CMSC 451: Subset Sum & Knapsack

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Based on Section 6.4 of *Algorithm Design* by Kleinberg & Tardos.

Subset Sum

Subset Sum

Given:

- an integer bound W, and
- a collection of n items, each with a positive, integer weight w_i,

find a subset S of items that:

$$\text{maximizes } \sum_{i \in S} w_i \text{ while keeping } \sum_{i \in S} w_i \leq W.$$

Motivation: you have a CPU with W free cycles, and want to choose the set of jobs (each taking w_i time) that minimizes the number of idle cycles.

Assumption

We assume W and each w_i is an integer.

Optimal Notation

Notation:

- Let S^* be an optimal choice of items (e.g. a set $\{1,4,8\}$).
- Let $OPT(n, W)$ be the value of the optimal solution.
- We design an dynamic programming algorithm to compute $OPT(n, W)$.

Subproblems:

- To compute $OPT(n, W)$: We need the optimal value for subproblems consisting of the first j items for every knapsack size $0 \leq w \leq W$.
- Denote the optimal value of these subproblems by $OPT(j, w)$.

Recurrence

Recurrence: How do we compute $OPT(j, w)$ given solutions to smaller subproblems?

$$OPT(j, W) = \max \begin{cases} OPT(j-1, W) & \text{if } j \notin S^* \\ w_j + OPT(j-1, W - w_j) & \text{if } j \in S^* \end{cases}$$

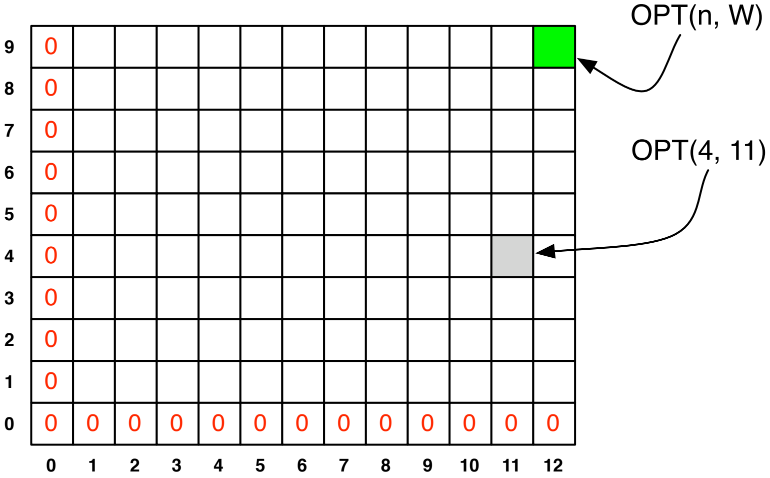
Special case: if $w_j > W$ then $OPT(j, W) = OPT(j-1, W)$.

Another way to write it...

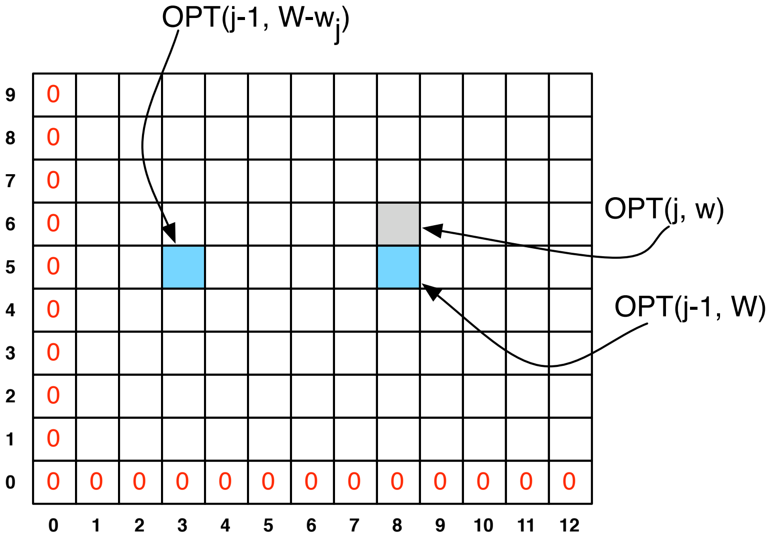
$$OPT(j, W) = \begin{cases} OPT(j-1, W) & \text{if } w_j > W \\ \max \begin{cases} OPT(j-1, W) \\ w_j + OPT(j-1, W - w_j) \end{cases} & \begin{array}{l} \text{if } j \notin S^* \\ \text{if } j \in S^* \end{array} \end{cases}$$

Note: Because we don't know the answer to the blue questions, we have to try both.

