Kruskal's Minimum Spanning Tree Algorithm & Union-Find Data Structures

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Reading: AD 4.5–4.6

Greedy minimum spanning tree rules

All of these greedy rules work:

1. Starting with any root node, add the frontier edge with the smallest weight. (**Prim's Algorithm**)

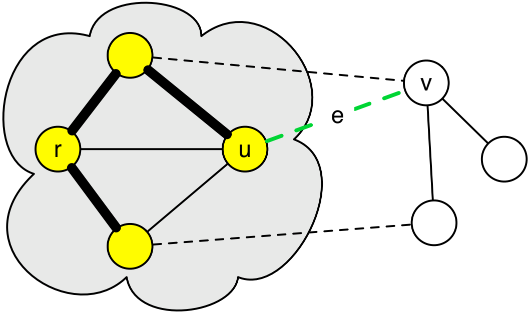
2. Add edges in increasing weight, skipping those whose addition would create a cycle. (**Kruskal's Algorithm**)

3. Start with all edges, remove them in decreasing order of weight, skipping those whose removal would disconnect the graph. (**"Reverse-Delete" Algorithm**)

Prim's Algorithm

Prim's Algorithm: Starting with any root node, add the frontier edge with the smallest weight.

Theorem. *Prim's algorithm produces a minimum spanning tree.*



S = set of nodes already in the tree when e is added

Cycle Property

Theorem (Cycle Property). *Let C be a cycle in G. Let $e = (u, v)$ be the edge with maximum weight on C. Then e is not in any MST of G.*

Suppose the theorem is false. Let T be a MST that contains e.

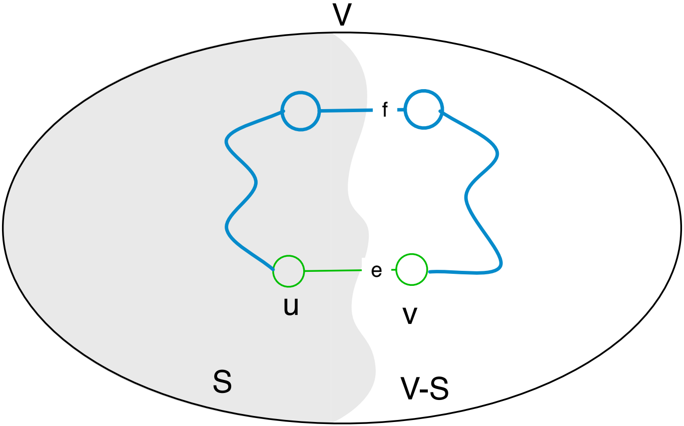
Deleting e from T partitions vertices into 2 sets:

$$S \text{ (that contains } u\text{) and } V - S \text{ (that contains } v\text{).}$$

Cycle C must have some *other* edge f that goes from S and $V - S$.

Replacing e by f produces a lower cost tree, contradicting that T is an MST.

Cycle Property, Picture



MST Property Summary

1. Cut Property: The smallest edge crossing any cut must be in all MSTs.

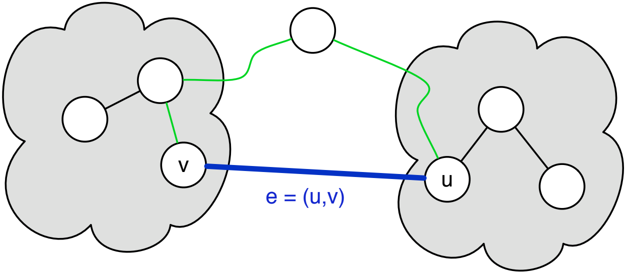
2. Cycle Property: The largest edge on any cycle is never in any MST.

Reverse-Delete Algorithm

Reverse-Delete Algorithm: Remove edges in decreasing order of weight, skipping those whose removal would disconnect the graph.

