $\Theta(n^d)$.

Solution: A similar argument to example 1 shows that $p(n)$ is $O(n^d)$. For all $n \ge 1$:

$$p(n) = \sum_{i=0}^{d} \alpha_i n^i \le \sum_{i=0}^{d} |\alpha_i| n^i \le \sum_{i=0}^{d} |\alpha_i| n^d = n^d \sum_{i=0}^{d} |\alpha_i|. \tag{6}$$

So take $c = \sum_{i=0}^{d} |\alpha_i|$ and $n_0 = 1$.
We now show that $p(n)$ is $\Omega(n^d)$. For $n \ge 1$, we have:

$$cn^d \le \sum_{i=0}^{d} \alpha_i n_i \iff c \le \sum_{i=0}^{d} \alpha_i n^i / n^d = \alpha_d + \sum_{i=0}^{d-1} \alpha_i n^{i-d} = \alpha_d + \sum_{i=0}^{d-1} \frac{\alpha_i}{n^{d-i}}. \tag{7}$$

Let $N = \{i : \alpha_i < 0\}$. Then we have:

$$\begin{aligned} \alpha_d + \sum_{i=0}^{d-1} \frac{\alpha_i}{n^{d-i}} &\ge \alpha_d + \sum_{i \in N} \frac{\alpha_i}{n^{d-i}} && \text{b/c we threw out the positive terms} && (8)\\ &= \alpha_d - \sum_{i \in N} \frac{|\alpha_i|}{n^{d-i}} && && (9)\\ &\ge \alpha_d - \frac{1}{n} \sum_{i \in N} |\alpha_i| && \text{b/c } 1/n \text{ is the largest dependence on } n && (10) \end{aligned}$$

We have $\frac{1}{n}\sum_{i\in N}|\alpha_i| < \alpha_d$ when $n > \frac{\sum_{i\in N}|\alpha_i|}{\alpha_d}$ and at that n and larger, (10) is strictly greater than 0. So we choose n_0 equal to that n and any c between 0 and $\alpha_d - \frac{1}{n_0}\sum_{i\in N}|\alpha_i|$. □

Example 3 (Transitivity)**.** Show that if $f(n) = O(g(n))$ and $g = O(h(n))$ then $f(n) = O(h(n))$.

Solution: By the assumptions, there are constants c_f, n_f, c_g, n_g such that $f(n) \leq c_f g(n)$ and $g(n) \leq c_g h(n)$ for all $n \geq \max\{n_f, n_g\}$. Therefore

$$f(n) \leq c_f g(n) \leq c_f c_g h(n) \tag{11}$$

and setting $c = c_f c_g$ and $n_0 = \max\{n_f, n_g\}$. □

Example 4 (Transitivity of Ω)**.** Show that if $f = \Omega(g)$ and $g = \Omega(h)$ then $f = \Omega(h)$.

Solution: a very similar argument to example 3: there are c_f, n_f such that $f(n) \geq c_f g(n)$ and there are c_g, n_g such that $f(n) \geq c_g h(n)$. Substituting in for $g(n)$ we get $f(n) \geq c_f c_g h(n)$ for all $n \geq \max\{n_f, n_g\}$. □

Example 5. For positive functions $f(n)$, $f'(n)$, $g(n)$, $g'(n)$, if $f(n) = O(f'(n))$ and $g(n) = O(g'(n))$ then $f(n)g(n) = O(f'(n)g'(n))$.

Solution: There are constants c_f, n_f, c_g, n_g as specified in the definition of O-notation:

$$f(n) \leq c_f f'(n) \qquad \text{for all } n > n_f \tag{12}$$

$$g(n) \leq c_g g'(n) \qquad \text{for all } n > n_g \tag{13}$$

So for $n > \max\{n_f, n_g\}$ both statements above hold, and we have:

$$f(n)g(n) \leq c_f f'(n) c_g g'(n) = c_f c_g f'(n) g'(n). \tag{14}$$

Taking $c = c_f c_g$ and $n_0 = \max\{n_f, n_g\}$ yields the desired statement. □

Example 6. Show that $\max\{f(n), g(n)\} = \Theta(f(n) + g(n))$ for any $f(n), g(n)$ that eventually become and stay positive.

Solution: We have $\max\{f(n), g(n)\} = O(f(n)+g(n))$ by taking $c = 1$ and n_0 equal to be the point where they become positive. We have $\max\{f(n), g(n)\} = \Omega(f(n) + g(n))$ taking the same n_0 and $c = 1/2$ since the average of positive functions $f(n)$ and $g(n)$ is always less than the max. □

Example 7. Show that $(n + a)^d = \Theta(n^d)$ when $d > 0$.

Solution: $(n + a)^d = \sum_{i=0}^{d}\binom{d}{i}a^{d-i}n^i$ by the binomial theorem. This equals $\sum_{i=0}^{d}\alpha_i n^i$ for $\alpha_i = \binom{d}{i}a^{d-i}$, which is a degree-$d$ polynomial, which is $\Theta(n^d)$ by example 2. □