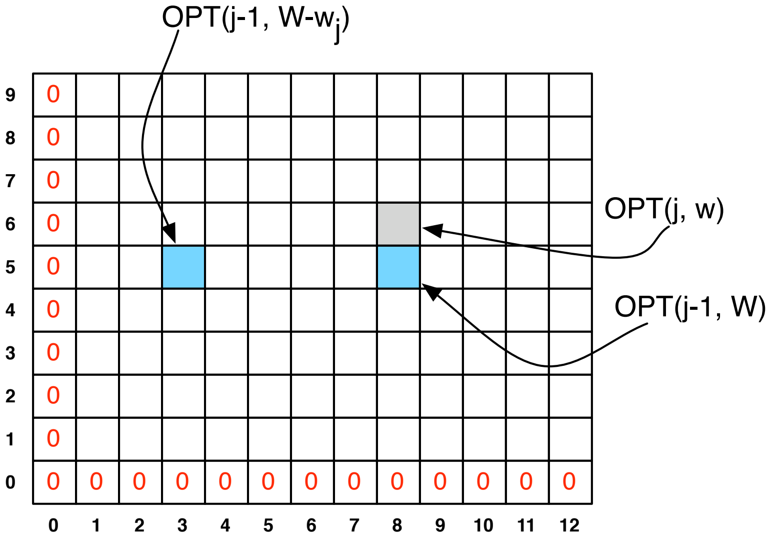
The table of solutions



Filling in a box using smaller problems

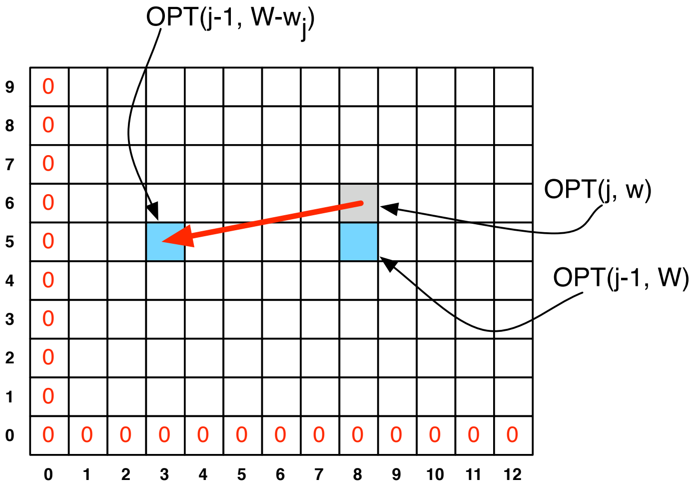


Filling in a box using smaller problems



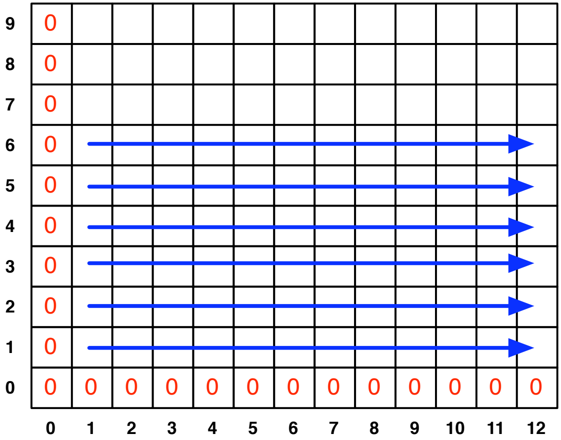
Remembering Which Subproblem Was Used

When we fill in the gray box, we also record which subproblem was chosen in the maximum:



Filling in the Matrix

Fill matrix from bottom to top, left to right.



When you are filling in box, you only need to look at boxes you've already filled in.

