Kruskal’s Minimum Spanning Tree Algorithm & Union-Find Data Structures

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Reading: AD 4.5–4.6
Greedy minimum spanning tree rules

All of these greedy rules work:

1. Starting with any root node, add the frontier edge with the smallest weight. (Prim’s Algorithm)

2. Add edges in increasing weight, skipping those whose addition would create a cycle. (Kruskal’s Algorithm)

3. Start with all edges, remove them in decreasing order of weight, skipping those whose removal would disconnect the graph. (“Reverse-Delete” Algorithm)
**Prim’s Algorithm**

**Prim’s Algorithm**: Starting with any root node, add the frontier edge with the smallest weight.

**Theorem.** *Prim’s algorithm produces a minimum spanning tree.*

\[ S = \text{set of nodes already in the tree when } e \text{ is added} \]
**Cycle Property**

**Theorem (Cycle Property).** Let $C$ be a cycle in $G$. Let $e = (u, v)$ be the edge with maximum weight on $C$. Then $e$ is not in any MST of $G$.

Suppose the theorem is false. Let $T$ be a MST that contains $e$.

Deleting $e$ from $T$ partitions vertices into 2 sets:

$$S \text{ (that contains } u\text{)} \text{ and } V - S \text{ (that contains } v\text{)}.$$

Cycle $C$ must have some other edge $f$ that goes from $S$ and $V - S$.

Replacing $e$ by $f$ produces a lower cost tree, contradicting that $T$ is an MST.
Cycle Property, Picture

\[ V \rightarrow S \rightarrow V - S \]

\[ u \rightarrow e \rightarrow v \]

\[ f \]
MST Property Summary

1. **Cut Property:** The smallest edge crossing any cut must be in all MSTs.

2. **Cycle Property:** The largest edge on any cycle is never in any MST.
Reverse-Delete Algorithm

**Reverse-Delete Algorithm**: Remove edges in decreasing order of weight, skipping those whose removal would disconnect the graph.

**Theorem.** *Reverse-Delete algorithm produces a minimum spanning tree.*

Because removing e won't disconnect the graph, there must be another path between u and v.

Because we're removing in order of decreasing weight, e must be the largest edge on that cycle.
Kruskal’s Algorithm

**Kruskal’s Algorithm**: Add edges in increasing weight, skipping those whose addition would create a cycle.

**Theorem.** *Kruskal’s algorithm produces a minimum spanning tree.*

**Proof.** Consider the point when edge \( e = (u, v) \) is added:

\[ S = \text{nodes to which } v \text{ has a path just before } e \text{ is added} \]

\[ u \text{ is in } V-S \] (otherwise there would be a cycle)

\[ e = (u,v) \]
Example run of Kruskal’s
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Example run of Kruskal’s
Example run of Kruskal’s
Example run of Kruskal’s algorithm.
Another example
**Data Structure for Kruskal’s Algorithm**

**Kruskal’s Algorithm**: Add edges in increasing weight, **skipping** those whose addition would create a cycle.

How would we check if adding an edge \( \{u, v\} \) would create a cycle?
Data Structure for Kruskal’s Algorithm

**Kruskal’s Algorithm**: Add edges in increasing weight, skipping those whose addition would create a cycle.

How would we check if adding an edge \( \{u, v\} \) would create a cycle?

- **Would create a cycle if** \( u \) and \( v \) are already in the same component.
**Data Structure for Kruskal’s Algorithm**

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How would we check if adding an edge \( \{u, v\} \) would create a cycle?

- Would create a cycle if \( u \) and \( v \) are already in the same component.
- We start with a component for each node.
Data Structure for Kruskal’s Algorithm

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How would we check if adding an edge \( \{u, v\} \) would create a cycle?

- Would create a cycle if \( u \) and \( v \) are already in the same component.
- We start with a component for each node.
- Components merge when we add an edge.
Data Structure for Kruskal’s Algorithm

Kruskal’s Algorithm: Add edges in increasing weight, skipping those whose addition would create a cycle.

How would we check if adding an edge \{u, v\} would create a cycle?

- Would create a cycle if \(u\) and \(v\) are already in the same component.
- We start with a component for each node.
- Components merge when we add an edge.
- Need a way to: check if \(u\) and \(v\) are in same component and to merge two components into one.
Union-Find Abstract Data Type

The Union-Find abstract data type supports the following operations:

- **UF.create(S)** — create the data structure containing $|S|$ sets, each containing one item from $S$.

- **UF.find(i)** — return the “name” of the set containing item $i$.

- **UF.union(a,b)** — merge the sets with names $a$ and $b$ into a single set.
A Union-Find Data Structure

UF Items:

UF Sizes:

UF Sets Array:
Implementing the union & find operations

\textbf{make\_union\_find}(S) Create data structures on previous slide. Takes time proportional to the size of S.

\textbf{find}(i) Return UF.sets[i]. Takes a constant amount of time.

\textbf{union}(x,y) Use the “size” array to decide which set is smaller. Assume x is smaller.
Walk down elements i in set x, setting sets[i] = y. Set size[y] = size[y] + size[x].
Runtime of array-based Union-Find

**Theorem.** Any sequence of $k$ union operations on a collection of $n$ items takes time at most proportional to $k \log k$.

**Proof.** After $k$ unions, at most $2k$ items have been involved in a union. (Each union can touch at most 2 new items).

We upper bound the number of times set[$v$] changes for any $v$:

- Every time set[$v$] changes, the size of the set that $v$ is in at least doubles. why?
- So, set[$v$] can have changed at most $\log_2(2k)$ times.

At most $2k$ items have been modified at all, and each updated at most $\log_2(2k)$ times $\implies 2k \log_2(2k)$ work.
Running time of Kruskal’s algorithm

Sorting the edges: \( \approx m \log m \) for \( m \) edges.

\[ m \leq n^2, \text{ so } \log m < \log n^2 = 2 \log n \]

Therefore sorting takes \( \approx m \log n \) time.

At most \( 2m \) “find” operations: \( \approx 2m \) time.

At most \( n - 1 \) union operations: \( \approx n \log n \) time.

\[ \implies \text{Total running time of } \approx m \log n + 2m + n \log n. \]

The biggest term is \( m \log n \) since \( m > n \) if the graph is connected.
Another way to implement Union-Find
Another way to implement Union-Find

union(1,2)
Tree-based Union-Find

\textit{make\_union\_find}(S) Create $|S|$ trees each containing a single item and size 1. Takes time proportional to the size of $S$.

\textit{find}(i) Follow the pointer from $i$ to the root of its tree.

\textit{union}(x,y) If the size of set $x$ is $<$ that of $y$, make $y$ point to $x$. Takes constant time.
Runtime of tree-based Find

**Theorem.** *find(i)* takes time $\approx \log n$ in a tree-based union-find data structure containing $n$ items.

**Proof.** The depth of an item equals the number of times the set it was in was renamed.

The size of the set containing $v$ at least doubles every time the name of the set containing $v$ is changed.

The largest number of times the size can double is $\log_2 n$. □
Running time of Kruskal’s algorithm using tree-based union-find

Same running time as using the array-based union-find:

- Sorting the edges: \( \approx m \log n \) for \( m \) edges.
- At most \( 2m \) “find” operations: \( \approx \log n \) time each.
- At most \( n - 1 \) union operations: \( \approx n \) time.

\[ \implies \text{Total running time of } \approx m \log n + 2m \log n + n. \]

The biggest term is \( m \log n \) since \( m > n \) if the graph is connected.