Theorem. *Reverse-Delete algorithm produces a minimum spanning tree.*

Because removing e won't disconnect the graph,
there must be another path between u and v



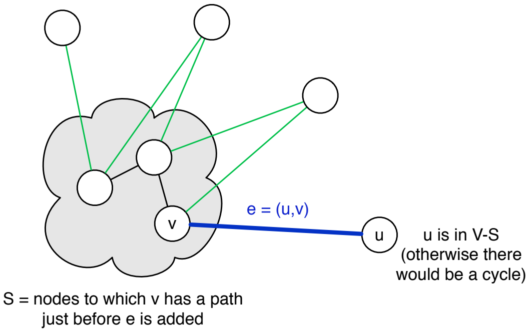
Because we're removing in order of decreasing weight,
e must be the largest edge on that cycle.

Kruskal's Algorithm

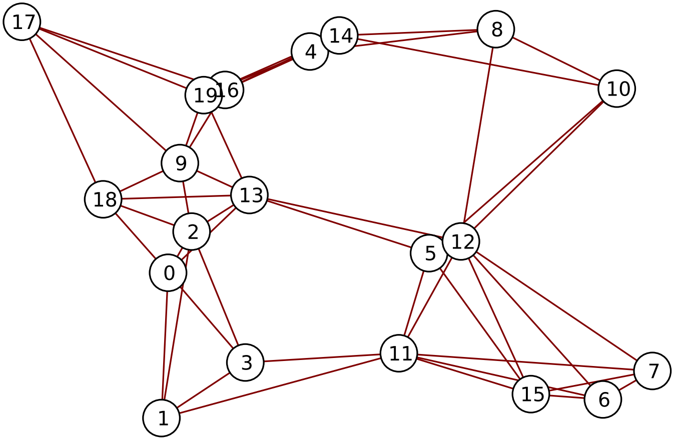
Kruskal's Algorithm: Add edges in increasing weight, skipping those whose addition would create a cycle.

Theorem. *Kruskal's algorithm produces a minimum spanning tree.*

Proof. Consider the point when edge $e = (u, v)$ is added:



Example run of Kruskal's



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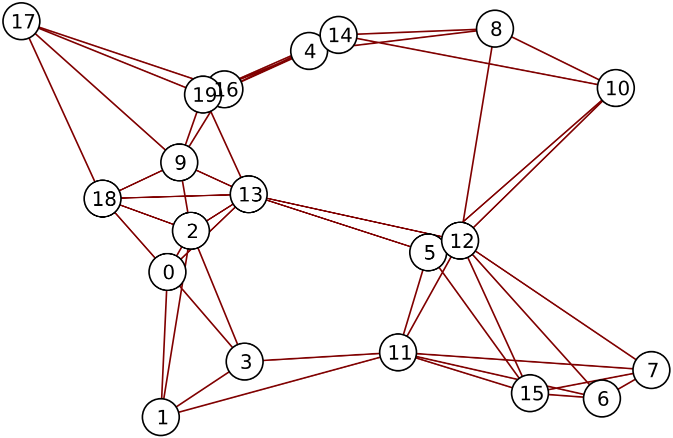

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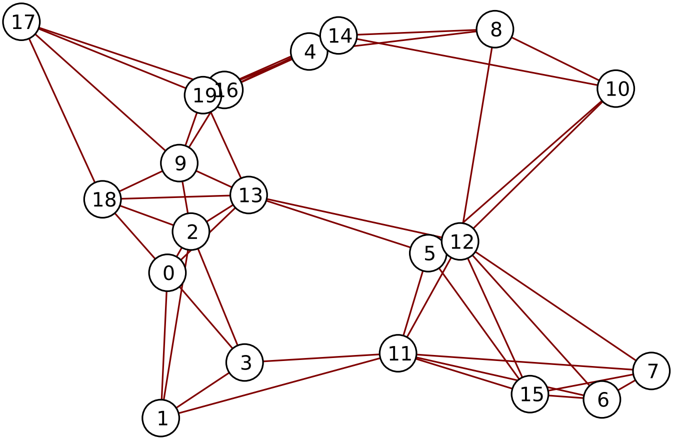