Pseudocode

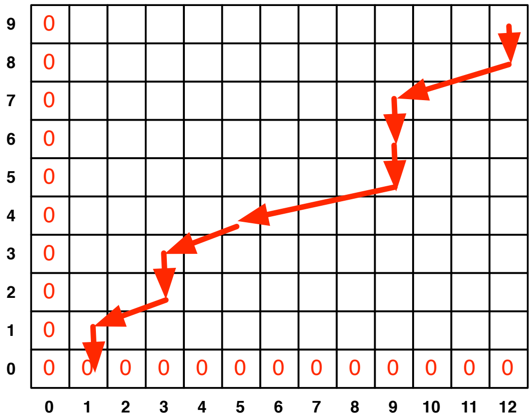
```
SubsetSum(n, W):
   Initialize M[0,w] = 0 for each w = 0,...,W
   Initialize M[i,0] = 0 for each i = 1,...,n

   For i = 1,...,n:               for every row
      For w = 0,...,W:            for every column
         If w[i] > w:             case where item can't fit
            M[i,w] = M[i-1,w]

         M[i,w] = max(            which is best?
            M[i-1,w],
            w[j] + M[i-1, W-w[j]]
         )
   Return M[n,W]
```

Finding The Choice of Items

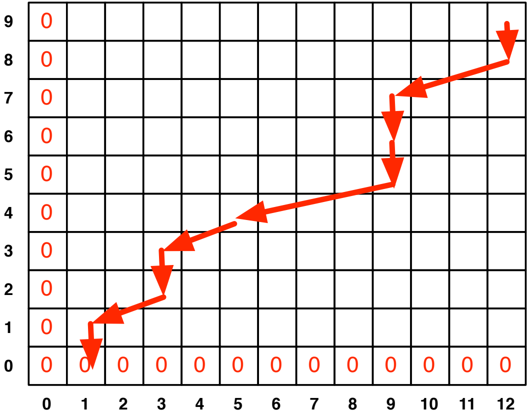
Follow the arrows backward starting at the top right:



Which items does this path imply?

Finding The Choice of Items

Follow the arrows backward starting at the top right:



Which items does this path imply? 8, 5, 4, 2

Runtime

Runtime:

- $O(nW)$ to fill in the matrix.
- $O(n)$ time to follow the path backwards.
- Total running time is $O(nW)$.

This is pseudo-polynomial because it depends on the size of the input numbers.

Knapsack

Knapsack

Given:

- a bound W, and
- a collection of n items, each with a weight w_i,
- a value v_i for each weight

Find a subset S of items that:

$$\text{maximizes } \sum_{i \in S} v_i \text{ while keeping } \sum_{i \in S} w_i \leq W.$$

Difference from Subset Sum: want to maximize value instead of weight.

How can we solve Knapsack?

Knapsack

Subset Sum:

$$OPT(j, W) = \max \begin{cases} OPT(j-1, W) & \text{if } j \notin S^* \\ w_j + OPT(j-1, W - w_j) & \text{if } j \in S^* \end{cases}$$

Knapsack

Subset Sum:

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Knapsack:

$$OPT(j, W) = \max \begin{cases} OPT(j-1, W) & \text{if } j \notin S^* \\ v_j + OPT(j-1, W - w_j) & \text{if } j \in S^* \end{cases}$$

Fractional Knapsack

0-1 Knapsack

You're presented with n, where item i has value v_i and size w_i. You have a knapsack of size W, and you want to take the items S so that

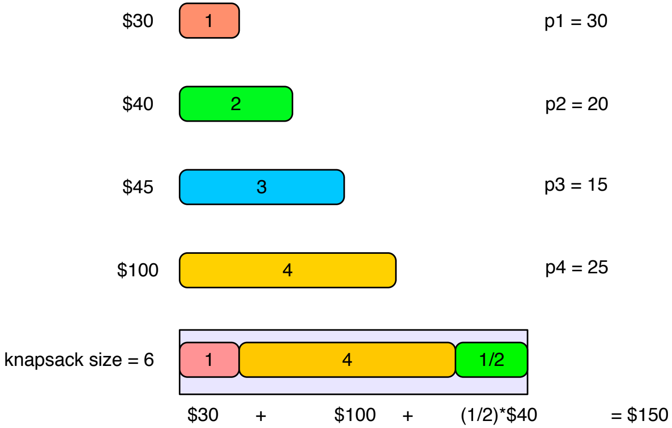
- $\sum_{i \in S} v_i$ is maximized, and
- $\sum_{i \in S} w_i \leq W$.

This is a hard problem. However, if we are allowed to take fractions of items we can do it with a simple greedy algorithm:

- Value of a fraction f of item i is $f \cdot v_i$
- Weight of a fraction f is $f \cdot w_i$.

Knapsack Example

Idea: Sort the items by $p_i = v_i / w_i$
Larger v_i is better, smaller w_i is better.



0-1 Knapsack

This greedy algorithm doesn't work for 0-1 knapsack, where we can't take part of an item: