# *CMSC 451: Reductions & NP-completeness*


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Based on Section 8.1 of *Algorithm Design* by Kleinberg & Tardos.

# Reductions as tool for hardness

We want prove some problems are computationally difficult.

As a first step, we settle for relative judgements:

Problem $X$ is at least as hard as problem $Y$

To prove such a statement, we reduce problem $Y$ to problem $X$:

*If you had a black box that can solve instances of problem $X$, how can you solve any instance of $Y$ using polynomial number of steps, plus a polynomial number of calls to the black box that solves $X$?*

# Polynomial Reductions

- If problem $Y$ can be reduced to problem $X$, we denote this by $Y \leq_P X$.

- This means "$Y$ is polynomal-time reducible to $X$."

- It also means that $X$ is at least as hard as $Y$ because if you can solve $X$, you can solve $Y$.

- Note: We reduce *to* the problem we want to show is the harder problem.

# Polynomial Problems

Suppose:

- $Y \leq_P X$, and
- there is an polynomial time algorithm for $X$.

Then, there is a polynomial time algorithm for $Y$.

Why?

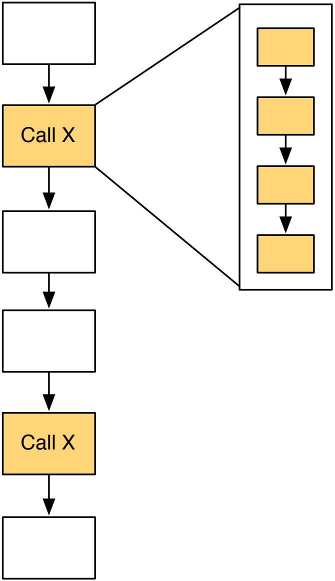
# Polynomial Problems

Suppose:

- $Y \leq_P X$, and
- there is an polynomial time algorithm for $X$.

Then, there is a polynomial time algorithm for $Y$.

Why? Because polynomials compose.



# We've Seen Reductions Before

Examples of Reductions:

- Max Bipartite Matching $\leq_P$ Max Network Flow.
- Image Segmentation $\leq_P$ Min-Cut.
- Survey Design $\leq_P$ Max Network Flow.
- Disjoint Paths $\leq_P$ Max Network Flow.

# Reductions for Hardness

**Theorem**

*If $Y \leq_P X$ and $Y$ cannot be solved in polynomial time, then $X$ cannot be solved in polynomial time.*

Why? If we *could* solve $X$ in polynomial time, then we'd be able to solve $Y$ in polynomial time using the reduction, contradicting the assumption.

So: If we could find one hard problem $Y$, we could prove that another problem $X$ is hard by reducing $Y$ to $X$.

# Vertex Cover

**Def.** A vertex cover of a graph is a set $S$ of nodes such that every edge has at least one endpoint in $S$.

In other words, we try to "cover" each of the edges by choosing at least one of its vertices.

**Vertex Cover**

Given a graph $G$ and a number $k$, does $G$ contain a vertex cover of size at most $k$.

# Independent Set to Vertex Cover

**Independent Set**

Given graph $G$ and a number $k$, does $G$ contain a set of at least $k$ independent vertices?

Can we reduce independent set to vertex cover?

**Vertex Cover**

Given a graph $G$ and a number $k$, does $G$ contain a vertex cover of size at most $k$.

# Relation btw Vertex Cover and Indep. Set

**Theorem**

*If $G = (V, E)$ is a graph, then $S$ is an independent set $\iff$ $V - S$ is a vertex cover.*

*Proof.* $\implies$ Suppose $S$ is an independent set, and let $e = (u, v)$ be some edge. Only one of $u, v$ can be in $S$. Hence, at least one of $u, v$ is in $V - S$. So, $V - S$ is a vertex cover.

$\impliedby$ Suppose $V - S$ is a vertex cover, and let $u, v \in S$. There can't be an edge between $u$ and $v$ (otherwise, that edge wouldn't be covered in $V - S$). So, $S$ is an independent set. □

# Independent Set $\leq_P$ Vertex Cover

Independent Set $\leq_P$ Vertex Cover

To show this, we change any instance of Independent Set into an instance of Vertex Cover:

- Given an instance of Independent Set $\langle G, k \rangle$,
- We ask our Vertex Cover black box if there is a vertex cover $V - S$ of size $\leq |V| - k$.

By our previous theorem, $S$ is an independent set iff $V - S$ is a vertex cover. If the Vertex Cover black box said:

*yes: then $S$ must be an independent set of size $\geq k$.*
*no: then there is no vertex cover $V - S$ of size $\leq |V| - k$, hence there is no independent set of size $\geq k$.*

# Vertex Cover $\leq_P$ Independent Set

Actually, we also have:

Vertex Cover $\leq_P$ Independent Set

*Proof.* To decide if $G$ has an vertex cover of size $k$, we ask if it has an independent set of size $n - k$. $\square$

So: Vertex Cover and Independent Set are equivalently difficult.