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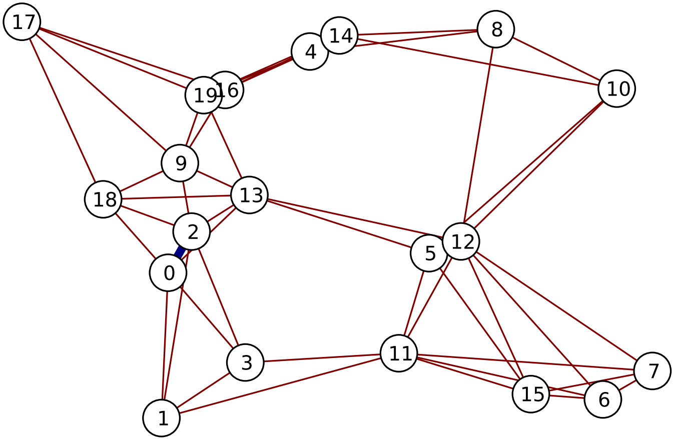

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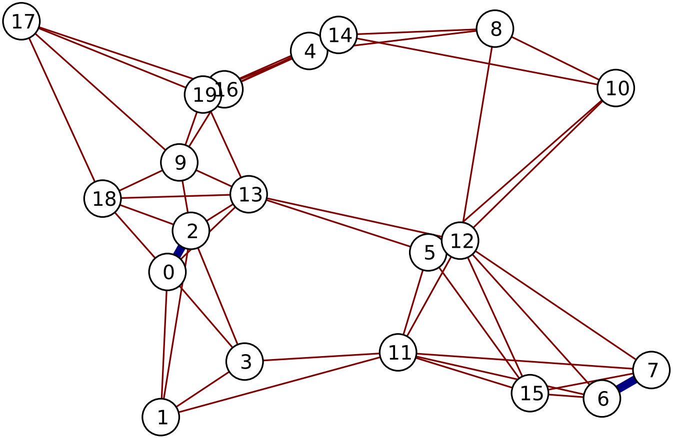

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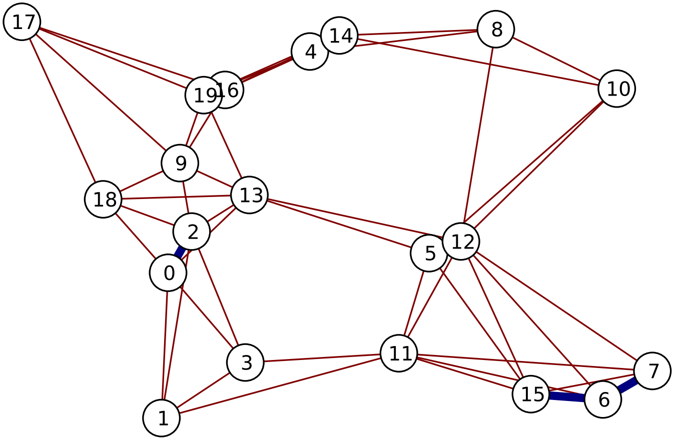

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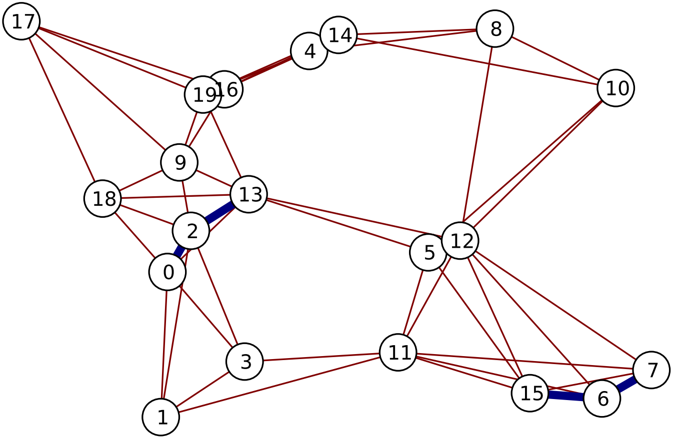

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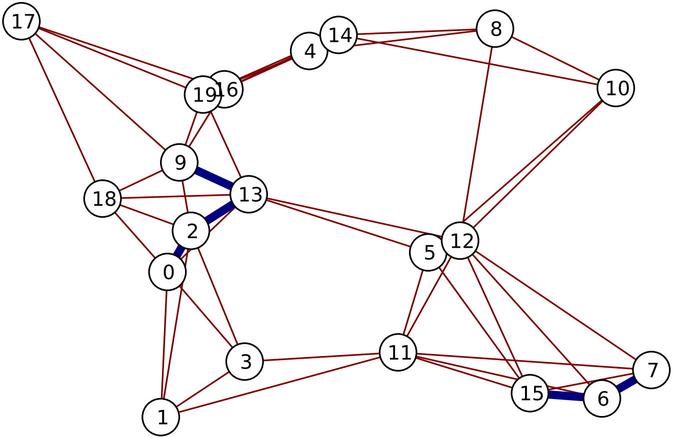

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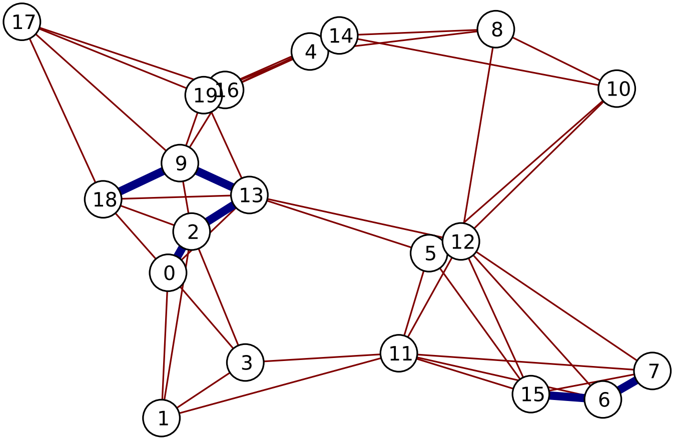

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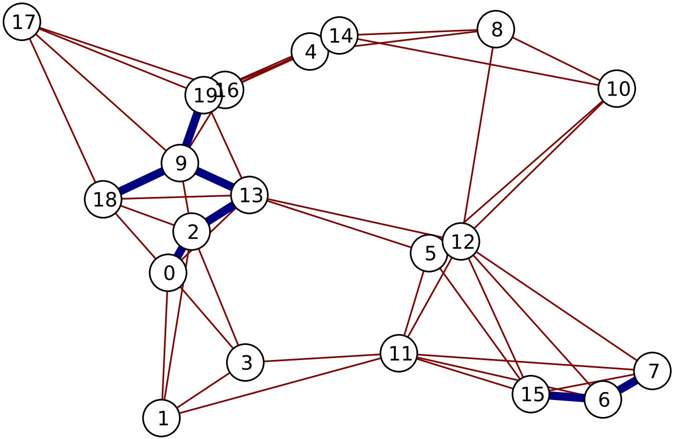

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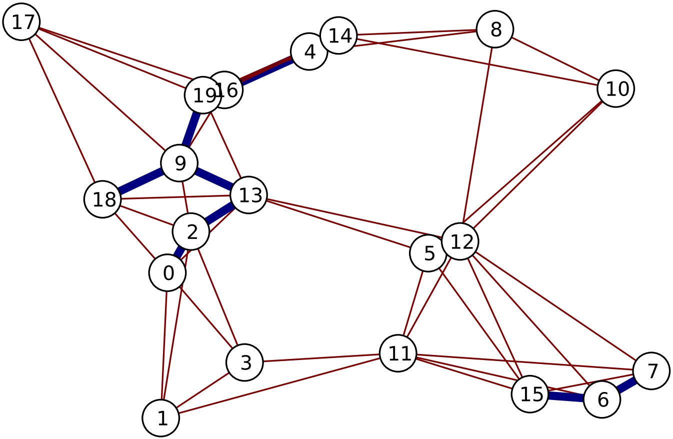

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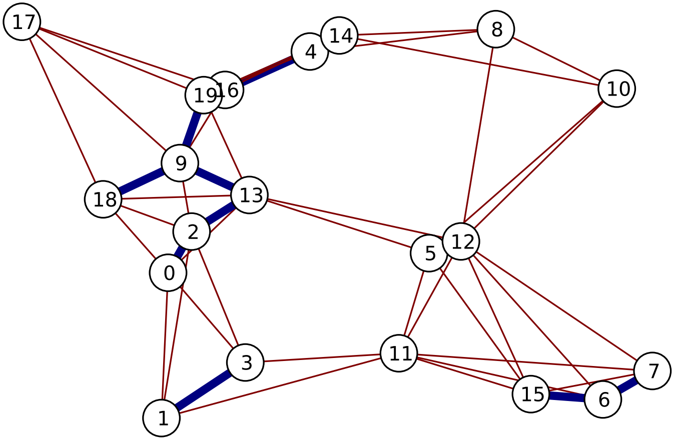

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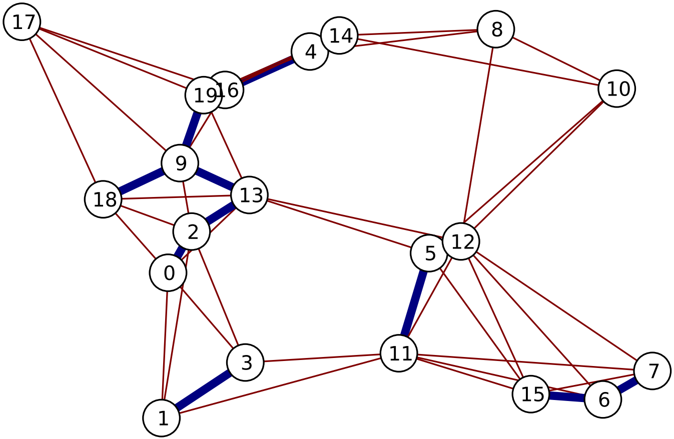

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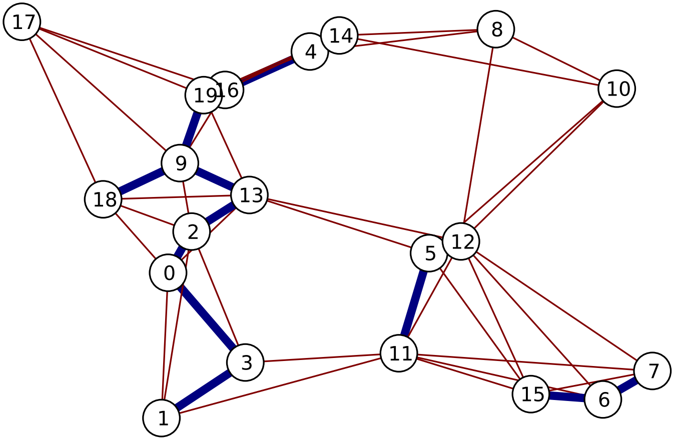

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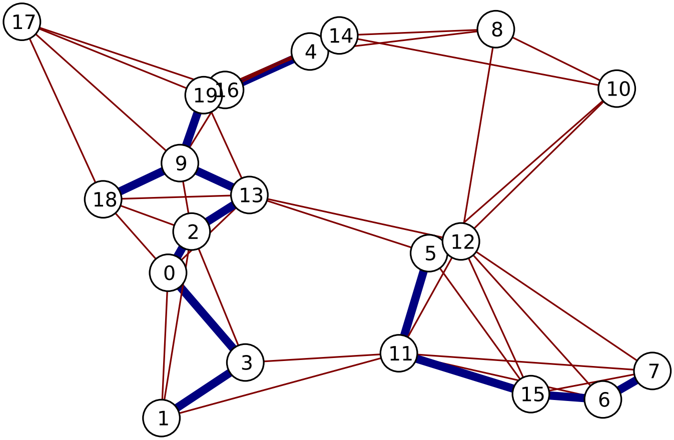

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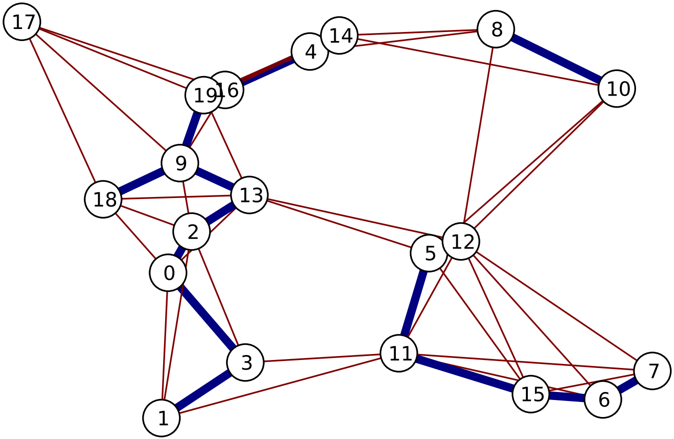

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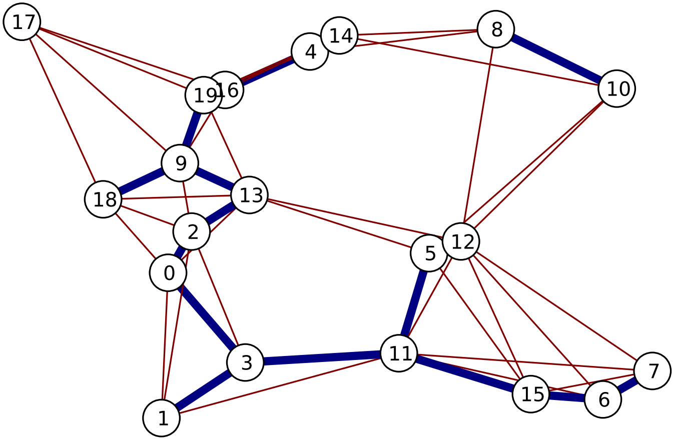

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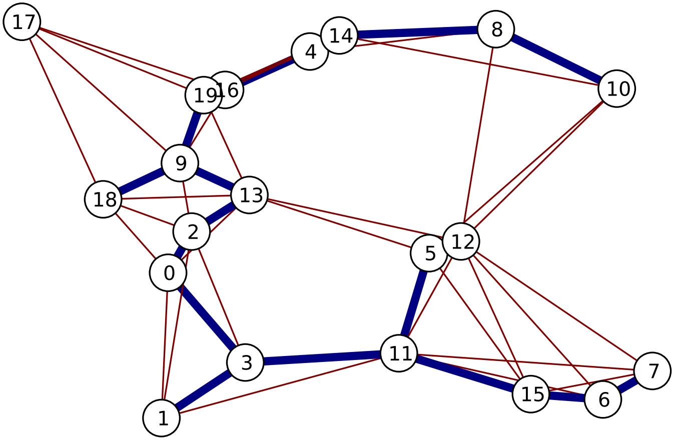