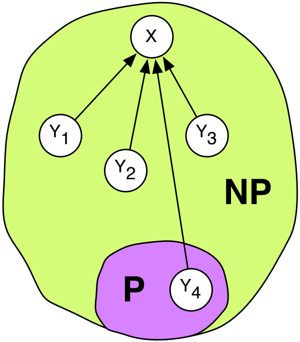
# NP-completeness

**Def.** We say $X$ is NP-complete if:

- $X \in \mathbf{NP}$
- for all $Y \in \mathbf{NP}$, $Y \leq_P X$.

If these hold, then $X$ can be used to solve every problem in **NP**.

Therefore, $X$ is definitely at least as hard as every problem in **NP**.



# NP-completeness and P=NP

**Theorem**

*If $X$ is NP-complete, then $X$ is solvable in polynomial time if and only if* $\mathbf{P} = \mathbf{NP}$.

*Proof.* If $\mathbf{P} = \mathbf{NP}$, then $X$ can be solved in polytime.

Suppose $X$ is solvable in polytime, and let $Y$ be any problem in **NP**. We can solve $Y$ in polynomial time: reduce it to $X$.

Therefore, every problem in **NP** has a polytime algorithm and $\mathbf{P} = \mathbf{NP}$.

# Reductions and NP-completeness

**Theorem**

*If $Y$ is NP-complete, and*

1. *$X$ is in NP*
2. *$Y \leq_P X$*

*then $X$ is NP-complete.*

In other words, we can prove a new problem is NP-complete by reducing some other NP-complete problem to it.

*Proof.* Let $Z$ be any problem in **NP**. Since $Y$ is NP-complete, $Z \leq_P Y$. By assumption, $Y \leq_P X$. Therefore: $Z \leq_P Y \leq_P X$. □

# Some First NP-complete problem

We need to find some first NP-complete problem.

Finding the first NP-complete problem was the result of the Cook-Levin theorem.

We'll deal with this later. For now, trust me that:

- Independent Set is a *packing problem* and is NP-complete.
- Vertex Cover is a *covering problem* and is NP-complete.

# Set Cover

Another very general and useful covering problem:

**Set Cover**

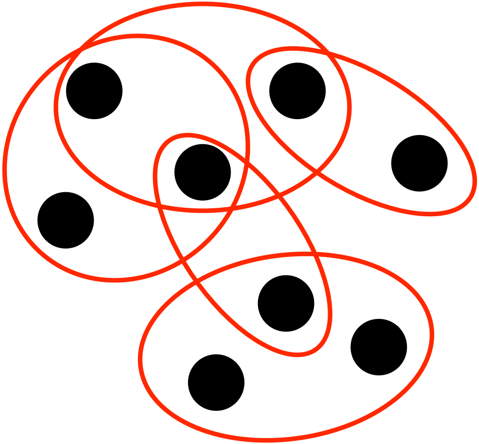
Given a set $U$ of elements and a collection $S_1, \ldots, S_m$ of subsets of $U$, is there a collection of at most $k$ of these sets whose union equals $U$?

We will show that

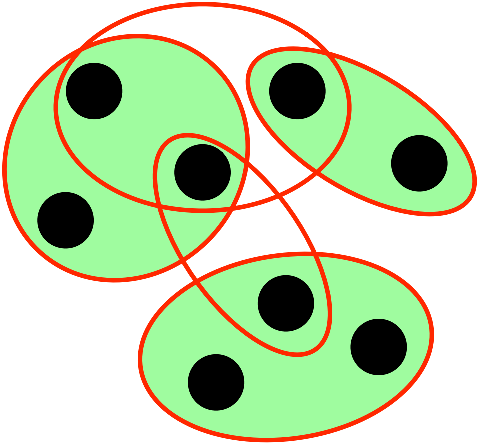
$$\textsc{Set Cover} \in NP$$
$$\textsc{Vertex Cover} \leq_P \textsc{Set Cover}$$

And therefore that Set Cover is NP-complete.

# Set Cover, Figure



# Set Cover, Figure



# Vertex Cover $\leq_P$ Set Cover

**Thm.** Vertex Cover $\leq_P$ Set Cover

*Proof.* Let $G = (V, E)$ and $k$ be an instance of VERTEX COVER. Create an instance of SET COVER:

- $U = E$
- Create a $S_u$ for for each $u \in V$, where $S_u$ contains the edges adjacent to $u$.

$U$ can be covered by $\leq k$ sets iff $G$ has a vertex cover of size $\leq k$.

Why? If $k$ sets $S_{u_1}, \ldots, S_{u_k}$ cover $U$ then every edge is adjacent to at least one of the vertices $u_1, \ldots, u_k$, yielding a vertex cover of size $k$.

If $u_1, \ldots, u_k$ is a vertex cover, then sets $S_{u_1}, \ldots, S_{u_k}$ cover $U$. $\square$

# Last Step:

We still have to show that Set Cover is in **NP**!

The certificate is a list of $k$ sets from the given collection.

We can check in polytime whether they cover all of $U$.

Since we have a certificate that can be checked in polynomial time, Set Cover is in **NP**.

# Summary

You can prove a problem is NP-complete by reducing a known NP-complete problem to it.

We know the following problems are NP-complete:

- Vertex Cover
- Independent Set
- Set Cover

Warning: You should reduce the *known* NP-complete problem to the problem you are interested in. (You *will* mistakenly do this backwards sometimes.)