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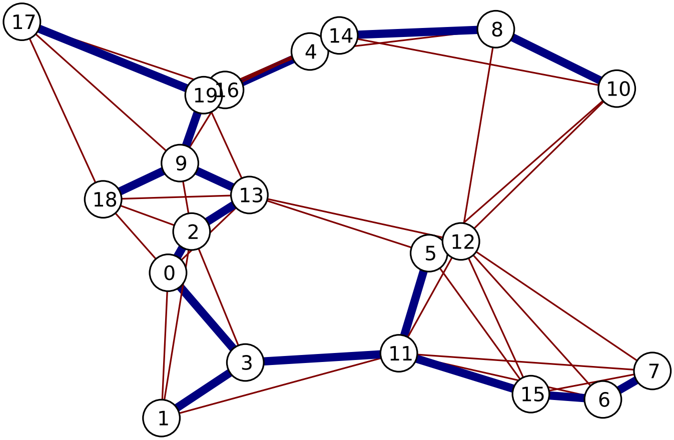
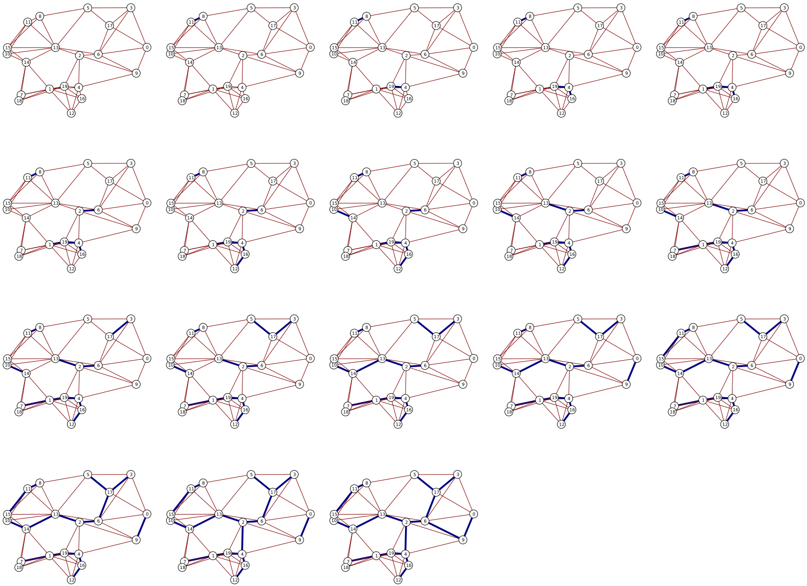


Another example



Data Structure for Kruskal's Algorithm

Kruskal's Algorithm: Add edges in increasing weight, skipping those whose addition would create a cycle.

How would we check if adding an edge $\{u, v\}$ would create a cycle?

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- Would create a cycle if u and v are already in the same component.

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- Would create a cycle if u and v are already in the same component.
- We start with a component for each node.

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- We start with a component for each node.
- Components merge when we add an edge.

Data Structure for Kruskal's Algorithm

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How would we check if adding an edge $\{u, v\}$ would create a cycle?

- Would create a cycle if u and v are already in the same component.
- We start with a component for each node.
- Components merge when we add an edge.
- Need a way to: check if u and v are in same component and to merge two components into one.

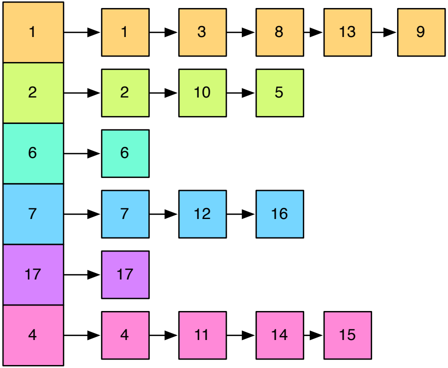
Union-Find Abstract Data Type

The Union-Find abstract data type supports the following operations:

- UF.create(S) — create the data structure containing $|S|$ sets, each containing one item from S.
- UF.find(i) — return the "name" of the set containing item i.
- UF.union(a,b) — merge the sets with names a and b into a single set.

A Union-Find Data Structure

UF Items:



UF Sizes:

1	5
2	3
6	1
7	3
17	1
4	4

UF Sets Array:

1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17
1	2	1	4	2	6	7	1	1	2	4	7	1	4	4	7	17

Implementing the union & find operations

make_union_find(S) Create data structures on previous slide.
Takes time proportional to the size of S.

find(i) Return UF.sets[i].
Takes a constant amount of time.

union(x,y) Use the "size" array to decide which set is smaller.
Assume x is smaller.
Walk down elements i in set x, setting sets[i] $=$ y.
Set size[y] $=$ size[y] $+$ size[x].

Runtime of array-based Union-Find

Theorem. *Any sequence of k union operations on a collection of n items takes time at most proportional to $k \log k$.*

Proof. After k unions, at most $2k$ items have been involved in a union. (Each union can touch at most 2 new items).

We upper bound the number of times set[v] changes for any v:

- Every time set[v] changes, the size of the set that v is in at least doubles. why?
- So, set[v] can have changed at most $\log_2(2k)$ times.

At most $2k$ items have been modified at all, and each updated at most $\log_2(2k)$ times $\implies 2k \log_2(2k)$ work. $\square$

Running time of Kruskal's algorithm

Sorting the edges: $\approx m \log m$ for m edges.

$$m \leq n^2, \text{ so } \log m < \log n^2 = 2 \log n$$

Therefore sorting takes $\approx m \log n$ time.

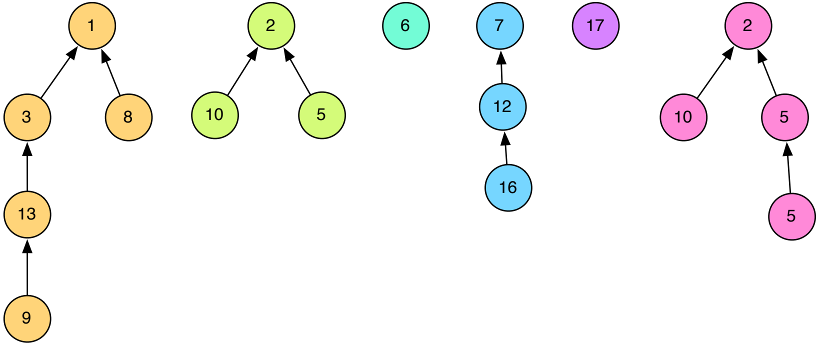
At most $2m$ "find" operations: $\approx 2m$ time.

At most $n - 1$ union operations: $\approx n \log n$ time.

$\implies$ Total running time of $\approx m \log n + 2m + n \log n$.

The biggest term is $m \log n$ since $m > n$ if the graph is connected.

Another way to implement Union-Find



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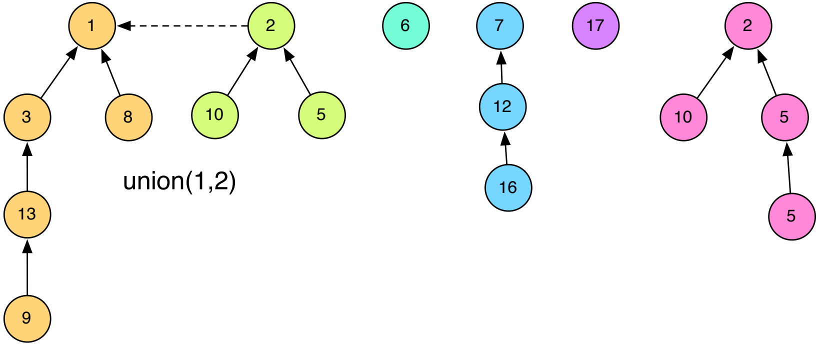

Tree-based Union-Find

make_union_find(S) Create $|S|$ trees each containing a single item and size 1. Takes time proportional to the size of S.

find(i) Follow the pointer from i to the root of its tree.

union(x,y) If the size of set x is $<$ that of y, make y point to x. Takes constant time.

Runtime of tree-based Find

Theorem. *find(i) takes time $\approx \log n$ in a tree-based union-find data structure containing n items.*

Proof. The depth of an item equals the number of times the set it was in was renamed.

The size of the set containing v at least doubles every time the name of the set containing v is changed.

The largest number of times the size can double is $\log_2 n$. □

Running time of Kruskal's algorithm using tree-based union-find

Same running time as using the array-based union-find:

- Sorting the edges: $\approx m \log n$ for m edges.
- At most $2m$ "find" operations: $\approx \log n$ time each.
- At most $n - 1$ union operations: $\approx n$ time.

$\Longrightarrow$ Total running time of $\approx m \log n + 2m \log n + n$.

The biggest term is $m \log n$ since $m > n$ if the graph is connected.